Asymptotic Behavior of Scattering Amplitudes in the Relativistic Eikonal Approximation*

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Using techniques developed in an earlier paper, a new one-parameter eikonal representation is derived and its high-energy behavior is studied. It is shown that by also taking t large, a Regge-like behavior is obtained in the crossed channel. The procedure of first making s large and then t large is investigated more closely by comparing the exact fourth-order amplitudes with those obtained in the relativistic eikonal approximation. It is found that the latter predicts the correct behavior in the region of large s, for all values of t. When a similar procedure is applied to the full $M_{\rm eik}(s,t)$, one obtains in an appropriate domain an asymptotic amplitude having a dual character: The bound states appear as poles in the s channel and lie on the Regge trajectory in the t channel. In particular, for electron-positron scattering the corresponding energy levels coincide with those obtained recently by Brezin *et al.* The asymptotic behavior of the eikonal function itself is also studied.

I. INTRODUCTION AND SUMMARY

In an earlier paper[1] it was shown how a covariant generalization of the eikonal approximation, long familiar in potential scattering,[2] could be derived from quantum field theory. A number of other authors have obtained similar results, using different methods.[3–5] More recently, some progress has been made in including radiative corrections to relativistic eikonal-type approximations,[6] and in applying such approximations to the calculation of bound states.[7]

In the present paper we consider an alternative form of relativistic eikonal approximation (REA) which has the virtue of involving an auxiliary integration over a single parameter (in contrast to the four-dimensional integration over a space-time point x, encountered in I) and which facilitates study of questions regarding asymptotic behavior. In addition, we study some aspects of the high-energy behavior of the amplitudes obtained with the REA in perturbation theory, where the results can be compared with the exact asymptotic behavior.[8] Such a study also sheds some light on the possibility of obtaining Regge-like behavior in the crossed channel by starting with an eikonal form in the direct channel.

We first review briefly the results of I which are of interest here. Consider the amplitude for scattering of spinless particles a and b, both of which can emit scalar mesons of mass μ with coupling constant g. Let M_n denote the sum of all those Feynman diagrams in which exactly n mesons are exchanged, i.e., $M_n(s,t)$ is the sum of all diagrams of order $2n$ which involve no radiative corrections. It can be written as an integral over virtual meson momenta k_i ($i=1, 2, \ldots, n$) whose integrand, apart from a factor $\delta(q-\sum k_i)$, is a sum of products of meson propagators $(k_i^2-m^2+i\epsilon)^{-1}$ and particle propagators $[(p\pm K)^2-m^2+i\epsilon]^{-1}$. Here p is an external four-momentum and $K=\sum' k_j$, where the prime denotes a partial sum. If the integrand is written in a suitably symmetrized form, and if in the resulting expression we drop terms in the propagators $[(p\pm K)^2-m^2+i\epsilon]^{-1}=(\pm 2p\cdot K+K^2+i\epsilon)^{-1}$ which are quadratic in the internal momenta, i.e., make the replacement

$$(\pm 2p\cdot K+K^2+i\epsilon)^{-1}\to(\pm 2p\cdot K+i\epsilon)^{-1}, \quad (1.1)$$

then the corresponding value of the integral, designated by $M_n^{\rm eik}$, may be written in a compact form and the sum on n may be carried out to yield

$$M_{\rm eik}(s,t)=g^2\int d^4x\, e^{-iq\cdot x}\Delta_F(x;\mu)\frac{e^{i\chi}-1}{\chi}. \quad (1.2)$$

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[8] After completing this work, we received a preprint by G. Tiktopoulos and S. Treiman [Phys. Rev. D **2**, 805 (1970)], who compare, in each order of perturbation theory, the asymptotic behavior of each Feynman graph, with and without the eikonal approximation. These authors show that the REA gives, for each graph, the correct leading high-s behavior. In the sum of the nth-order graphs, the leading logarithmic contributions cancel; although for each individual graph the REA gives the remaining $s^{-(n-1)}$ term with the wrong coefficient, it predicts correctly the coefficient of $s^{-(n-1)}$ in the sum. [See also B. M. Barbashov and V. V. Nesterenko, Dubna Report No. PZ-4900, 1970 (unpublished).]

Here Δ_F is the meson propagator and

$$\chi=\frac{g^2}{(2\pi)^4}\int d^4k\,\frac{e^{ik\cdot x}}{k^2-\mu^2+i\epsilon}\times\left(\frac{1}{-2p_a\cdot k+i\epsilon}+\frac{1}{+2p_a'\cdot k+i\epsilon}\right)\times\left(\frac{1}{+2p_b\cdot k+i\epsilon}+\frac{1}{-2p_b'\cdot k+i\epsilon}\right),\quad(1.3)$$

with p_a, p_b the initial momenta, p_a', p_b' the final momenta, $q=p_a-p_a'$, $t=q^2$, and $s=(p_a+p_b)^2$.

As already mentioned in I, the computation goes through without any essential change if the diagonal terms in $K^2=\sum' k_i\cdot k_j=\sum_{i\neq j}' k_i\cdot k_j+\sum_i' k_i^2$ are kept, i.e., if we make the replacement

$$(\pm 2p\cdot K+K^2+i\epsilon)^{-1}\rightarrow(\pm 2p\cdot K+\textstyle\sum' k_i^2+i\epsilon)^{-1}=[\textstyle\sum'(\pm 2p\cdot k_i+k_i^2)+i\epsilon]^{-1}.\quad(1.1')$$

The resulting approximation $M_{\text{eik}}'(s,t)$ has a form analogous to (1.2):

$$M_{\text{eik}}'(s,t)=g^2\int d^4x\,e^{-iq\cdot x}\Delta_F(x;\mu)\frac{\exp i\chi'-1}{\chi'},\quad(1.2')$$

where

$$\chi'=\frac{g^2}{(2\pi)^4}\int d^4k\,\frac{e^{ik\cdot x}}{k^2-\mu^2+i\epsilon}\times\left(\frac{1}{-2p_a\cdot k+k^2+i\epsilon}+\frac{1}{2p_a'\cdot k+k^2+i\epsilon}\right)\times\left(\frac{1}{2p_b\cdot k+k^2+i\epsilon}+\frac{1}{-2p_b'\cdot k+k^2+i\epsilon}\right).\quad(1.3')$$

