# Mass-Dispersion Relations and Asymptotic Constraints on Form Factors

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The mass extrapolation procedure of Fubini and Furlan and of de Alfaro and Rossetti, together with current algebra and partial conservation of axial-vector current, is used to derive asymptotic relations between the vector and axial-vector form factors of the nucleon. These relations are compared with the ones obtained recently by Gordon and Peccei. We are led to the nonrenormalization of the axial-vector coupling constant, provided that the time derivative of the divergence of axial-vector current is assumed to transform like the time component of the axial-vector current.

## I. INTRODUCTION

IN the last few years, current algebra supplemented with the hypothesis of a partially conserved axial-vector current (PCAC) has been extensively used to derive a large variety of relations for pions of zero mass (soft pions).[1] In order to make these relations valid for physical pions (nonzero-mass pions) one must do an extrapolation in the pion mass. A natural and simple procedure to carry out the mass extrapolation has been suggested by Fubini and Furlan, and de Alfaro and Rossetti (FFAR).[2] The essential point of the FFAR formalism is to write dispersion relations in a suitable variable which is related to the pion mass. The dispersion relations are then evaluated at the pion mass and one obtains relations valid for physical pions. The FFAR mass-dispersion relations have already been studied in a number of problems with useful results.[3]

In a recent publication, Gordon and Peccei[4] have pointed out a rather interesting application of the FFAR mass-dispersion relations; these dispersion relations can be used to derive asymptotic relations between various form factors. In particular, these authors derive the relation

$$G_A(t) \underset{t\to-\infty}{=} G_A(0)G_M(t)\,, \tag{1.1}$$

where $G_{\rm A}(t)$ and $G_M(t)$ are, respectively, the axial-vector and the Sachs magnetic, isovector form factors of the nucleon. To start with, they obtain relations for the nucleon vector and axial-vector form factors in the limit of invariant momentum transfer $t\to-\infty$, in terms of single-pion electroproduction amplitudes. The unknown electroproduction amplitudes are then eliminated by means of a link to double-pion electroproduction and Eq. (1.1) follows. It becomes pertinent to inquire what relations obtain if one considers the single and double weak pion production resulting from only the axial-vector part of the hadronic current. Of course, one expects to get some kind of consistency relations that go with Eq. (1.1). We carry out this program in the present paper and obtain the two asymptotic relations

$$G_M(t)=G_{\rm A}(0)G_{\rm A}(t)\,, \tag{1.2}$$

$$g_{\rm A}(t)=(2m/t)G_{\rm A}(t)\,, \tag{1.3}$$

in the limit $t\to-\infty$. On combining (1.1) and (1.2), we get

$$G_{\rm A}{}^2(0)=1\,. \tag{1.4}$$

Equation (1.4) will be recognized as the result of exact $SU(2)\otimes SU(2)$. Equations (1.2)–(1.4) may be regarded as the main results of this paper.

The plan of the paper is as follows: In Sec. II, we study the weak single-pion production (due to the axial-vector part of the weak hadronic current) with a view to writing once-subtracted dispersion relations for suitable amplitudes. The Low representation for the weak production amplitude is examined in some detail, and the asymptotic behavior in the dispersion variable is ascertained. The dispersion relations obtained are evaluated in the limit of large nucleon energies and one obtains relations expressing various nucleon form factors in terms of the amplitudes of weak pion production. In Sec. III, we carry out a similar analysis for the weak double-pion production. The one interesting phenomenon here is that the nucleon pole contribution to this five-point function (which, apart from the nucleon propagator, is the product of the axial-vector form factors of the nucleon with the weak single-pion production amplitude) becomes, in the limit of nucleon energy going to infinity, identical with a diagram which is the product of axial-vector nucleon form factors with the pion-nucleon scattering amplitude. Thus the asymptotic relations of Sec. III are expressions giving the nucleon form factors in terms of the pion-nucleon and weak single-pion production amplitudes. One can now eliminate the unwanted weak single-pion production amplitudes from the asympotic relations of Secs. II and III, and one obtains relations purely between the form factors. A discussion of these relations is presented in Sec. IV.

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Cluster decompositions of the four- and five-point functions are given in Appendices A and B, respectively.

## II. WEAK PION PRODUCTION FROM NUCLEONS —FOUR-POINT FUNCTION

We begin by considering the matrix element of the time-ordered product of two axial-vector currents between nucleons:

$$M_{\mu\nu}{}^{\beta\alpha}=-i\int d^4z \times e^{iqz}\langle N(p')|T(A_\mu{}^\beta(z)A_\nu{}^\alpha(0))|N(p)\rangle. \quad (2.1)$$

From (2.1) follows the Ward identity

$$q_\mu M_{\mu\nu}{}^{\beta\alpha}=U_\nu{}^{\beta\alpha}+C_\nu{}^{\beta\alpha}, \quad (2.2)$$

where

$$U_\nu{}^{\beta\alpha}=\int d^4z\, e^{iqz}\langle N(p')|T(D^\beta(z)A_\nu{}^\alpha(0))|N(p)\rangle, \quad (2.3)$$

$$C_\nu{}^{\beta\alpha}=\int d^4z\, e^{iqz}\delta(z_0)\langle N(p')|[A_0{}^\beta(z),A_\nu{}^\alpha(0)]|N(p)\rangle$$
$$=i\epsilon_{\beta\alpha\gamma}\langle N(p')|V_\nu{}^\gamma(0)|N(p)\rangle. \quad (2.4)$$

Here $V_\nu{}^\gamma$ is the weak vector current and $D^\beta=\partial_\mu A_\mu{}^\beta$.

