$$\times M_j^2[j,m+1|\ddot{V}|j,m-1]-\{4C_+^2(j,,m+1)+4C_+^2(j,m)+C_+^2(j,m-1)-1\}[j,m+1|\ddot{V}|j,m+1]$$
$$+\{C_+^2(j,m+1)+4C_+^2(j,m)+4C_+^2(j,m-1)-1\}[j,m|\ddot{V}|j,m]$$
$$-3C_+^2(j,m+1)C_+^2(j,m+2)M_j^2[j,m+3|\ddot{V}|j,m+1]-3C_+^2(j,m-1)[j,m-1|\ddot{V}|j,m-1]$$
$$+3C_+^2(j,m+1)[j,m+2|\ddot{V}|j,m+2]. \quad \text{(A5)}$$

Using (A2) and (A4) gives Eq. (23).

For low angular momentum where $m'-m=\pm 3, \pm 5$ is not physically accessible, the equations above have to be modified as follows. For $j=\frac{1}{2}$, the equal-mass angular condition [Eq. (17)] is identically satisfied. Therefore, it gives no restrictions. For $j=\frac{3}{2}$, $m'-m=\pm 5$ is not accessible. For $j=\frac{3}{2}$, $m'-m=3$, i.e., $m'=\frac{3}{2}$ and $m=-\frac{3}{2}$, Eq. (17) is satisfied identically. For $j=\frac{3}{2}$, $m'-m=1$ and $m=\frac{1}{2}$ or $m=-\frac{3}{2}$, we obtain Eq. (23).

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Unitarity Constraint on a Regge Trajectory in a Model Amplitude*

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A representation is constructed for the elastic scattering of neutral scalar bosons with crossing symmetry, direct-channel poles, polynomial residues, and Regge behavior based on an arbitrary Regge trajectory. It is shown, within the context of a specific model, how the amplitude can approximately satisfy elastic unitarity. A purely linear trajectory is inconsistent with the unitarity constraint unless the amplitude is zero, while the addition of a complex threshold term of definite strength enables elastic unitarity to be numerically approximated.

I. INTRODUCTION

MUCH work[1] has been carried out, spurred on by the introduction of the closed solutions[2] to the finite-energy sum rules,[3] in attempts to construct unitary, crossing-symmetric amplitudes, necessarily accommodating nonlinear Regge trajectories. The nonlinearity of the statement of unitarity and the necessary inclusion of all inelastic channels are among the features that make unitarity a particularly difficult constraint to impose on a scattering amplitude. In this paper we propose a model representation for the elastic scattering of neutral scalar bosons (σ-σ scattering) with crossing symmetry, Regge asymptotic behavior, direct-channel poles, and satisfying an approximate form of unitarity.

In a previous paper,[4] hereafter referred to as I, the advantage of constructing a crossing-symmetric amplitude with Regge asymptotic behavior and direct-channel poles based initially on an *arbitrarily* specified trajectory was pointed out. It was shown how sequences of such amplitudes could be constructed. The proposed amplitudes were then analyzed in the complex angular momentum plane, and it was found that any member of the sequence contains a leading finite array of simple poles spaced by two units followed by additional nonleading arrays of higher-order singularities.

A central feature in I was the realization that crossing symmetry could be built into an amplitude, without introducing new kinematic singularities, by using the three elementary symmetry functions[5] of the Mandelstam variables s, t, and u. The function of weight 1 is the constant

$$c=s+t+u\,, \tag{1.1}$$

while the other two functions are conveniently defined by

$$x=\tfrac{1}{2}(s^2+t^2+u^2)\,, \tag{1.2}$$

$$y=-stu\,. \tag{1.3}$$

Such elementary symmetric functions were previously used by Wanders,[6] who investigated Mandelstam analyticity in them and then derived some π-π sum

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rules from crossing symmetry, analyticity, and unitarity.