In Sec. II A we show that a considerably simpler looking result is obtained if in the meson propagator $[(q-\sum k_i)^2-\mu^2+i\epsilon]^{-1}$ entering the expression for $M_n(s,t)$, the off-diagonal terms $k_i\cdot k_j$ are also dropped, i.e., the replacement

$$[(q-\textstyle\sum k_i)^2-\mu^2+i\epsilon]^{-1}\rightarrow[q^2+\textstyle\sum(-2q\cdot k_i+k_i^2)-\mu^2+i\epsilon]^{-1}\quad(1.4)$$

is made. If the approximation (1.4) is combined with (1.1), the sum on n—call it $\bar{M}_{\text{eik}}$—can be performed without further approximation and we obtain

$$\bar{M}_{\text{eik}}(s,t)=g^2\int_0^\infty da\,e^{ia(t-\mu^2)}\frac{\exp i\bar{\chi}-1}{\bar{\chi}},\quad(1.5)$$

where

$$\bar{\chi}=\frac{g^2}{(2\pi)^4}\int d^4k\,\frac{e^{ia(k^2-2q\cdot k)}}{k^2-\mu^2+i\epsilon}\left(\frac{1}{-2p_a\cdot k+i\epsilon}+\frac{1}{2p_a'\cdot k+i\epsilon}\right)\times\left(\frac{1}{2p_b\cdot k+i\epsilon}+\frac{1}{-2p_b'\cdot k+i\epsilon}\right).\quad(1.6)$$

Combining (1.4) with (1.1′), one obtains of course an analogous approximation

$$\bar{M}_{\text{eik}}'(s,t)=g^2\int_0^\infty da\,e^{ia(t-\mu^2)}\frac{\exp i\bar{\chi}'-1}{\bar{\chi}'},\quad(1.5')$$

where $\bar{\chi}'$ is obtained from $\bar{\chi}$ by replacing the quantities $\pm 2p\cdot k$ by $\pm 2p\cdot k+k^2$ in (1.6). In Sec. II B we obtain the high-s behavior of $\bar{\chi}$ [Eq. (2.16)] and exhibit the Regge-like behavior of $\bar{M}_{\text{eik}}$ in the domain $s\gg t\gg$masses [Eq. (2.21)].

In Sec. III we compare the asymptotic behavior of the exact fourth-order amplitude $M^{(2)}=M^a+M^b$, where M^a and M^b denote the contribution of the ladder diagram and crossed-ladder diagram, respectively, with $M_{\text{eik}}^{(2)}=M_{\text{eik}}^a+M_{\text{eik}}^b$ obtained in the REA. As is well known,[9] with t fixed and $s\rightarrow\infty$,

$$M^a\sim(\text{const})\bar{\alpha}(t;\mu^2)\ln s/s,\quad(1.7)$$

where

$$\bar{\alpha}(t;\mu^2)=\frac{g^2}{16\pi^2}\int_0^1\frac{dz}{\mu^2-tz(1-z)}.\quad(1.8)$$

It is readily shown that M_{eik}^a has precisely the same asymptotic behavior as M^a, with the *same* coefficient $\bar{\alpha}(t)$ which appears in (1.7). Furthermore, since the crossing relation $M^b(s,t)=M^a(u,t)$ with $u=(p_a-p_b')^2$ is not destroyed by (1.1), we have $M_{\text{eik}}^b(s,t)=M_{\text{eik}}^a(u,t)$, so that the familiar cancellation of logarithms in the direct channel continues to hold in fourth order in the REA: $M_{\text{eik}}^{(2)}(s,t)\sim\bar{\alpha}(t)/s$, just as for $M^{(2)}(s,t)$. Also interesting is the fact which emerges from (1.7) and the observation that, ignoring constant factors, for $t\rightarrow\infty$,

$$\bar{\alpha}(t)\sim\ln t/t,\quad(1.9)$$

namely, that the analytic continuation of the right-hand side of the equation

$$M_{\text{eik}}^a\sim\frac{\ln t}{t}\frac{\ln s}{s},\quad(1.10)$$

valid for $s\gg|t|\gg$all masses, to the region $t\gg|s|\gg$all masses, yields the correct behavior for large t in the crossed (annihilation) channel. However, it should be noted that a corresponding result does not hold for either M^b or M_{eik}^b.

In Sec. IV we discuss the relation of our work with that of other authors. In particular, we show that the REA (1.2) coincides with that of Ref. 3 in the forward direction and describe how the formulas are related for $t\neq 0$. We also discuss the asymptotic behavior of χ and M_{eik} and write down the analog of the one-parameter representation for potential scattering.

To conclude this section, we remark that although the eikonal approximation is designed to be useful for

[9] See, e.g., R. J. Eden, *High Energy Collisions of Elementary Particles* (Cambridge U. P., Cambridge, England, 1967), and references quoted therein.

large s, part of the purpose of this paper is to demonstrate that one can still get interesting results by looking at the domain $\mu^2 \ll |t| \ll |s-4m^2|$; this does not require s to be large. A rather striking example of this fact is provided by examination of an approximate form of $M_{\rm eik}$, considered in Sec. IV, appropriately modified for electron-positron scattering. In this case we have, on replacing α_0 by β_0 in (4.9),

$$M_{\rm eik} \sim \frac{1}{\lambda^2} \frac{\Gamma(-\beta_0+1)}{\Gamma(\beta_0)} \left(\frac{-t}{4\lambda^2}\right)^{\beta_0-1}, \tag{1.11}$$

where λ^2 is a small photon mass and

$$\beta_0(s) = \frac{e^2}{8\pi} \frac{2s-4m^2}{[s(4m^2-s)]^{1/2}}. \tag{1.12}$$

The quantity β_0 is obtained from (4.13) for α_0, by changing g^2 to e^2 and multiplying by the factor $4p_a \cdot p_b = 2s-4m^2$, corresponding to the exchange of photons.

