the right-hand side of (41′) is given by $\rho_V^{\pi\pi}$, Eq. (55). Since F^* appears in the discontinuity of all three terms of (41′), we have the result that $M_{\pi\pi}^{11}(t)$, $M_{\pi A}^{11}(t)$, and $F(t)$ are *linearly related* in the region $(2m_\pi)^2<t<(4m_\pi)^2$. Since this is a relation among analytic functions of t, it holds throughout their common domain of analyticity. This result cannot be obtained in any way from conventional dispersion theory alone. Given the usual hypotheses of current algebra, it is an *exact* result, obtained from three-point function Ward identities and unitarity. A similar relation can also be deduced in a current algebraic analysis of four-point functions.[30] In this case the relation is among $M_{\pi\pi}^{T=1}(s,t)$, $M_{\pi A}^{T=1}(s,t)$, where $s=-(q+p_2)^2$, and $F(t)$, and the soft-pion limit, p or $q\to 0$, is taken to eliminate $M_{\pi A}$. The Veneziano representation,[31] used off the pion mass shell, for $M_{\pi\pi}$ then in principle provides a determination of $F(t)$. Our relation gives $F(t)$ in terms of partial-wave amplitudes. If we wished to invoke A_1 dominance in p^2 and q^2 for $M_{\pi A}^{11}$, we would have a prescription for obtaining $F(t)$ from on-shell $\pi\pi$ p-wave elastic scattering and from the $T=J=1$ amplitude for $\pi\pi$ on shell $\to A_1A_1$ off shell. We regard this result as an interesting example of what can be learned from the simultaneous application of the constraints of current algebra and of unitarity. Further consequences that it may have towards improving our knowledge of the t-dependent form factors and phase shifts for the $A_1\rho\pi$ system are actively being examined.

[30] P. Nath, R. Arnowitt, and M. Friedman, Phys. Rev. D 1, 1813 (1970).

[31] G. Veneziano, Nuovo Cimento 57A, 190 (1968).

ACKNOWLEDGMENT

Part of the work of one of us (E.G.) was carried out at the Aspen Center for Physics. Their hospitality is gratefully acknowledged.

PHYSICAL REVIEW D VOLUME 2, NUMBER 8 15 OCTOBER 1970

Pion-Pion Dynamics in the σ Model

J. L. Basdevant

Centre d'Etudes Nucléaires de Saclay, Service de Physique Théorique, CEN-Saclay BP. 2-91-Gif-sur-Yvette, France
and
Laboratoire de Physique Théorique et Hautes Energies, Orsay, France*

AND

B. W. Lee†

Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790

(Received 30 April 1970)

We propose an unambiguous way of constructing amplitudes which satisfy both unitarity and the current-algebra constraints. This consists in working out higher-order corrections on a Lagrangian which produces the correct soft-pion limit in the tree approximation. We consider $\pi\pi$ scattering in the σ model, and we compute the perturbation series up to second order. The renormalization procedure preserves the partially conserved axial-vector current condition and the current-algebra constraints at each order. In order to sum the strong-coupling perturbation series, we use the Padé-approximation technique. Thereby, our partial-wave amplitudes satisfy unitarity. The ρ and f_0 resonances are generated, although they were not present in the Lagrangian. Our unitary amplitudes satisfy crossing symmetry to a very good accuracy, showing the consistency of the results. Our results are in agreement with the "up-down" solution of the $I=0$, s-wave $\pi\pi$ phase shift, with a very broad σ resonance; the $I=2$ s-wave phase shift is repulsive, and agrees very well with experiment.

I. INTRODUCTION

ALTHOUGH current algebra has been successful in in describing low-energy pion processes, the predictive power of the theory in the form used so far becomes weakened as soon as the energy increases beyond the threshold, since the unitarity is not taken into account in the usual treatments. With the help of chiral Lagrangians, one can realize the results of current algebra within the framework of Lagrangian field theory; based on this observation, we have proposed[1] an unambiguous way of unitarizing the current-algebra amplitude. This consists in taking a Lagrangian which is renormalizable and which produces the correct soft-pion limit, and in computing higher-order corrections and summing the presumably divergent perturbation series by the Padé algorithm.