Following FFAR, we aim at constructing a once-subtracted dispersion relation in the pion mass variable. The subtraction point is chosen to correspond to the zero mass of the pion. The subtraction constant is then simply the result obtained from current algebra plus massless pions. As emphasized by Fubini and Furlan, a convenient frame in which to consider mass-dispersion relations is the Breit frame, i.e., $\mathbf{p}'+\mathbf{p}=0$. Furthermore, we wish to work with collinear dispersion relations; that is to say, we set

$$q=xP=\tfrac{1}{2}x(p+p'). \quad (2.5)$$

Thus,

$$\mathbf{q}=0,\quad q_0=xP_0=xE, \quad (2.6)$$

where we have noted that $p_0'=p_0=E$ in the Breit frame. $x$ is our dispersion variable. The Ward identity, (2.2), becomes

$$xP_\mu M_{\mu\nu}{}^{\beta\alpha}=U_\nu{}^{\beta\alpha}+C_\nu{}^{\beta\alpha}. \quad (2.7)$$

To proceed further, we carry out the cluster decomposition of $M_{\mu\nu}{}^{\beta\alpha}$ and $U_\nu{}^{\beta\alpha}$ in the usual fashion. The results of this decomposition are given in Appendix A. In the limit $x\to 0$ (which, of course, is the usual current-algebra point), we obtain from (2.7)

$$U_\nu{}^{\beta\alpha}(0)=\mathfrak{M}_\nu{}^{\beta\alpha}-C_\nu{}^{\beta\alpha}, \quad (2.8)$$

where

$$\mathfrak{M}_\nu{}^{\beta\alpha}=\lim_{x\to 0}[xP_\mu M_{\mu\nu}{}^{\beta\alpha}]. \quad (2.9)$$

Notice that only the nucleon intermediate state contributes to $\mathfrak{M}_\nu{}^{\beta\alpha}$ since it is the only intermediate state degenerate in mass with the external states. More precisely,

$$\mathfrak{M}_\nu{}^{\beta\alpha}=-\frac{1}{E}\sum_{\text{spin}}\{\langle N(p')|P_\mu A_\mu{}^\beta|N(p')\rangle \times\langle N(p')|A_\nu{}^\alpha|N(p)\rangle-\langle N(p')|A_\nu{}^\alpha|N(p)\rangle \times\langle N(p)|P_\mu A_\mu{}^\beta|N(p)\rangle\}, \quad (2.10)$$

where the summation is over the intermediate nucleon spin. The explicit evaluation of $U_\nu{}^{\beta\alpha}(0)$ is easily obtained on using the matrix elements

$$\langle N(p')|A_\nu{}^\alpha|N(p)\rangle =i(m/E)\bar{u}(p')[G_A(t)\gamma_\nu+ig_A(t)\Delta_\nu] \times\gamma_5\tfrac{1}{2}\tau_\alpha u(p), \quad (2.11)$$

$$\langle N(p')|V_\nu{}^\alpha|N(p)\rangle =i(m/E)\bar{u}(p')[F_1(t)\gamma_\nu-[F_2(t)/2m]\sigma_{\nu\mu}\Delta_\mu] \times\tfrac{1}{2}\tau_\alpha u(p)$$
$$=i\left(\frac{m}{E}\right)\bar{u}(p')\left[G_M(t)\gamma_\nu+\frac{i}{m}\frac{G_M(t)-G_E(t)}{1-t/4m^2}P_\nu\right] \times\tfrac{1}{2}\tau_\alpha u(p), \quad (2.12)$$

where $G_{E,M}(t)$ are the Sachs isovector electric and magnetic form factors of the nucleon and in terms of $F_1$ and $F_2$ are given by

$$G_E(t)=F_1(t)+(t/4m^2)F_2(t),\quad G_M(t)=F_1(t)+F_2(t). \quad (2.13)$$

Also,

$$\Delta=p'-p,\quad t=-\Delta^2.$$

Now we isolate the single-pion pole contribution in $U_\nu{}^{\beta\alpha}$. From Appendix A we immediately find that

$$U_\nu{}^{\beta\alpha}(\pi)=\frac{i}{E}\left(\frac{1}{x-x_0}\right) \times\langle N(p')|A_\nu{}^\alpha|\pi_\beta(\mathbf{p}_n=0,E_n=\mu),N(p)\rangle \times\langle\pi_\beta(\mathbf{p}_n=0,E_n=\mu)|D^\beta|0\rangle, \quad (2.14)$$