Equation (1.11) has poles in s at $-\beta_0+1=-n+1$ or $\beta_0 = n$ with $n=1, 2, \ldots$; on solving for s, one recovers the energy-level formula of Brezin *et al.*[7] On the other hand, if we regard the exponent of t in (1.11) as a Regge trajectory and accordingly set $\beta_0 - 1 = l$, with $l = 0, 1, \ldots$, we recover precisely the same levels with $n = l+1$. Thus (1.11) exhibits a dual character with the bound states in the s channel lying on the Regge trajectory obtained from the asymptotic behavior in the t channel. Moreover, it has the feature that the partial-wave amplitude obtained by projecting (1.11) with $P_l(\cos\theta_s)$ has poles only for $0 \leq l \leq n-1$, so that the bound states are associated with the correct physical angular momenta. A comparison of $\alpha_0(s)$ and $\bar{\alpha}(s; m^2)$ [Eq. (1.8)] is made in Sec. IV.

II. MODIFIED EIKONAL APPROXIMATION

A. One-Parameter Representation

To obtain the one-parameter eikonal representation mentioned above, we recall first the expression for $M_{\rm eik}^{(n+1)}$ obtained in I, using the approximation (1.1) [viz., the Fourier transform of Eq. (3.17) of Ref. 1]:

$$M_{\rm eik}^{(n+1)} = \frac{-g^2}{(n+1)!} \int d^4k_1 \cdots d^4k_n \times \Big[\Big(q - \sum_{i=1}^{n} k_i\Big)^2 - \mu^2\Big]^{-1} \prod_{j=1}^{n} R(k_j), \tag{2.1}$$

where μ^2 has an infinitesimal negative imaginary part and

$$R(k) = \frac{ig^2}{(2\pi)^4} \frac{1}{k^2-\mu^2} \left(\frac{1}{-2p_a \cdot k + i\epsilon} + \frac{1}{2p_a' \cdot k + i\epsilon}\right) \times \left(\frac{1}{2p_b \cdot k + i\epsilon} + \frac{1}{-2p_b' \cdot k + i\epsilon}\right). \tag{2.2}$$

If, in a spirit similar to that used in arriving at (1.2′), we make the approximation (1.4) and drop the term $k_i \cdot k_j$ with $i \neq j$ in the propagator in (2.1), we obtain a modified approximation

$$\bar{M}_{\rm eik}^{(n+1)} = \frac{-g^2}{(n+1)!} \int \frac{d^4k_1 \cdots d^4k_n}{[q^2 - \mu^2 + \sum (k_i^2 - 2q \cdot k_i)]} \times \prod_{j=1}^{n} R(k_j). \tag{2.3}$$

Instead of the four-dimensional Fourier representation used in I for the denominator in (2.1), for the denominator in (2.3) we may use the one-dimensional integral representation

$$\frac{1}{q^2-\mu^2+\sum(k_i^2-2q\cdot k_i)} = -i \int_0^\infty da \exp\{ia[q^2-\mu^2+\sum(k_i^2-2q\cdot k_i)]\}. \tag{2.4}$$

Substitution of (2.4) into (2.3) yields a factorized integrand and

$$\bar{M}_{\rm eik}^{(n+1)} = \frac{ig^2}{(n+1)!} \int_0^\infty da\, e^{ia(q^2-\mu^2)} (i\bar{\chi})^n, \tag{2.5}$$

where

$$\bar{\chi} = -i \int d^4k\, e^{ia(k^2-2q\cdot k)} R(k). \tag{2.6}$$

Summation on n in (2.5) then gives

$$\bar{M}_{\rm eik}(s,t) = g^2 \int_0^\infty da\, e^{ia(t-\mu^2)} \frac{\exp i\bar{\chi} - 1}{\bar{\chi}}, \tag{2.7}$$

which coincides with Eq. (1.5).

Similarly, if we use (1.1′) and accordingly replace $R(k_j)$ in (2.1) by $R'(k_j)$, where

$$R'(k) = \frac{ig^2}{(2\pi)^4} \frac{1}{k^2-\mu^2+i\epsilon} \times \left(\frac{1}{-2p_a\cdot k + k^2 + i\epsilon} + \frac{1}{2p_a'\cdot k + k^2 + i\epsilon}\right) \times \left(\frac{1}{2p_b\cdot k + k^2 + i\epsilon} + \frac{1}{-2p_b'\cdot k + k^2 + i\epsilon}\right), \tag{2.2′}$$

and in the corresponding expression $M_{\rm eik}'^{(n+1)}$ make the approximation (1.4), we get, on summation over n, Eq. (1.5′) with

$$\bar{\chi}' = -i \int d^4k\, e^{ia(k^2-2q\cdot k)} R'(k). \tag{2.6′}$$

It should be noted that this new approximation has the same symmetry properties as the REA (1.2) or (1.2′) derived and discussed in I, namely, time-reversal invariance and crossing symmetry, and that it reduces to the Born approximation in the small g^2 limit. Also, it is clear from the derivation that $\bar{M}_{\rm eik}$ and $\bar{M}_{\rm eik}'$ are identical in fourth order with $M_{\rm eik}$ and $M_{\rm eik}'$, respectively, and that they differ only from the sixth order on. Furthermore, reasoning along the same lines as those which lead to the derivation of $M_{\rm eik}$, we expect $\bar{M}_{\rm eik}$ to be very close to $M_{\rm eik}$ if t is not *too* small, in other words, if at high energies the scattering angle is small but is not zero.