The σ model of Gell-Mann and Lévy[2] is ideally suited for implementing this program. The Lagrangian

* Laboratoire associé au CNRS.

† Supported in part by U. S. Atomic Energy Commission under Contract No. AT(30-1)3668B.

[1] B. W. Lee, Nucl. Phys. B9, 649 (1969); see also J. L. Gervais and B. W. Lee, *ibid.* B12, 627 (1969).

[2] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

of the σ model is known to satisfy all the hypotheses of current algebra: The vector and axial-vector currents of the model satisfy the chiral commutation relations and the hypothesis of the partially conserved axial-vector currents (PCAC) holds in the model. Furthermore, the chiral partner of the π meson, the σ meson, may in fact exist in nature in the form of a broad resonance about 700 MeV. These facts give some credence to the supposition that the σ model may be a reasonable model for low-energy $\pi\pi$ scattering. In Ref. 1 we have shown that the σ model can be renormalized in such a way as to preserve the current-algebra condition and PCAC, owing to the fact that the divergent parts of higher-order terms are strictly chiral symmetric.

Once renormalization is performed, one has still to cope with the problem of nonconvergence of the perturbation series. In recent years much progress has been made in the use of the Padé approximations to surmount this difficulty.[3–5] While there exists no proof that the Padé approximants converge to the true solution in field theory, this method is a particularly simple one among all techniques of summing divergent series, with the additional virtue that the method yields amplitudes which are exactly unitary when applied to partial waves. Meson dynamics has been studied in the Φ^4 theory,[3–4] using the Padé technique. The results indicate that (1) the approximants appear to converge rather rapidly in practice, and (2) higher partial-wave resonances like ρ and f_0 are generated even in lower-order approximations, but that (3) the s-wave $\pi\pi$ phase shifts so computed are in disagreement with experiment. Since various tests of consistency of the Padé amplitudes (such as the stability of the amplitudes as the order of approximation is increased, and the extent to which the amplitudes are crossing symmetric) indicate the reliability of the approximation scheme, the failure of the Φ^4 theory to give the correct s-wave $\pi\pi$ phase shifts seems to imply the inadequacy of the dynamical input of this particular model. The approximate chiral symmetry and the particular role the pions play in the symmetry scheme may well be the missing dynamical input in the previous study of the meson dynamics based on the Padé technique.

In this paper we present the results of a calculation on low-energy $\pi\pi$ scattering ($\lesssim 1$ BeV) carried out along the lines indicated above. The $\pi\pi$ scattering amplitude is computed from the truncated σ model (i.e., with the neglect of the nucleon contribution) up to the second order in a perturbation expansion which maintains PCAC (and therefore the correct soft-pion limit) in each order. A Padé approximant is constructed from the perturbation series. As we indicated before, the Padé amplitude so constructed is automatically unitary. A preliminary account of this calculation has been given elsewhere.[6] The present article gives a detailed exposition of the rational and method of the calculation, and a fuller description of the results obtained. In addition, the present discussion contains various tests of consistency of the amplitudes so constructed.

One of these tests consists in checking crossing symmetry of the resulting amplitudes. There are several reasons for this. One is that since our amplitudes are exactly unitary, we must be certain that crossing relations are not too badly violated in the course of enforcing unitarity. Another reason is that in discussing crossing properties we can clearly see the difference between our amplitudes and previous attempts at unitarizing current-algebra amplitudes.[7] Previous attempts have suffered some ambiguities stemming from the facts that (1) only the s and p waves were treated (these are the only waves present in the Weinberg amplitudes[8] that are linear in s, t, and u) and (2) they involve some arbitrary meromorphic functions of energy (this is essentially the Castillejo-Dalitz-Dyson ambiguity). One may believe that a way of removing these ambiguities is to impose crossing symmetry on the resulting unitary amplitudes. Recently, Roskies[9] has shown that, in the simple parametrization of Brown and Goble,[7] for instance, the present of both the σ resonance in the s wave and the ρ resonance in the p wave leads to a sizable violation of crossing symmetry. On the contrary, we shall demonstrate that in the present calculation crossing is well satisfied by the unitary amplitudes.