where $x_0=\mu/E$. The residue of the pole at $x=x_0$ is thus proportional to the weak pion production amplitude. The residue is explicitly evaluated on using the definitions

$$\langle\pi_\beta(p_n)|D^\beta|0\rangle=(2\mu)^{-1/2}\mu^2 f_\pi, \quad (2.15)$$

$$\langle N(p')|A_\nu{}^\alpha|\pi_\beta(p_n),N(p)\rangle$$

$$=\frac{i}{(2\mu)^{1/2}}\left(\frac{m}{E}\right)\bar{u}(p')[iT_1{}^{\alpha\beta}\gamma_\nu+T_2{}^{\alpha\beta}P_\nu+T_3{}^{\alpha\beta}\Delta_\nu]u(p)\,, \quad (2.16)$$

$$T_i{}^{\alpha\beta}=T_i{}^{(+)}\delta_{\alpha\beta}+T_i{}^{(-)}\times\tfrac{1}{2}[\tau_\alpha,\tau_\beta],\quad i=1,2,3. \quad (2.17)$$

The amplitudes $T_i{}^{(\pm)}(x)$ have the following crossing properties under $x\to -x$:

$$\begin{array}{llll} T_1{}^{(-)}(x)\,, & T_2{}^{(-)}(x)\,, & T_3{}^{(+)}(x) & \text{even}\,,\\ T_1{}^{(+)}(x)\,, & T_2{}^{(+)}(x)\,, & T_3{}^{(-)}(x) & \text{odd}\,.\end{array} \quad (2.18)$$

Next, we examine the asymptotic behavior of $U_\nu{}^{\beta\alpha}(x)$ as $x\to\infty$. Following Bjorken's[5] arguments, we find that

$$U_\nu{}^{\beta\alpha}(x)\underset{x\to\infty}{\longrightarrow}\frac{C}{x}+O\left(\frac{1}{x^2}\right) \quad (2.19)$$

and

$$C=\frac{i}{E}\int d^4z\, e^{iqz}\delta(z_0)\times\langle N(p')|[D^\beta(z),A_\nu{}^\alpha(0)]|N(p)\rangle. \quad (2.20)$$

Finally, we introduce a function $F_\nu{}^{\beta\alpha}(x)$ defined by

$$F_\nu{}^{\beta\alpha}(x)=(1-x^2/x_0{}^2)U_\nu{}^{\beta\alpha}(x). \quad (2.21)$$

We see that

$$F_\nu{}^{\beta\alpha}(0)=U_\nu{}^{\beta\alpha}(0) \quad (2.22)$$

and

$$F_\nu{}^{\beta\alpha}(x_0)=\lim_{x\to x_0}\left(1-\frac{x^2}{x_0{}^2}\right)U_\nu{}^{\beta\alpha}(x)$$
$$=(m/E)f_\pi\bar{u}(p')[iT_1{}^{\alpha\beta}(x_0)\gamma_\nu+T_2{}^{\alpha\beta}(x_0)P_\nu+T_3{}^{\alpha\beta}(x_0)\Delta_\nu]u(p). \quad (2.23)$$

Also,

$$F_\nu{}^{\beta\alpha}(x)\underset{x\to\infty}{\longrightarrow}-\left(\frac{C}{x_0{}^2}\right)x+\text{const.} \quad (2.24)$$

In the following we want to write dispersion relations for the even parts of $F_\nu{}^{\beta\alpha}(x)$. From (2.24), we see that the even parts approach a constant for large values of $x$. Hence, we can write once-subtracted dispersion relations for the amplitudes $F_{1,2}{}^{(-)}(x)$ and $F_3{}^{(+)}(x)$, where $F_i{}^{(\pm)}$ are the invariant amplitudes in the decomposition of $F_\nu{}^{\beta\alpha}(x)$. At the point $x=x_0$, we get

$$f_\pi T_1{}^{(-)}(x_0)=\tfrac{1}{2}[G_M(t)-G_A(0)G_A(t)]+\delta F_1{}^{(-)}\,, \quad (2.25a)$$

$$f_\pi T_2{}^{(-)}(x_0)=(m/2E^2)\{G_A(0)G_A(t)-[G_M(t)-G_E(t)]\}+\delta F_2{}^{(-)}\,, \quad (2.25b)$$

$$f_\pi T_3{}^{(+)}(x_0)=(1/4E^2)[mG_A(t)+2(E^2-m^2)g_A(t)]+\delta F_3{}^{(+)}\,, \quad (2.25c)$$

where

$$\delta F_i=\frac{2x_0{}^2}{\pi}\int_0^\infty dx\frac{\text{Im}F_i(x)}{x(x^2-x_0{}^2)}. \quad (2.25d)$$

Let us now take $E$ to infinity. Then $x_0=\mu/E\to 0$, so that $F_i(x_0)\to F_i(0)$ and, consequently, $\delta F_i\to 0$. We obtain (in the limit $E\to\infty$)

$$f_\pi T_1{}^{(-)}(0)=\tfrac{1}{2}[G_M(t)-G_A(0)G_A(t)]\,, \quad (2.26a)$$

$$f_\pi T_2{}^{(-)}(0)=(m/2E^2)\{G_A(0)G_A(t)-[G_M(t)-G_E(t)]\}\,, \quad (2.26b)$$

$$f_\pi T_3{}^{(+)}(0)=(m/4E^2)G_A(t)+\tfrac{1}{2}g_A(t). \quad (2.26c)$$

We note that $t=-(p'-p)^2=-4(E^2-m^2)\to-\infty$ when $E\to\infty$.