B. High-Energy Behavior

It is relatively easy to discuss the high-energy behavior of $\bar{\chi}$. Using notation similar to that used in (I), we rewrite $\bar{\chi}$ as

$$\bar{\chi}=-i\sum_{i=1}^{4}\bar{U}_i\,, \tag{2.8}$$

where

$$\bar{U}_1(a;p_a,p_b)=\frac{ig^2}{(2\pi)^4}\int\frac{d^4k}{k^2-\mu^2+i\epsilon}\times\frac{e^{ia(k^2-2q\cdot k)}}{(-2p_a\cdot k+i\epsilon)(2p_b\cdot k+i\epsilon)} \tag{2.9}$$

and

$$\bar{U}_2=\bar{U}_1(a;p_a,p_b')\,,\quad \bar{U}_3=\bar{U}_1(a;-p_a',p_b)\,,\quad \bar{U}_4=\bar{U}_1(a;-p_a',-p_b')\,.$$

We can then write

$$\bar{U}_1=\frac{-g^2}{(2\pi)^4}\int_0^1dz\int_0^\infty du\int_0^\infty dv\,v\int d^4k\,\exp iA(k)\,, \tag{2.10}$$

where $A(k)=a(k^2-2q\cdot k)+u(k^2-\mu^2)-2vk\cdot P_z$, with $P_z=p_az-p_b(1-z)$. After the usual displacement of the k variable, the k integration can be done by making use of the relation

$$\int e^{iak^2}d^4k=\frac{-i\pi^2\epsilon(a)}{a^2}\,, \tag{2.11}$$

and we obtain

$$\bar{U}_1=\frac{ig^2}{16\pi^2}\int_0^1dz\int_0^\infty du\int_0^\infty dv\frac{v}{(a+u)^2}\times\exp\left\{\frac{-i}{a+u}[v^2P_z^2+a(v+a)t-iu\mu^2]\right\}\,. \tag{2.12}$$

Since $P_z^2\sim s$ $[P_z^2=m^2-z(1-z)s$ for $m_a=m_b=m]$, only small values of v are important for large s and we may neglect v compared to a in the coefficient of t on the right-hand side of Eq. (2.12). The v integration then becomes trivial, and we obtain

$$\bar{U}_1\simeq\tfrac{1}{2}\bar{\alpha}(s)F(a,t)\,, \tag{2.13}$$

where, as in I,

$$\bar{\alpha}(s)=\frac{g^2}{16\pi^2}\int_0^1\frac{dz}{P_z^2}\sim\frac{g^2}{16\pi^2}\frac{\ln s}{s} \tag{2.14}$$

and

$$F(a,t)=\int_0^\infty\frac{du}{a+u}\exp\left(-iu\mu^2-\frac{ia^2t}{a+u}\right)\,. \tag{2.15}$$

It follows from (2.8) and (2.13) that the complete modified eikonal $\bar{\chi}(a)$ is given, for large s, by

$$\bar{\chi}(a)\sim -i[\bar{\alpha}(s)+\bar{\alpha}(u)]F(a,t)\,, \tag{2.16}$$

which exhibits at high energy, $s\simeq -u\gg|t|$, the usual cancellation of the $\ln s$ dependence. Of course, since $\bar{\chi}\to 0$ as $s\to\infty$, $\bar{M}_{\rm eik}\to M_{\rm Born}$ for large s.

However, it is instructive in the spirit of our discussion of the fourth-order graphs in Sec. I to consider the high-t behavior of $\bar{M}_{\rm eik}(s,t)$. To do this, we rewrite Eq. (2.7) as follows, by changing a into η/t:

$$\bar{M}\simeq\frac{g^2}{t}\int_0^\infty d\eta\,e^{i\eta}\frac{\exp i\bar{\chi}-1}{\bar{\chi}}=\frac{ig^2}{t}\int_0^1d\beta\int_0^\infty d\eta\,\exp(i\eta+i\beta\bar{\chi})\,, \tag{2.17}$$

and u into $a(\xi-1)$ in F:

$$F(\eta,t)\simeq\int_1^\infty\frac{d\xi}{\xi}e^{-i(\mu^2\eta\xi/t)-i(\eta/\xi)}\,. \tag{2.18}$$

Introducing an auxiliary parameter $\xi_{\rm max}$ such that $\eta\ll\xi_{\rm max}\ll|t|/\mu^2\eta$, we then write, for the high-t behavior of F,

$$F(\eta,t)\simeq\int_1^{\xi_{\rm max}}\frac{d\xi}{\xi}e^{-i\eta/\xi}+\int_{\xi_{\rm max}}^\infty\frac{d\xi}{\xi}e^{-i\mu^2\eta\xi/t}\,.$$

For large t, we therefore find

$$F(\eta,t)=\ln(t/\mu^2)+\phi(\eta)+O(1/t)\,, \tag{2.19}$$

where

$$\phi(\eta)=\lim_{\xi_{\rm max}\to\infty}\left(\int_1^{\xi_{\rm max}}\frac{d\xi}{\xi}e^{-i\eta/\xi}-\ln\eta\xi_{\rm max}-C-\tfrac{1}{2}i\pi\right)\,, \tag{2.20a}$$

with C = Euler's constant or, alternatively,

$$\phi(\eta)=\int_0^1\frac{d\xi}{\xi}(e^{-i\eta\xi}-1)-\ln\eta-C-\tfrac{1}{2}i\pi\,. \tag{2.20b}$$

On putting $\bar{\alpha}(s)+\bar{\alpha}(u)=\nu$, we can write, using (2.16), (2.17), and (2.19),

$$\bar{M}_{\rm eik}\simeq i(g^2/\mu^2)\int_0^1 d\beta(t/\mu^2)^{\beta\nu-1}N(\beta\nu)\,, \qquad (2.21)$$

where

$$N(\beta\nu)=\int_0^\infty d\eta\; e^{i\eta+\beta\nu\phi(\eta)}\,. \qquad (2.22)$$

In the high-t limit, the scattering amplitude in the new approximation therefore exhibits the Regge-like behavior encountered in I in connection with part of the relativistic eikonal which had a $\ln x^2$ singularity; the form (2.21) corresponds to a cut in the complex-angular-momentum l plane.