Our application of the elementary symmetric functions, both in I and in the present paper, is essentially as follows. A non-crossing-symmetric phenomenological Regge amplitude may be taken as a starting point. Such a one-channel amplitude may then be symmetrized into the full three-channel amplitude by approximating it with functions built up from the symmetric functions x and y. This symmetrization procedure has the immediate advantage over the usual additive symmetrization procedure, i.e.,

$$f(s,t,u)\rightarrow f(s,t,u)+f(s,u,t)+\cdots, \qquad (1.4)$$

in that Regge behavior is automatically preserved. On the other hand, the procedure is not unique and undesirable analytic properties may be introduced. The nonuniqueness of the symmetrization procedure can, however, be put to some advantage in helping to satisfy unitarity. Also the undesirable analytic properties can be controlled to a large extent by taking further advantage of the nonuniqueness, as shown in Sec. III.

The non-crossing-symmetric phenomenological amplitude used in I as a starting point and generalized in this paper is the factorized form

$$\beta(1+e^{-i\pi\alpha(s)})\Gamma(-\alpha(s))(\tau/\tau_0)^{\alpha(s)}, \qquad (1.5)$$

where $\alpha(s)$ is a Regge trajectory function and τ is linear in t. In Sec. II a superposition of such forms is introduced that display in a natural way branch points which simulate the elastic and inelastic unitarity thresholds. In Sec. IV these ideas are put into concrete form with a simple σ-σ scattering model containing the elastic and first inelastic thresholds. Finally, of particular interest, the constraints imposed by elastic unitarity on the model trajectory are investigated.

II. FORM OF AMPLITUDE

The crossing-symmetric amplitude considered here is a superposition of the factorized forms introduced in I. The amplitude here, as in I, is derived from a family of Regge trajectories, of which the leading trajectory is an arbitrarily specified function $\alpha(s)$. The amplitude exhibits the features already demonstrated in I, such as Regge asymptotic behavior, direct-channel poles, and arrays of daughter poles spaced by two units in the complex angular momentum plane. Specifically, the structure of the amplitude now under consideration is

$$A=(1+e^{-i\pi\alpha(\sigma)})\times\Gamma(-\alpha(\sigma))\sum_i\int_{a_i}^{\infty}da\,\beta_i(a)\left[\frac{\tau(a,x,y)}{\tau_i(a)}\right]^{\alpha(\sigma)}, \qquad (2.1)$$

comprised of a product of a signature factor, a pole factor, and a superposition of terms carrying the high-energy Regge power behavior.

As in I, the variable σ is a member of an infinite set of functions of x and y that approach s with arbitrary accuracy in two limiting domains A and B of the combined s,t complex planes. In s-channel language, where z_s is the cosine of the c.m. scattering angle, domain A is centered about $z_s=0$ and $s=s_A$. The parameter s_A is ideally situated somewhere in the low-energy resonance region. Domain B is defined by letting $|t|$ go to infinity in the complex t plane for a fixed value of s in the neighborhood of $s=s_B$. Under some circumstances, as shown in the example of Sec. III, domain B may include *any* fixed value of s. These are the physically important domains in any representation of the amplitude derived from Regge trajectories. Domain A is the physically important region for exhibiting the effects of direct channel poles, while domain B exhibits the characteristic Regge-power behavior.

It is easily seen that up to an over-all multiplicative constant the most general form of the function $\tau(a,x,y)$, which ensures that the integrand of Eq. (2.1) is a polynomial in z_s of degree α whenever α is an even integer, is

$$\tau=(y-ax+b)^{1/2}, \qquad (2.2)$$

where b may be any function of a. However, if we choose

$$b=a^3-a^2c+\tfrac{1}{2}ac^2, \qquad (2.3)$$

then

$$\tau^\alpha=(a-s)^{\alpha/2}(a-t)^{\alpha/2}(a-u)^{\alpha/2}, \qquad (2.4)$$

which displays thresholdlike singularities in the Mandelstam variables. The sum in Eq. (2.1) can therefore simulate the occurrence of unitarity cuts with branch points at the thresholds a_i when appropriate choices of the otherwise arbitrary functions $\beta_i(a)$ and $\tau_i(a)$ are taken.