## III. WEAK PION PRODUCTION FROM NUCLEONS —FIVE-POINT FUNCTION

Here we work with the five-point amplitude

$$T_{\mu\nu}{}^{\beta\alpha}=i\int d^4z\, e^{-iqz}\langle N(p')|T(A_\mu{}^\beta(z)A_\nu{}^\beta(0))\times|N(p);\pi_\alpha(k)\rangle, \quad (3.1)$$

where $\pi_\alpha(k)$ denotes a $\pi$ meson of isospin index $\alpha$ and four-momentum $k$. From (3.1) we get the Ward identity

$$q_\mu T_{\mu\nu}{}^{\beta\alpha}=W_\nu{}^{\beta\alpha}\,, \quad (3.2)$$

where

$$W_\nu{}^{\beta\alpha}=\int d^4z\, e^{-iqz}\langle N(p')|T(D^\beta(z)A_\nu{}^\beta(0))\times|N(p);\pi_\alpha(k)\rangle. \quad (3.3)$$

Notice that the equal-time commutator term is absent from the right-hand side of (3.2) since the two axial-vector currents in (3.1) carry the same isospin index.

As in Sec. II, we want to write a mass-dispersion relation for suitably chosen amplitudes. A convenient frame in which to work here is defined by

$$\mathbf{Q}=0,\quad\text{where}\quad Q=\tfrac{1}{2}(p+p'+k).$$

We take the initial pion to be at rest, i.e., $\mathbf{k}=0$, $k_0=\mu$. Then $\mathbf{p}=-\mathbf{p}'$, $p_0=p_0'=E$, and $Q_0=E+\frac{1}{2}\mu=E'$. Again we use the collinear frame, i.e., $q=yQ$, so that $\mathbf{q}=0$, $q_0=yE'$. The Ward identity (3.2) becomes

$$yQ_\mu T_{\mu\nu}{}^{\beta\alpha}=W_\nu{}^{\beta\alpha}. \quad (3.4)$$

At this stage we carry out the cluster decomposition of the amplitudes $T_{\mu\nu}{}^{\beta\alpha}$ and $W_\nu{}^{\beta\alpha}$. The results are given in Appendix B. Taking the limit $y\to 0$, we obtain from (3.4)

$$W_\nu{}^{\beta\alpha}(0)=\lim_{y\to 0}[yQ_\mu T_{\mu\nu}{}^{\beta\alpha}]. \quad (3.5)$$

From Eq. (B1) we see that only the nucleon intermediate states contribute to tke right-hand side of (3.5).

[5] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

Actually, we get

$$W_\nu^{\beta\alpha}(0)=\frac{1}{E'}\sum_{\text{spin}}\{\langle N(p')|A_\nu^\beta|N(p);\pi_\alpha(k)\rangle\times\langle N(p)|Q_\mu A_\mu^\beta|N(p)\rangle-\langle N(p')|Q_\mu A_\mu^\beta|N(p')\rangle\times\langle N(p')|A_\nu^\beta|N(p);\pi_\alpha(k)\rangle\}. \quad (3.6)$$

Notice that (3.6) involves the weak pion production amplitude we have already made use of [cf. Eqs. (2.14) and (2.16)]. Using the matrix elements (2.11) and (2.16), we can evaluate (3.6) explicitly.

Next, we isolate the double-pion pole contribution in the amplitude $W_\nu^{\beta\alpha}$. By looking at the Low representation of $W_\nu^{\beta\alpha}$ (see Appendix B), we immediately obtain

$$W_\nu^{\beta\alpha}(\text{double pion pole})=\frac{-i\mu^2 f_\pi}{(y_0+y)(y_0-y)^2E'^3}\sum_{\text{spin}}\{\langle N(p')|J_\pi^\beta|N(p');\pi_\alpha(k)\rangle\times\langle N(p')|A_\nu^\beta|N(p)\rangle-\langle N(p')|A_\nu^\beta|N(p)\rangle\times\langle N(p)|J_\pi^\beta|N(p);\pi_\alpha(k)\rangle\}, \quad (3.7)$$

where $y_0=\mu/E'$ and we have used $D^\beta=\mu^2 f_\pi\varphi_\pi^\beta$. Equation (3.7) involves the pion-nucleon scattering amplitude $T^{\beta\alpha}$,

$$\langle N(p')|J_\pi^\beta|N(p');\pi_\alpha(k)\rangle=\frac{1}{(2\mu)^{1/2}}\left(\frac{m}{E}\right)\bar{u}(p')T^{\beta\alpha}(s,0)u(p), \quad (3.8)$$

$$T^{\beta\alpha}=\delta_{\beta\alpha}T^{(+)}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]T^{(-)},$$

where $s=-(p'+k)^2=m^2+2\mu E'$ and the invariant momentum transfer is zero. We may note here that $s\to\infty$ as $E'\to\infty$. On using (2.11) and (3.8), we obtain an explicit evaluation of (3.7).