However, one should note that if instead of averaging over the $(n+1)$, alternative forms of M^{n+1} (as discussed in I), we had summed over them—a procedure which, for large t, is advocated by Schiff[10] and by Sugar and Blankenbecler,[11] at least in the case of potential scattering—we would have obtained a one-parameter eikonal representation of the form

$$\tilde{M}\simeq ig^2\int_0^\infty e^{ia(t-\mu^2)}\exp i\bar{\chi}(a)du\,, \qquad (2.23)$$

the asymptotic behavior of which is

$$\tilde{M}\sim i(g^2/\mu^2)(t/\mu^2)^{\nu-1}N(\nu)\,. \qquad (2.24)$$

This is then a pure Regge behavior, corresponding to a pole in the l plane.[12]

It should be realized that we have obtained the results (2.21) and (2.24) only by making s large first, and then t, so that we are only able to describe the asymptotic behavior of the amplitude defined by (2.7) in the sector $s\gg t\gg$all masses.

III. REA IN FOURTH ORDER

The amplitude M^a, corresponding to the fourth-order ladder diagram, is given by

$$M^a=\frac{ig^2}{(2\pi)^4}\int d^4k\frac{1}{k^2-\mu^2+i\epsilon}\frac{1}{k'^2-\mu^2+i\epsilon}\times\frac{1}{(p_a-k)^2-m_a^2+i\epsilon}\frac{1}{(p_b+k)^2-m_b^2+i\epsilon}, \qquad (3.1)$$

with $q=p_a-p_a'=k+k'$. On introducing Feynman parameters, we may rewrite M^a in the form

$$M^a=({\rm const})\int\prod_{i=1}^4 d\alpha_i\frac{\delta(1-\sum\alpha_j)}{(\alpha_2\alpha_4 s+d+i\epsilon)^2}\,, \qquad (3.2)$$

where

$$d(t;\alpha_1,\alpha_3;\alpha_2,\alpha_4)=\alpha_1\alpha_3 t+(\alpha_1+\alpha_3)\times(\alpha_2 m_a^2+\alpha_4 m_b^2-\mu^2)-\alpha_2 m_a^2-\alpha_4 m_b^2. \qquad (3.3)$$

As is well known,[9] the asymptotic behavior of M^a for $s\to\infty$ can be found by replacing d by $d(t;\alpha_1,\alpha_3;0,0)$ and $\delta(1-\sum\alpha_i)$ by $\delta(1-\alpha_1-\alpha_3)$ since only small values of α_2 and α_4 are important for large s.

The quantity $M_{\rm eik}{}^a$ is obtained by dropping the k^2 terms in the particle propagators in (3.1). On introducing Feynman parameters in the same way as before, one obtains instead of (3.2) the result

$$M_{\rm eik}{}^a=({\rm const})\int\prod_i d\alpha_i\frac{\delta(1-\sum\alpha_j)}{(\alpha_2\alpha_4 s+d_{\rm eik})^2}\,, \qquad (3.4)$$

where

$$d_{\rm eik}=d-(\alpha_2+\alpha_4)[\alpha_3 t-(\alpha_1+\alpha_3)\mu^2]\,. \qquad (3.5)$$

Since

$$d_{\rm eik}(t;\alpha_1\alpha_3;0,0)=d(t;\alpha_1,\alpha_3;0,0)\,, \qquad (3.6)$$

it follows that the asymptotic behavior of $M_{\rm eik}{}^a$ for $s\to\infty$, t fixed, is exactly the same as that of M^a given by (1.7).[8]

The amplitude M^b, corresponding to the fourth-order cross-ladder diagram, can be obtained from M^a by the transformation $s\to u$, $t\to t$ and similarly for $M_{\rm eik}{}^a$, i.e., as already mentioned, $M_{\rm eik}{}^b(s,t)=M_{\rm eik}{}^a(u,t)$. Thus, $M_{\rm eik}{}^b$ has the same asymptotic behavior as M^b.

On the other hand, it can be inferred from Eqs. (3.3) and (3.5) that the asymptotic behavior of $M_{\rm eik}{}^a$ for $t\to\infty$, s fixed, is *not* the same as that of M^a because the REA destroys the essential s,t symmetry which $M^a(s,t)$ has for large s and t. The reason is that the coefficient of t in the denominator of (3.4) is $\alpha_3(\alpha_1-\alpha_2-\alpha_4)$ instead of $\alpha_3\alpha_1$ as it is in the exact d. Consequently, for large t, the dominant contribution no longer comes from the neighborhood $\alpha_1=\alpha_3=0$ and the $(\ln t)/t$ behavior of M^a is not reproduced by $M_{\rm eik}{}^a$. Of course, this result is not surprising since the REA is not expected to be valid for $t\gg s$. Nevertheless, it is worth noting that once the high-s behavior has been obtained, one may let t become large and still get the behavior

$$M_{\rm eik}{}^a(s,t)\sim\frac{\ln s}{s}\frac{\ln t}{t} \qquad (3.7)$$

for $s\gg t\gg$masses. This is the *same* behavior as that of M^a, valid for large s and large t with either variable fixed and hence, in particular, for $t\gg s\gg$masses.

With regard to $M_{\rm eik}{}^b$, the crossing relation $M^b(s,t)=M^a(u,t)$ implies that for t large and s fixed $M^b\ll M^a$, and so is $M_{\rm eik}{}^b\ll M_{\rm eik}{}^a$ in this domain.

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[10] See L. I. Schiff, Ref. 2.

[11] R. Sugar and R. Blankenbecler, Phys. Rev. **183**, 1387 (1969).

[12] We have been informed that a similar result has been obtained by R. Blankenbecler, in *The Three Body Problem in Nuclear and Particle Physics*, edited by J. S. C. McKee and P. M. Rolph (North-Holland, Amsterdam, 1970), p. 448.