Resonance poles occur whenever $\alpha(\sigma)=n$, a nonnegative even integer. In the s channel, this actually happens in (2.1) at values of s which are very nearly independent of t, rather than at strictly fixed values of s, since $\sigma\approx s$. In fact, in the analytic neighborhood of $\alpha(\sigma)\approx n$, we may write

$$\alpha(\sigma)=n+\delta_n(s-s_n)+\epsilon_n(t-u)^2+\cdots, \qquad (2.5)$$

where ϵ_n is small in the sense that $|\epsilon_n s_n|\ll\delta_n$. The factor $(a-t)^{\alpha/2}(a-u)^{\alpha/2}$ in (2.4) is a polynomial in z_s of degree n whenever $\alpha(\sigma)=n$. However, the factor $(a-s)^{\alpha/2}$, evaluated at the value of s for which $\alpha=n$, has a slight t dependence according to (2.5). Thus for $n=0$, the residue of the resonance pole is correctly a polynomial of degree 0 in z_s, while for $n\geq 2$, the residue is a polynomial of degree n plus weak ancestor terms proportional to the small parameter ϵ_n. This departure from true polynomial residues for $n\geq 2$ is traced to the error in the relation $\sigma\approx s$, and has nothing to do with the analytic structure of $\alpha(s)$.

III. FUNCTION σ

The function σ, whose properties are specified in Sec. II, is generalized in this section from the form constructed in I primarily to introduce more control over its singularity structure. In I it was shown to be convenient to define a pair of new symmetric functions

$$p=\tfrac{4}{3}x-(2/9)c^2, \tag{3.1}$$

$$q=-4y+\tfrac{4}{3}cx-(10/27)c^3. \tag{3.2}$$

There σ was defined to be a member of a sequence of rational functions of p and q of the form

$$\sigma_{mn}=N'(p,q)p^{2-3(m+n)}, \tag{3.3}$$

where $N'(p,q)$ is a polynomial in p and q constructed so that $\sigma_{mn}\to s$ in domains A and B.

The generalization of form (3.3) considered here is the rational function

$$\sigma=\tfrac{1}{3}c+N(p,q)/D(p,q), \tag{3.4}$$

where D is *any* polynomial in p and q. The zeros of p on the right-hand side of Eq. (3.3) produce multiple-order poles in the combined s,t complex planes that come objectionably close to the physical region. The generalization to an arbitrary denominator polynomial introduces a flexibility into σ that can be used to relocate such singularities away from the regions of interest.

Given the arbitrary polynomial D, we attempt to find an N that is matched to D in some optimal way. The approach here is to discover what terms p^iq^j may be present in N for the given D and then to determine a set of optimal coefficients multiplying these terms. To facilitate taking the limits appropriate to the domains A and B, it is helpful to introduce new variables

$$v=s-\tfrac{1}{3}c, \tag{3.5}$$

$$z=(t-u)^2/3v^2, \tag{3.6}$$

in terms of which

$$p=v^2(1+z), \tag{3.7}$$

$$q=v^3(1-3z). \tag{3.8}$$

Domain A is attained when $z\to 0$ and $v\to v_A=s_A-\tfrac{1}{3}c$, while domain B is attained when $z\to\infty$ and $v\to v_B=s_B-\tfrac{1}{3}c$.

The rational function N/D may now be written in the form

$$N/D=\sum_{k,l}a_{kl}p^{3l-k}q^{k-2l}\Big/\sum_{k,l}b_{kl}p^{3l-k}q^{k-2l} \tag{3.9}$$

$$=\sum_{k,m}\bar{a}_{km}v^kz^m\Big/\sum_{k,m}\bar{b}_{km}v^kz^m, \tag{3.10}$$

where

$$\bar{a}_{km}=\sum_{l,j}a_{kl}(-3)^j\binom{3l-k}{m-j}\binom{k-2l}{j}, \tag{3.11}$$

$$\bar{b}_{km}=\sum_{l,j}b_{kl}(-3)^j\binom{3l-k}{m-j}\binom{k-2l}{j}. \tag{3.12}$$

TABLE I. Occurrence (X) of the values of k and l in Eq. (3.15) up to $k=13$.