We now turn our attention to the asymptotic behavior of $W_\nu^{\beta\alpha}(y)$ as $y\to\infty$. Again following Bjorken's method, we get

$$W_\nu^{\beta\alpha}(y)\to\frac{C_1}{y}+\frac{C_2}{y^2}+O\left(\frac{1}{y^3}\right), \quad (3.9)$$

where

$$C_1=\frac{1}{E'}\int d^4z\, e^{-iqz}\delta(z_0)\langle N(p')|[D^\beta(z),A_\nu^\beta(0)]\times|N(p);\pi_\alpha(k)\rangle, \quad (3.10)$$

$$C_2=\frac{1}{E'^2}\int d^4z\, e^{-iqz}\delta(z_0)\langle N(p')|\left[\frac{\partial}{\partial z_0}D^\beta(z),A_\nu^\beta(0)\right]\times N(p);\pi_\alpha(k)\rangle. \quad (3.11)$$

As in the case of the four-point function, we are interested here in only the even part $W_\nu^{\beta\alpha,\text{even}}$ of $W_\nu^{\beta\alpha}(y)$. The leading term in $W_\nu^{\beta\alpha,\text{even}}$ is seen from (3.9) to behave like $y^{-2}$, the coefficient being given by (3.11).

Finally, we introduce the function $G_\nu^{\beta\alpha}(y)$ defined by

$$G_\nu^{\beta\alpha}(y)=(1-y^2/y_0^2)^2W_\nu^{\beta\alpha}(y). \quad (3.12)$$

We have

$$G_\nu^{\beta\alpha}(0)=W_\nu^{\beta\alpha}(0), \quad (3.13)$$

$$G_\nu^{\beta\alpha}(y_0)=\lim_{y\to y_0}\left(1-\frac{y^2}{y_0^2}\right)^2W_\nu^{\beta\alpha}(y)=\frac{1}{(2\mu)^{1/2}}\left(\frac{m}{E}\right)\frac{2mf_\pi}{\mu E}T^{(-)}\times\bar{u}(p')[G_A(t)\gamma_\nu+ig_A(t)\Delta_\nu]\gamma_5(-\tau_\alpha+\delta_{\beta\alpha}\tau_\beta)u(p) \quad (3.14)$$

and

$$G_\nu^{\beta\alpha,\text{even}}\underset{y\to\infty}{\longrightarrow}-\left(\frac{C_2}{y_0^4}\right)y^2+\text{const.} \quad (3.15)$$

We shall assume now that the equal-time commutator $[\partial D^\beta(z)/\partial z_0,A_\nu^\beta(0)]\delta(z_0)$ is either a $c$ number or it transforms like an isovector. In the latter case it is identically zero, the two isospin indices being equal. With the above assumption, $G_\nu^{\beta\alpha,\text{even}}$ goes like a constant asymptotically. Thus we can write once-subtracted dispersion relations for the amplitudes $G_i^{\beta\alpha,\text{even}}$, where $G_i^{\beta\alpha}$ are the invariant amplitudes in the decomposition of $G_\nu^{\beta\alpha}$. That is,

$$G_i^{\beta\alpha,\text{even}}(y_0)=G_i^{\beta\alpha,\text{even}}(0)+\frac{2y_0^2}{\pi}\int_0^\infty dy\frac{\text{Im}G_i^{\beta\alpha,\text{even}}(y)}{y(y^2-y_0^2)}=G_i^{\beta\alpha,\text{even}}(0)+\delta G_i^{\beta\alpha,\text{even}}. \quad (3.16)$$

When evaluation is made of the relation (3.16), one obtains a set of sum rules,

$$\frac{2mf_\pi}{\mu E}G_A(t)T^{(-)}(s,0)=G_A(0)\left(1-\frac{m^2}{EE'}\right)T_1^{(-)}+\delta G_1, \quad (3.17\text{a})$$

$$\frac{2mf_\pi}{\mu E}g_A(t)T^{(-)}(s,0)=-\frac{m}{2EE'}G_A(0)T_1^{(-)}+\delta G_2, \quad (3.17\text{b})$$

$$\frac{2mf_\pi}{\mu E}g_A(t)T^{(-)}(s,0)=-G_A(0)\left[\frac{m}{2EE'}T_1^{(-)}+T_3^{(+)}\right]+\delta G_3, \quad (3.17\text{c})$$

plus some other sum rules which are identical to (3.17a)–(3.17c) in the limit $E'\to\infty$, or else do not yield any useful information in the present context. When we take the limit $E'\to\infty$, then the corrections $\delta G_i\to 0$ (as explained in the last section), and we obtain

$$\frac{2mf_\pi}{\mu E}G_A(t)T^{(-)}(s,0)=G_A(0)T_1^{(-)}(0)\,, \qquad (3.18a)$$

$$\frac{2mf_\pi}{\mu E}g_A(t)T^{(-)}(s,0)=\frac{-m}{2E^2}G_A(0)T_1^{(-)}(0)\,, \qquad (3.18b)$$

$$\frac{2mf_\pi}{\mu E}g_A(t)T^{(-)}(s,0)=-G_A(0)\left(\frac{m}{2E^2}T_1^{(-)}(0)+T_3^{(+)}(0)\right). \qquad (3.18c)$$

The variables $s$ and $t$ in Eqs. (3.18) have assumed the asymptotic values $s\to\infty$ and $t\to-\infty$.