IV. CONCLUDING DISCUSSION

In the preceding sections we have seen how a modified REA, $\bar{M}_{\text{eik}}$ or $\bar{M}_{\text{eik}}'$, may be obtained, which involves an integration over a single parameter; in fourth order, $\bar{M}_{\text{eik}}$ and $\bar{M}_{\text{eik}}'$ coincide, respectively, with M_{eik} and M_{eik}'. Although, no doubt, $\bar{M}_{\text{eik}}$ will differ from M_{eik} in sixth order, we expect this difference to be small if t is not too small—it would seem worthwhile to investigate this point. Another, perhaps more important, reason for studying the sixth-order graphs is in connection with the appropriate form of an eikonal expression to be used for large s and fixed $\theta\neq 0$, in particular, the question of whether one should use in this case the factor $e^{i\chi}$ rather than $(e^{i\chi}-1)/i\chi$ in the REA.

It should be emphasized that one must distinguish between the two situations: (i) $(|t|/s)\ll 1$, but $|t|$ large and fixed, for which our discussion of the fourth-order amplitude already shows that the REA (1.2) gives the correct asymptotic behavior—the replacement of $(e^{i\chi}-1)/i\chi$ by $e^{i\chi}$ would give in fourth order a contribution too large by a factor of 2; (ii) s and $|t|$ large but $|t|/s$ fixed, i.e., s large and fixed, angle different from zero, for which the asymptotic behavior has apparently not been studied.[13]

In conclusion we wish to discuss a number of topics, related to the above and to the work of other authors.

A. REA in Forward Direction

For scattering in the forward direction, we may put $p_a'=p_a$, $p_b'=p_b$ in Eq. (1.3). Then $\chi\to\chi_0$ with

$$\chi_0=\frac{-g^2}{4\pi^2}\int d^4k\,\frac{e^{ik\cdot x}}{k^2-\mu^2+i\epsilon}\delta(2p_a\cdot k)\delta(2p_b\cdot k). \quad (4.1)$$

To evaluate $M_{\text{eik}}(s,0)$, it is convenient to work in the c.m. system, with $p_a=(E_a,0,0,p)$, $p_b=(E_b,0,0,-p)$; the δ functions may then be used to eliminate k_0 and k_3, so that

$$\chi_0=\frac{g^2}{16\pi^2 p\sqrt{s}}\int d^2k_\perp\frac{e^{-ik_\perp\cdot x_\perp}}{k_\perp^2+\mu^2}=\frac{g^2}{8\pi p\sqrt{s}}\int_0^\infty dk_\perp\frac{k_\perp J_0(k_\perp x_\perp)}{k_\perp^2+\mu^2},$$

where $k_\perp=(k_1,k_2)$ and $x_\perp=(x_1,x_2)$, or

$$\chi_0=\frac{g^2}{8\pi p\sqrt{s}}K_0(\mu x_\perp). \quad (4.2)$$

[13] After this was written, we became aware of the work of I. G. Halliday, Ann. Phys. (N. Y.) 28, 370 (1964), and that of J. L. Cardy [Cambridge University report (unpublished)], who have studied the fixed-angle problem.

Writing $d^4x=d^2x_\perp dx_0dx_3$ in Eq. (1.2) and noting that

$$\int dx_0dx_3\,\Delta_F(x;\mu)=\frac{-i}{4\pi^2}\int d^2k\frac{e^{-ik_\perp\cdot x_\perp}}{k_\perp^2+\mu^2}=\frac{-i}{2\pi}K_0(\mu x_\perp), \quad (4.3)$$

we see that

$$M_{\text{eik}}(s,0)=-4ip(\sqrt{s})\int d^2x_\perp(e^{i\chi_0}-1), \quad (4.4)$$

which coincides with the result of Ref. 3 for $t=0$.

So far we have made no approximation outside of the REA itself; the latter is, however, expected to be a good approximation to $M(s,t)$ at high s and fixed t, where it actually gives the correct asymptotic behavior in each order of perturbation theory, for the sum of generalized ladder graphs.[8] On the other hand, an expression *identical* with the forward REA can be derived[3,7] for any s, for the most singular part of $M(s,0;\mu^2)$, corresponding to the same graphs, in the limit $\mu\to 0$. That is, putting $\mu\mathbf{x}_\perp=\mathbf{y}_\perp$ in Eq. (4.4), we have

$$\lim_{\mu\to 0}\mu^2M(s,0;\mu^2)=\mu^2M_{\text{eik}}(s,0;\mu^2) \quad (4.5a)$$

$$=-4ip(\sqrt{s})\int d^2y_\perp\left\{\exp\left[\frac{g^2}{8\pi p\sqrt{s}}K_0(y_\perp)\right]-1\right\}. \quad (4.5b)$$

The validity of (4.5a) is easily seen by making the substitution $k_i=\mu k_i'$ for the virtual meson momenta entering the Feynman integrals for the relevant graphs. (This procedure does not work for integrals more singular than $1/\mu^2$, e.g., for diagrams involving vertex-type radiative corrections.)

In the case of quantum electrodynamics, the location of the poles of $\mu^2M_{\text{eik}}(s,0;\mu)$ (which is independent of μ^2) corresponds rather accurately to the energy levels of positronium.[7,1] To understand this more fully, it would be desirable to prove that any pole of the function $M(s,0;\mu)-M_{\text{eik}}(s,0;\mu^2)$ is also a pole of $M_{\text{eik}}(s,0;\mu^2)$. It is clear that more work must be done on the bound-state problem in the limit $\mu\to 0$.

Finally, we remark that if in Eq. (1.3) we replace both p_a and p_a' by the average $P_a=\frac{1}{2}(p_a+p_a')$, and similarly p_b and p_b' by $P_b=\frac{1}{2}(p_b+p_b')$, the REA (1.2) can in the same way be put into a two-dimensional form, and the amplitude becomes equal to

$$-4i\tilde{p}(\sqrt{s})\int d^2x_\perp\,e^{iq_\perp\cdot x_\perp}(\exp i\tilde{\chi}_0-1), \quad (4.6)$$

where $\tilde{\chi}_0$ is obtained from (4.2) by replacing p by $\tilde{p}=\frac{1}{2}[-(P_a-P_b)^2]^{1/2}$; Eq. (4.5) coincides with the result of Ref. 3 for all t.