k	l	$m=0$	1	2	3	4	5	6
0	0	X						
2	1	X	X					
3	1	X	X					
4	2	X	X	X				
5	2	X	X	X				
6	2	X	X	X				
6	3	X	X	X	X			
7	3	X	X	X	X			
8	3	X	X	X	X			
8	4	X	X	X	X	X		
9	3	X	X	X	X			
9	4	X		X	X	X		
10	4	X	X	X	X	X		
10	5	X	X	X	X	X	X	
11	4	X	X	X	X	X		
11	5	X	X	X	X	X	X	
12	4	X	X	X	X	X		
12	5	X	X	X	X	X	X	
12	6	X	X	X	X	X	X	X
13	5	X	X	X	X	X	X	
13	6	X	X	X	X	X	X	X

In order that negative powers of p and q not be present in N and D, the coefficients a_{kl} and b_{kl} must be zero for those values of k and l not satisfying

$$0\leq\tfrac{1}{3}k\leq l\leq\tfrac{1}{2}k. \tag{3.13}$$

The problem now is to find a_{kl} for the given b_{kl} such that

$$N/D=v+\text{small error} \tag{3.14}$$

in domains A and B. Setting the right-hand side of Eq. (3.10) equal to v requires that

$$\sum_k\bar{a}_{km}v^k=\sum_k\bar{b}_{km}v^{k+1}, \tag{3.15}$$

which of course cannot be an identity for all m because v is not a symmetric function.

Both sides of Eq. (3.15) are polynomials in v which are to be equated to each other in some approximation in domains A and B. In domain B the highest power m of z is most important and it is sometimes possible to satisfy Eq. (3.15) identically for the highest values of m. As m decreases from these highest values, identical matching eventually becomes impossible and the best that can be done is to match the polynomials to some degree of approximation in the neighborhood of v_B. However, such identical matching is to little avail when $v=0$, unless we take $a_{00}=0$. This is immediately seen from the right-hand side of Eq. (3.9) which reduces to a_{00}/b_{00} when $v\to 0$.

Table I indicates which values of k and l are present in each set of equations corresponding to a given power m of z for all $k\leq 13$. For example, consider a denominator polynomial containing a set of given b_{kl} with $k\leq 12$, $l\leq 6$. The corresponding unknowns a_{kl} are 20 in number, after setting $a_{00}=0$, with $k\leq 13$, $l\leq 6$. In domain B, the most important power of z is $m=6$. In

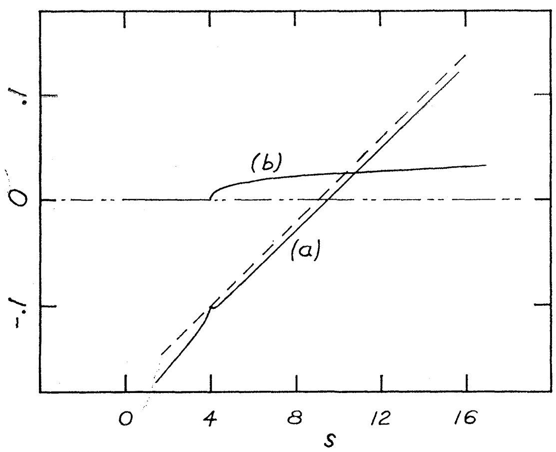

FIG. 1. (a) rms of Δ and (b) percent error plotted against α_2. Units of α_2 are such that $m=1$. The straight dashed line is $\alpha(s)$ when $\alpha_2=0$.

the last column of Table I, the two entries correspond to the unknowns $a_{12,6}$ and $a_{13,6}$, which can therefore be determined by this set of equations. Moreover, the resulting equation is an identity in v since just two powers of v are present in Eq. (3.15) for $m=6$. In the column labeled $m=5$, four additional unknowns occur and since just four powers of v are involved, an identity is again possible. The additional unknowns that can be determined from this set of equations are $a_{13,5}$, $a_{12,5}$, $a_{11,5}$, and $a_{10,5}$. In the column labeled $m=4$, six powers of v but only five additional unknowns occur. Hence an identity is not possible and just the five lowest powers of $v-v_B$ can be matched in Eq. (3.15). Such a matching could determine the unknowns a_{84}, a_{94}, $a_{10,4}$, $a_{11,4}$, and $a_{12,4}$, in which case we could write the behavior of N/D in domain B as

$$(N/D)_B=v+O\big((v-v_B)^5z^{-2}\big)+O(z^{-3}). \quad (3.16)$$

Thus far 11 unknowns have been determined, and so it seems reasonable to determine the remaining 10 by turning to domain A. An equitable way to distribute the errors between powers of z and v in domain A would be to carry out a least-squares fit between the left- and right-hand sides of Eq. (3.14) in domain A, thereby determining the best set of remaining unknown coefficients. This is a straightforward task since N is linear in the unknown coefficients.