On eliminating $T_1^{(-)}$ and $T_3^{(+)}$ from Eqs. (2.26) and (3.18), we get

$$\frac{2mf_\pi^2}{\mu E}T^{(-)}(s,0)=\tfrac{1}{2}G_A(0)\left(\frac{G_M(t)}{G_A(t)}-G_A(0)\right), \qquad (3.19)$$

$$g_A(t)/G_A(t)=-m/2E^2\,. \qquad (3.20)$$

It is well known that the antisymmetric pion-nucleon amplitude $T^{(-)}(s,0)$ has the asymptotic behavior

$$T^{(-)}(s,0)\sim s^{\alpha_\rho(0)}\,,$$

where $\alpha_\rho(0)$ is the intercept of the $\rho$ trajectory at zero momentum transfer. Remembering that $s\sim 2\mu E$ and $t\sim -4E^2$ in the limit $E\to\infty$, we obtain from (3.19) and (3.20) the relations

$$\lim_{t\to-\infty}\frac{G_M(t)}{G_A(t)}=G_A(0) \qquad (3.21)$$

and

$$\lim_{t\to-\infty}\left(g_A(t)-\frac{2m}{t}G_A(t)\right)=0\,. \qquad (3.22)$$

If one considers the time-ordered product of a vector and an axial-vector current corresponding to (2.1) and (3.1) one obtains, as shown by Gordon and Peccei, the result

$$\lim_{t\to-\infty}\frac{G_M(t)}{G_A(t)}=G_A^{-1}(0)\,. \qquad (3.23)$$

When (3.21) is read in conjuction with (3.23), we have

$$G_A^2(0)=1\,. \qquad (3.24)$$

## IV. DISCUSSION

We started out with the aim of supplementing the asymptotic relations between form factors derived by Gordon and Peccei. The required relations are given in Eqs. (3.22) and (3.24). Equation (3.24) is obviously the result of exact $SU(2)\times SU(2)$ while Eq. (3.22) is the statement of conservation of axial-vector current for large values of the momentum transfer. For this latter result, note that it is valid for all values of $t$ provided that the axial-vector current is exactly conserved.

It will be recalled that the derivation of our results depends on the possibility of our being able to write once-subtracted dispersion relation in the variables $x$ and $y$ in Secs. II and III, respectively. This, of course, means that we must determine the asymptotic behavior of the four- and five-point amplitudes in these variables. While no essential difficulty is encountered in the case of four-point function, one is faced with the problem of knowing the nature of the equal-time commutator

$$\mathcal{A}_\nu\equiv[\partial D^\beta(x)/\partial x_0,A_\nu^\beta(0)]\delta(x_0)$$

in the case of the five-point function. In the work of Gordon and Peccei the corresponding commutator is

$$\mathcal{U}_\nu\equiv[\partial D^\beta(x)/\partial x_0,V_\nu^\beta]\delta(x_0)$$

and they make the assumption that it is either a $c$ number (in which case its matrix element between two different states is zero) or it transforms like an isovector so that the commutator itself is zero, the two isospin indices being equal. We make the same assumptions with regard to our commutator. One can inquire whether these assumptions are valid for $\mathcal{U}_\nu$ in the first place; if so, are they also valid for $\mathcal{A}_\nu$? The answer to this is model dependent. Specifically, we need information on $\partial D^\beta(x)/\partial x_0=-i[H,D^\beta(x)]$. If this object transforms like the time component of the axial-vector current, the commutators $\mathcal{A}_\nu$ and $\mathcal{U}_\nu$ are both isovector. This would, for instance, be the case in a simple quark model in which $H$ is proportional to the scalar quark current density and $D$ to the pseudoscalar density. If, on the other hand, $[H,D^\beta(x)]$ is proportional to the pseudoscalar current density, the two commutators $\mathcal{A}_\nu$ and $\mathcal{U}_\nu$ have different isospin character, being symmetric and antisymmetric in the isospin indices, respectively. In such an eventuality (barring the possibility that $\mathcal{A}_\nu$ is a $c$ number) we would not be able to write once-subtracted dispersion relations in the case of the five-point function and Eqs. (3.22) and (3.24) would not follow. In Gordon and Peccei's case, however, their asymptotic relations would still follow since $\mathcal{U}_\nu$ would transform like an isovector irrespective of whether $\partial D^\beta(x)/\partial x_0$ is proportional to a pseudoscalar current density or to the time component of an axial-vector current density.