B. REA for $|t|\gg\mu^2$

We can get some information on the high-t behavior of the eikonal amplitude, after taking the high-s limit, in the same spirit as in our discussion of the one-parameter REA and the fourth-order scattering amplitudes. Since $q \simeq q_\perp$ for high s, we can write, using (4.2) and (4.3), the approximate equation

$$M_{\text{eik}} \simeq -2is \int d^2x_\perp \, e^{iq_\perp \cdot x_\perp} (\exp i\chi_0 - 1)$$
$$= -2is \int_0^\infty dx_\perp \, x_\perp J_0(q_\perp x_\perp)(\exp i\chi_0 - 1) \,, \quad (4.7)$$

where χ_0 is given by Eq. (4.2). The behavior for $|t| \gg \mu^2$ is now governed by the singularity of χ_0 for small $x_\perp^2$. This is given by

$$\chi_0 \simeq (-g^2/8\pi p\sqrt{s}) \ln|\mu x_\perp| \,, \quad (4.8)$$

so that for high $t \simeq -q_\perp^2$,

$$M_{\text{eik}} \propto \frac{\Gamma(-\alpha_0+1)}{\mu^2 \Gamma(\alpha_0)} (-t/4\mu^2)^{-1+\alpha_0(s)} \,, \quad (4.9)$$

where we have put $\alpha_0(s) = ig^2/16\pi p\sqrt{s}$.

Equation (4.9) can also be obtained with the same technique used in deriving (4.5), i.e., *without* assuming that s is large, by examining the limit $\mu^2 \to 0$. For $t \neq 0$, the transformation $k_i = \mu^2 k_i'$ leads to the result for $\mu^2 \to 0$

$$M(s,t;\mu^2) \simeq \frac{g^2}{\mu^2} \int d^4x \, \Delta_F(x;1) e^{i(q\cdot x/\mu)} \frac{e^{i\chi(x;1)}-1}{\chi(x;1)} \,, \quad (4.10)$$

where $\chi(x;1)$ is given by (1.3) with $\mu^2 = 1$. The behavior of the right-hand side of (4.10) in the domain

$$\mu^2 \ll |t| \ll |s-4m^2| \quad (4.11)$$

is readily found. Using the c.m. system as before with

$$p_a' = (E, 0, p\sin\theta, p\cos\theta) \,,$$
$$p_b' = (E, 0, -p\sin\theta, -p\cos\theta) \,,$$

one has

$$2p_a' \cdot k = 2p_a \cdot k \{1 + O[(\sqrt{t})/(s-4m^2)^{1/2}]\} \,,$$

and similarly for particle b. It follows that for $|t| \ll |s-4m^2|$, the factors in parentheses in (1.3) can be replaced by δ functions and $\chi(x;1) \simeq \chi_0(x_\perp;1)$. In a similar way in (4.10), we may let $q \to (0, \mathbf{q}_\perp)$ so that

$$M(s,t;\mu^2) \simeq \frac{-8\pi i p\sqrt{s}}{\mu^2} \int_0^\infty dx_\perp \, x_\perp J_0\left(\frac{q_\perp x_\perp}{\mu}\right)$$
$$\times [e^{i\chi_0(x_\perp;1)} - 1] \,. \quad (4.12)$$

For $\mu \to 0$, the major contribution to the last integral comes from small $x_\perp$ so that (4.8) can be used (with $\mu = 1$), and the right-hand side of (4.12) reduces to the right-hand side of (4.9). We therefore see that in the domain (4.11) the exact $M(s,t;\mu)$, corresponding to the sum of generalized ladder graphs, has the dual character described in Sec. I. Since we have not assumed that s is large, the equation may be used in the bound-state region where

$$\alpha_0(s) = (g^2/8\pi)[(4m^2-s)s]^{-1/2} \,; \quad (4.13)$$

the corresponding poles are essentially those obtained in Ref. 7.

It should be noted that $\alpha_0(s)$ differs from $\bar{\alpha}(s;m^2)$, which is the Regge trajectory obtained from the ladder approximation in the crossed channel. It turns out, however, that the two functions have approximately the same behavior for $s \lesssim 4m^2$. In fact it follows from (1.8) that, for $0 \leq s \leq 4m^2$,

$$\bar{\alpha}(s) = \alpha_0(s) \left[\frac{2}{\pi} \tan^{-1} \left(\frac{s}{4m^2 - s} \right)^{1/2} \right] .$$

Since the factor in brackets is 1 for $s = 4m^2$ and a very slowly varying function of s near $s = 4m^2$, one can see why the two "trajectories" predict the same levels to high accuracy.[14] Note, incidentally, that $\alpha_0(s) = \operatorname{Im}\bar{\alpha}(s;m)^2$ for $s > 4m^2$.

C. Asymptotic Behavior of Eikonal Function

As is well known, in perturbation theory leading contributions coming from individual graphs may cancel (see, e.g., Sec. II); in the eikonal approximation such cancellations are reflected in similar cancellations occurring among the terms contributing to χ itself. This can be seen in more detail by study of the behavior, for fixed x, of the basic function [Eq. (3.15) of I]

$$U_\pm(x;p,p') = \frac{ig^2}{16\pi^4} \int \frac{d^4k}{k^2 - \mu^2 + i\epsilon}$$
$$\times \frac{e^{ik\cdot x}}{(-2k\cdot p + i\epsilon)(2p'\cdot k \pm i\epsilon)} \,, \quad (4.14)$$

with p^0 and $p'^0 \geq 0$. If for simplicity we consider the case of zero external masses ($p^2 = p'^2 = 0$), and choose, without loss of generality, a frame in which $p = (\omega, 0, 0, \omega)$ and $p' = (\omega, 0, 0, -\omega)$, the integration over k_0 and k_3 is readily carried out, via contour integration. On changing to polar coordinates for the k_1, k_2 integration, one then finds the result

$$U_\pm = \frac{\pm ig^2}{16\pi\omega^2} \theta\left(\mp \frac{p\cdot x}{2\omega}\right) \theta\left(\frac{p'\cdot x}{2\omega}\right) K_0(\mu x_\perp) + \frac{ig^2}{32\pi\omega^2}$$
$$\times \int_0^\infty dk \frac{kJ_0(kx_\perp)}{k^2+\mu^2} H_0^{(2)}[(x_L^2(k^2+\mu^2))^{1/2}] \,, \quad (4.15)$$