Table I is easily extended to handle polynomials of higher degree but it probably contains enough low-order cases of practical interest. The table is primarily an aid in determining how best to distribute the errors among the powers of z and v in domains A and B.

IV. THREE-PARAMETER MODEL

In this section we construct a simple hypothetical model, describing σ-σ scattering, as an application of the general representation (2.1) based on three adjustable parameters. Other parameters in the model will be fixed at physically reasonable values while the adjustable parameters will be used to satisfy the constraint of elastic unitarity. The first inelastic threshold is assumed to be that of $2\sigma\rightarrow 4\sigma$.

The function σ in this model is derived from a denominator polynomial $D=(p+d^2)^5$, containing the fixed parameter d. From Table I it is apparent that the numerator polynomial contains at most 15 coefficients ranging from a_{21} to $a_{11,5}$. Six of these are immediately determined by identical matching of the left- and right-hand sides of Eq. (3.15) for the two highest powers of z. Hence the domain B result is

$$(N/D)_B=v+O(z^{-2}). \quad (4.1)$$

The remaining nine coefficients together with the denominator parameter d are determined by a least-squares fit to the equation $(N/D)_A=v$ in the s,t plane over the physical region $4m^2\leq s\leq 16m^2$, where m is the mass of the σ. The significant numerical results of the fit are that the rms of the difference $\sigma-s$ over the physical region is $0.13m^2$, while the best value of d determined this way is $17.4m^2$. The denominator polynomial is actually just a slight modification of the one proposed in I and is perhaps the simplest such modification giving some promise of pushing undesirable singularities away from the physical region. The relatively large size of d as compared to the extent of that part of the physical region under consideration is therefor a reassuring result.

The nonlinear form

$$\alpha(s)=\alpha_0+\alpha_1(s-4m^2)-\alpha_2(4m^2-s)^{\alpha_0+1/2}, \quad (4.2)$$

possessing the correct threshold behavior,[7] is assumed for the trajectory function. The parameters α_0 and α_1, defining the linear part of the trajectory, are to have fixed values, while only the real coefficient α_2 multiplying the nonlinear term is considered as an adjustable parameter. In this way, we can isolate the effect of varying the nonlinear part on the unitarity requirement.

For the sum in Eq. (2.1) we assume in this model the occurrence of just two discrete terms, associated with the elastic and first inelastic thresholds. Thus the amplitude is

$$A=(1+e^{-i\pi\alpha(\sigma)})\Gamma(-\alpha(\sigma))\left[\beta_1\left(\frac{\tau(4m^2,x,y)}{\tau_1}\right)^{\alpha(\sigma)}+\beta_2\left(\frac{\tau(16m^2,x,y)}{\tau_2}\right)^{\alpha(\sigma)}\right], \quad (4.3)$$

where the real coefficients β_1 and β_2 are the remaining adjustable parameters, and the scaling parameters τ_1 and τ_2 are to have fixed positive values.

The trajectory parameters are set at the values $\alpha_0=-0.1$ and $\alpha_1=0.02m^{-2}$. These choices fix the linear

[7] A. O. Barut and D. E. Zwanziger, Phys. Rev. 127, 974 (1962).

part of the trajectory to have a slope comparable to the slopes observed in nature if $m \approx m_\pi$ and to give a hypothetical spin-0 resonance at $m_r \approx 3m$, approximately in the middle of the elastic region. The scaling parameters are arbitrarily set at $\tau_1 = (2m)^3$ and $\tau_2 = (4m)^3$.