To conclude, we are led to the nonrenormalization of the axial-vector coupling constant when we simultaneously consider the asymptotic form-factor relations following from single- and double-pion weak and electroproduction. This result, however, hinges on the important assumption that $\partial D^\beta(x)/\partial x_0$ transforms like the time component of an axial-vector current density.

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## APPENDIX A

Here we give the Low representation for the amplitude $M_{\mu\nu}{}^{\alpha\beta}$. The collinear Breit frame (i.e., $\mathbf{p}'=-\mathbf{p}$, $p_0'=p_0=E$, $\mathbf{q}=0$, $q_0=xE$) is used.

$$
\begin{aligned}
(2\pi)^{-3}M_{\mu\nu}{}^{\beta\alpha}=&\sum_n\left[\frac{\delta^3(\mathbf{p}'-\mathbf{p}_n)}{-xE+E-E_n}\langle N(p')|A_\mu{}^\beta|n\rangle\langle n|A_\nu{}^\alpha|N(p)\rangle+\text{c.t.}\right]\\
&+\sum_m\left[\frac{\delta^3(\mathbf{p}'-\mathbf{p}-\mathbf{p}_m)}{-xE-E_m}\langle N(p')|A_\mu{}^\beta|m,N(p)\rangle\langle m|A_\nu{}^\alpha|0\rangle+\text{c.t.}\right.\\
&\left.+\frac{\delta^3(\mathbf{p}'+\mathbf{p}-\mathbf{p}_m)}{-xE-E_m}\langle 0|A_\mu{}^\beta|m\rangle\langle m,N(p')|A_\nu{}^\alpha|N(p)\rangle+\text{c.t.}\right]\\
&-\sum_l\left[\frac{\delta^3(\mathbf{p}'-\mathbf{p}_l)}{-xE-E-E_l}\langle 0|A_\mu{}^\beta|l,N(p)\rangle\langle l,N(p')|A_\nu{}^\alpha|0\rangle+\text{c.t.}\right].
\end{aligned}\tag{A1}
$$

The crossed term (c.t.) is obtained from the direct term by the replacement $\mathbf{p}'\leftarrow D$ in the $\delta$ function and $x\rightarrow -x$ in the denominator, and the interchange $A_\mu{}^\beta\leftrightarrow A_\nu{}^\alpha$. As one can see from (A1), the three types of intermediate states $n$, $m$, $l$ have baryon numbers $+1$, $0$, $-1$, respectively. Terms with these intermediate states correspond, respectively, to fully connected, semidisconnected, and $Z$ graphs.

To obtain the Low representation for $U_\nu{}^{\beta\alpha}$, make the replacement

$$A_\mu{}^\beta\rightarrow iD^\beta.\tag{A2}$$

## APPENDIX B

Given below is the Low representation of the amplitude $T_{\mu\nu}{}^{\beta\alpha}$ in the collinear Breit frame ($\mathbf{k}=0$, $k_0=\mu$, $\mathbf{p}'=-\mathbf{p}$, $p_0'=p_0=E$, $\mathbf{q}=0$, $q_0=yE'$):

$$
\begin{aligned}
-(2\pi)^3T_{\mu\nu}{}^{\beta\alpha}=&\sum_n\left[\frac{\delta^3(\mathbf{p}'-\mathbf{p}_n)}{yE'+E-E_n}\langle N(p')|A_\mu{}^\beta|n\rangle\langle n|A_\nu{}^\beta|N(p),\pi_\alpha(k)\rangle+\text{c.t.}\right.\\
&\left.+\frac{\delta^3(\mathbf{p}'-\mathbf{p}_n)}{yE'+E-\mu-E_n}\langle N(p')|A_\mu{}^\beta|n,\pi_\alpha(k)\rangle\langle n|A_\nu{}^\beta|N(p)\rangle+\text{c.t.}\right]\\
&+\sum_m\left[\frac{\delta^3(\mathbf{p}'-\mathbf{p}-\mathbf{p}_m)}{yE'-E_m}\langle N(p')|A_\mu{}^\beta|m,N(p)\rangle\langle m|A_\nu{}^\beta|\pi_\alpha(k)\rangle+\text{c.t.}\right.\\
&+\frac{\delta^3(\mathbf{p}'-\mathbf{p}-\mathbf{p}_m)}{yE'-\mu-E_m}\langle N(p')|A_\mu{}^\beta|m,N(p)\pi_\alpha(k)\rangle\langle m|A_\nu{}^\beta|0\rangle+\text{c.t.}\\
&+\frac{\delta^3(\mathbf{p}'+\mathbf{p}-\mathbf{p}_m)}{yE'-E_m}\langle 0|A_\mu{}^\beta|m\rangle\langle m,N(p')|A_\nu{}^\beta|N(p),\pi_\alpha(k)\rangle+\text{c.t.}\\
&\left.+\frac{\delta^3(\mathbf{p}'+\mathbf{p}-\mathbf{p}_m)}{yE'-\mu-E_m}\langle 0|A_\mu{}^\beta|m,\pi_\alpha(k)\rangle\langle m,N(p')|A_\nu{}^\beta|N(p)\rangle+\text{c.t.}\right]\\
&-\sum_l\left[\frac{\delta^3(\mathbf{p}'-\mathbf{p}_l)}{yE'-E-E_l}\langle 0|A_\mu{}^\beta|l,N(p)\rangle\langle l,N(p')|A_\nu{}^\beta|\pi_\alpha(k)\rangle+\text{c.t.}\right.\\
&\left.+\frac{\delta^3(\mathbf{p}'-\mathbf{p}_l)}{yE'-E-\mu-E_l}\langle 0|A_\mu{}^\beta|l,N(p),\pi_\alpha(k)\rangle\langle l,N(p')|A_\nu{}^\beta|0\rangle+\text{c.t.}\right].
\end{aligned}\tag{B1}
$$