[14] It should be remarked that $\bar{\alpha}(s) + \bar{\alpha}(u) = \alpha_0(s)$ for $t = 0$. It follows that Eq. (4.9) is an accurate representation of M_{eik} in the region $|s-4m^2| \gg |t| \gg \mu^2$. This region is particularly interesting in quantum electrodynamics because of the zero mass of the photon.

where

$$\omega=\omega(p,p')=[(p+p')^2/4]^{1/2},$$

$$x_L{}^2=x_0{}^2-x_3{}^2=(p\cdot x)(p'\cdot x)/\omega^2,$$

and

$$x_\perp{}^2=x_1{}^2+x_2{}^2=x_L{}^2-x^2;$$

K, J, and H are the usual Bessel functions, and when $x_L{}^2<0$, the argument of $H_0{}^{(2)}$ must be replaced by $-i[-x_L{}^2(k^2+\mu^2)]^{1/2}$. It is easy to verify that the functions U_i of Eq. (3.16) in Ref. 1 can be expressed in terms of $U_\pm$ as follows:

$$U_1=U_+(x;p_a,p_b),\qquad U_2=-U_-(x;p_a,p_a'),$$
$$U_3=-U_-(-x;p_a',p_b),\quad U_4=U_+(-x;p_a',p_b').\qquad(4.16)$$

We also note that

$$\omega(p_a,p_b)=\omega(p_a',p_b')=\tfrac{1}{2}\surd s$$

and

$$\omega(p_a,p_b')=\omega(p_a',p_b)=\tfrac{1}{2}\surd(-u),$$

so that for fixed t and high s all the ω's become equal. Furthermore,

$$(p_a\cdot x)/\omega-(p_a'\cdot x)/\omega=(p_b'\cdot x)/\omega-(p_b\cdot x)/\omega=q\cdot x/\omega$$

tends to zero in this limit with x and $\hat{q}=q/\surd(-t)$ fixed. Therefore, using (4.15) and (4.16), the terms in the U_i which are integrals over two Bessel functions are seen to be equal in magnitude but opposite in sign for U_1 U_4 and U_2, U_3, respectively, and cancel in the sum $\chi=-i\sum_{i=1}{}^4 U_i$. On the other hand, the product of θ functions in the first term of the U_i add up to unity, so that we get

$$\chi\longrightarrow(ig^2/4\pi s)K_0(\mu x_\perp),\qquad(4.17)$$

in agreement with the discussion of Sec. IV B.

The slightly modified REA, defined by Eqs. (1.2′) and (1.3′), has recently been derived by Barbashov *et al.*[15] using the methods of functional integration. These authors have also investigated, using a dispersion-relation technique, the separate high-energy behavior of χ_1' and χ_2' defined by

$$\chi_1'=i(U_1'+U_3'),\quad \chi_2'=i(U_2'+U_4'),$$

where the U_i' are obtained from the U_i (defined in Sec. III of I) by adding k^2 to $\pm 2p_i\cdot k$ in the denominators. Since for $x\neq 0$ the asymptotic behavior of the χ_i' will be the same as that of the χ_i, we may compare our results with theirs.

They conclude that each of these eikonal functions behaves like $\pm s^{-1}[\ln(\pm s)]K_0(\mu|x_\perp|)$ so that the logarithm cancels in the sum, yielding the same result as our Eq. (4.17) above. However, our Eq. (4.15) does not have for fixed x and fixed $\hat{q}$ and t (or fixed $\hat{p}_i$) any term proportional to $\ln s$. The only way to obtain such terms appears to be by keeping $(p\cdot x)(p'\cdot x)$ small compared to s, which is not possible if x is fixed.

D. One-Parameter Representation in Potential Scattering

The analog of (1.5) in potential scattering is easily derived, following the methods of Sec. II of I. For a spherically symmetric potential $V=V(r)$, we obtain for the scattering amplitude f

$$\bar{f}_{\rm eik}=\frac{-m}{2\pi i}\int_0^\infty da\,\sigma(a)e^{iaq^2}\frac{\exp i\bar{\chi}_{\rm pot}-1}{\bar{\chi}_{\rm pot}},\qquad(4.18)$$

where

$$\bar{\chi}_{\rm pot}=-i[\bar{U}(a;\mathbf{p})+\bar{U}(a;-\mathbf{p}')],$$

with

$$\bar{U}_{(a;\,\mathbf{p})}=\frac{2m}{(2\pi)^3}\int dk\frac{\hat{V}(k)}{2\mathbf{p}\cdot\mathbf{k}+i\epsilon}e^{ia(\mathbf{k}^2-2\mathbf{q}\cdot\mathbf{k})}.$$

In (4.18), $\sigma(a)$ is related to $\hat{V}(k)$, the Fourier transform of $V(r)$, by the equation

$$V(k)=\int_0^\infty da\,\sigma(a)e^{iak^2},$$

with k^2 having an infinitesimal positive imaginary part. $V(r)$ can be expressed directly in terms of $\sigma(a)$ through the formula

$$V(r)=-e^{-i\pi/4}\int_0^\infty da\frac{\sigma(a)e^{-ir^2/4a}}{(4\pi a)^{3/2}}.$$

It would be interesting to study the accuracy of (4.18) relative to the more familiar eikonal approximation in potential scattering, with which (4.18) agrees to second order in V.

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[15] B. M. Barbashov, S. P. Kuleshov, V. A. Matveev, and A. N. Sissakian, Dubna Report No. E2-4692 (unpublished). We thank these authors for informative correspondence.