The statement of elastic unitarity is

$$\mathrm{Im}A(s,t) = \frac{1}{16\pi^2}\left(\frac{s-4m^2}{s}\right)^{1/2}\int_0^{\pi/2} d\phi \int_0^{\pi} d\theta \times \sin\theta \,\mathrm{Re}[A^*(s,t_+)A(s,t_-)], \quad (4.4)$$

where

$$t_\pm = -\tfrac{1}{2}(s-4m^2) + \tfrac{1}{2}(s-4m^2)^{1/2} \times [(-u)^{1/2}\cos\theta \pm (-t)^{1/2}\sin\theta\cos\phi]. \quad (4.5)$$

The unitarization procedure adopted here is to search in the space of β_1, β_2 and α_2 for the nontrivial minima of the rms of Δ in the s,t plane over the elastic region, where Δ is the difference between the left- and right-hand sides of Eq. (4.4). Only positive α_2 is considered since a negative value corresponds to a resonance pole on the physical sheet. The results of the search are as follows. For arbitrarily small values of α_2, the only rms minimum in the real β_1, β_2 subspace is the trivial solution $\beta_1 = \beta_2 = 0$. In the approximate range $0.005 < \alpha_2 < 0.025$ (in units where $m=1$), a nontrivial minimum occurs in the β_1, β_2 subspace. Figure 1 shows this becomes a minimum in the full β_1, β_2, α_2 space at $\alpha_2 = 0.013$. For higher values of α_2, other minima are not ruled out, but such possibilities were not explored in depth. The upper curve of Fig. 1 gives the ratio (in percent) of the rms of Δ to the maximum value of ImA in the physical region. This percent error curve goes through a minimum of 4.4% in very near coincidence with the rms minimum. In qualitative terms, unitarity is satisfied to a few percent in the neighborhood of the resonance when $\alpha_2 = 0.013$. The corresponding values of the other parameters at this minimum are $\beta_1 = -0.32$ and $\beta_2 = 1.11$.

V. CONCLUSION

The intent of the above model is to present a concrete example of the representation (2.1) based on an arbitrarily specified trajectory and then, more significantly, to show that a purely linear trajectory is unsatisfactory under the requirement of unitarity. In fact, the only solution in such a case turns out to be the trivial solution $0=0$. We have shown, in the context of an admittedly hypothetical model, that the constraint imposed by unitarity is certainly more consistent *with* a nonlinear cut in the trajectory, rather than without. The strength of this cut, for the best set of parameters determined in Sec. IV, is displayed in Fig. 2.

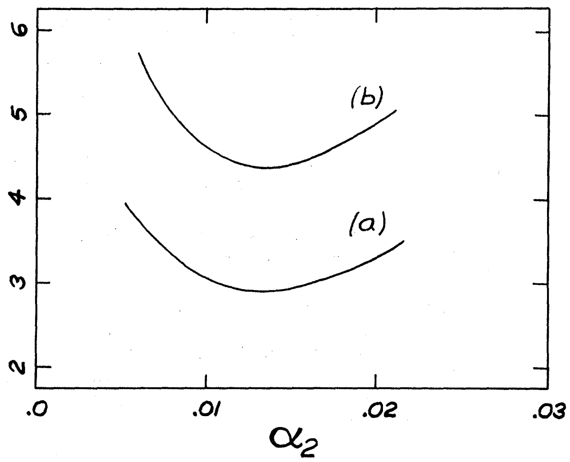

FIG. 2. (a) Re$\alpha(s)$ and (b) Im$\alpha(s)$ plotted against s when $\alpha_2 = 0.013$ in units such that $m=1$.

A number of immediately obvious improvements in the σ-σ scattering model are possible. For example, higher inelastic threshold terms with parameters β_i and τ_i could be included in the representation (4.3). Also a search in the full parameter space could be undertaken, the expectation being that the entire trajectory may be bootstrapped by the unitarity condition. However, the hypothetical nature of the scattering system suggests that such improvements are of doubtful value. The simple model has served its purpose and it is now of considerable interest to move on to a realistic application. Work has therefore started on an application of representation (4.3) to the process $\pi\pi \rightarrow \pi\omega$ in which the ρ should play a dominant role. A statement of elastic unitarity below the $\pi\omega$ threshold and based on the $T=1$ $\pi\pi$ amplitude could in principle bootstrap the imaginary part of the ρ trajectory once a suitable parametrization of the imaginary part is adopted.