The crossed term is obtained from the direct term by the replacement $\mathbf{p}' \to \mathbf{p}$ in the $\delta$ function and $yE' \to -yE'+\mu$ in the denominator, and the interchange $A_\mu{}^\beta \leftrightarrow A_\nu{}^\beta$. The right-hand side of (B1) is the sum of terms of three categories, namely with the intermediate states of baryon number $+1$, 0, and $-1$.

The Low representation for the amplitude $W_\nu{}^{\beta\alpha}$ is obtained on making the replacement

$$A_\mu{}^\beta \to -iD^\beta. \tag{B2}$$

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# Low-Momentum-Transfer Predictions of the Collinearized Current Algebra*

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We investigate the low-momentum-transfer predictions of the collinearized current algebra at $p=\infty$. The study is greatly facilitated by combining the Dashen–Gell-Mann angular conditions with current algebra in order to obtain a difference equation for the dipole-moment operator. In the resonance approximation an elementary solution is obtained.

## I. INTRODUCTION

THE problem of finding the representations of relativistic current algebra at finite momentum transfer has been of considerable interest to elementary-particle physicists.[1] However, a number of examples[2] and general arguments[3] seem to preclude the existence of physical solutions.[4] Nevertheless, since the answer to this problem is intimately linked to the spectrum and degeneracy of the space on which the solution is realized, a definitive answer is not yet available.

In contrast to the difficulties encountered above, the collinearized current algebra has been a useful scheme for relating the low-energy parameters of different resonances.[5] For instance, by saturating the isospin algebra with $N$ and $N^*(1236)$, some of the $SU(6)$ predictions are obtained.[6] A more elaborate saturation scheme which includes $N$, $N^*(1236)$, and $D_{13}(1525)$ as well as the parametrization of transitions to other states results in an excellent prediction of the $\gamma N D_{13}$ coupling.[7]

The results above were obtained by first realizing the dipole moment operator in the subspace of low-lying spin states and then calculating form factors. The requirement of Lorentz invariance was then imposed. This last step severely restricts the solution; even then the resulting form factors obey the algebra and covariance requirements at low momentum transfer only. However, this approach becomes increasingly involved with the enlargement of the subspace.

The method outlined below converts the low-energy requirement of current algebra and relativistic invariance into a nonlinear difference equation. It must be kept in mind that a solution to this problem on a set of resonances may not suffice to obtain form factors at finite momentum transfer. Then this scheme would only relate the static parameters of a large family of resonances, by analogy with the results referred to previously.

## II. KINEMATICS

At $p_z=\infty$, the $SU(3)$ current algebra, written in the Okubo notation, is

$$[V_\beta{}^\alpha(\mathbf{q})V_\delta{}^\gamma(\mathbf{q}')]=\delta_\delta{}^\alpha V_\beta{}^\gamma(\mathbf{q}+\mathbf{q}')-\delta_\beta{}^\gamma V_\delta{}^\alpha(\mathbf{q}+\mathbf{q}'). \tag{1}$$

The formal solution of this algebra is

$$V_\beta{}^\alpha(\mathbf{q})=\exp[-\mathbf{q}\cdot\mathbf{V}'_\alpha{}^\alpha(0)]V_\beta{}^\alpha(0)\times\exp[\mathbf{q}\cdot\mathbf{V}'_\alpha{}^\alpha(0)],\quad \alpha\neq\beta \tag{2}$$

where

$$\mathbf{V}'_\alpha{}^\alpha(0)=\boldsymbol{\nabla}V_\alpha{}^\alpha(\mathbf{q})\,|_{\mathbf{q}=0}. \tag{3}$$

From Eq. (1),

$$[V'_{x\alpha}{}^\alpha(0),V'_{y\alpha}{}^\alpha(0)]=0, \tag{4}$$

where $V'_{x\alpha}{}^\alpha(0)$ and $V'_{y\alpha}{}^\alpha(0)$ are the components of the dipole moment operator, $\mathbf{V}'_\alpha{}^\alpha(0)$. Since the dipole moment operator determines the form factor completely, its kinematic properties will be described first.

As a consequence of the reality properties of the charge density, it follows that

$$V_\beta{}^\alpha(\mathbf{q})^\dagger=V_\alpha{}^\beta(-\mathbf{q}). \tag{5}$$

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