From the structure of Feynman graphs, one can regard the σ model as the sum of Φ^4 and Φ^3 theories, the various couplings and subtraction constants being related by PCAC and the current algebra constraints. From this point of view, the σ model supplements the Φ^4 theory by adding a new short-range force (σ exchange) as well as an s-wave isospin-zero resonance in order to implement chiral symmetry. In this respect, one expects that the σ model will maintain most of the desirable features of the previous Φ^4 calculations and correct the bad ones, in particular, the behavior of s-wave phase shifts, and this is in fact what happens. A remarkable fact is that the value of the arbitrary coupling constant g, which in the present calculation is obtained by fitting the physical mass of the s-wave σ resonance, is very

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[7] L. Brown and R. Goble, Phys. Rev. Letters **20**, 346 (1968); University of Washington report, 1969 (unpublished); R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. Letters **20**, 475 (1968); A. M. S. Amatya, A. Pagnamenta, and B. Renner, Phys. Rev. **172**, 1755 (1968); A. Neveu and J. Scherk, Nuovo Cimento Letters **1**, 414 (1969); D. F. Greenberg, Phys. Rev. **184**, 1934 (1969).

[8] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[9] R. Roskies, Phys. Letters **30B**, 42 (1969); Nuovo Cimento **65A**, 467 (1970).

close to the value of the same parameter in pure Φ^4 calculations which generates higher partial-wave resonances like the ρ and f_0 so well. In the present calculation, we obtain reasonable s-wave phase shifts, with a very broad σ resonance around 700 MeV, and the ρ and f_0 resonances are generated, although they have not been inserted in the Lagrangian. However, in this second-order calculation, the only Padé approximant we can construct is the [1,1], and it turns out that for higher partial waves, this approximant is insufficient if one desires accurate numerical results. The reason for this is fully analyzed and stems from the fact that the Born term of the series contains only σ exchange in the crossed channels, which is relatively short range, while the two-pion-exchange contributions of longer range are contained in higher-order terms. As a consequence, in this approximation, the ρ and f_0 tend to be too strongly bound in general. However, since we have the results of the Φ^4 calculations at our disposal, we may may speculate about the outcome of higher-order calculations in the σ model. Most probably, the numerical results will be considerably improved in a third-order calculation, since the gross features of $\pi\pi$ scattering are already present here.

In summary, the results we obtain are $\pi\pi$ scattering amplitudes which satisfy the current-algebra soft-pion limit, and which are unitary and approximately crossing symmetric to a surprising degree. The $I=0$ wave shift exhibits a broad resonance around the ρ-meson mass in the manner of the so-called "up-down" solution. The $I=1$ p wave and the $I=0$ d wave contain resonances approximately at the masses of the ρ and f_0 mesons.

The paper is organized as follows. In Sec. II we recall the σ model, its renormalization, and the PCAC condition. In Sec. III we give the explicit renormalization at second order and we compute the amplitudes at that order. We also show how the parameters in the theory are fixed. In Sec. IV we give a general discussion of the Padé approximation and its application in the present calculation. Section V contains the numerical results. The discussion of crossing symmetry is given in Sec. VI. Finally, our concluding remarks are contained in Sec. VII, where we examine the possible extensions of the theory. In Appendix A, we discuss the Goldstone limit of the σ model. Computational details are relegated to Appendix B.

II. σ MODEL

In this section we shall summarize the results of Ref. 1 on the renormalization of the σ model. The Lagrangian of the σ model, first studied by Schwinger[10] and Gell-Mann and Lévy,[2] is, neglecting the nucleon fields,

$$\mathcal{L}=\tfrac{1}{2}[(\partial_\mu\sigma_0)^2+(\partial_\mu\boldsymbol{\pi}_0{}^2)]-\tfrac{1}{2}\mu^2(\sigma_0{}^2+\boldsymbol{\pi}_0{}^2)-\tfrac{1}{4}g_0(\sigma_0{}^2+\boldsymbol{\pi}_0{}^2)^2+c_0\sigma_0-\tfrac{1}{2}\delta\mu^2(\sigma_0{}^2+\boldsymbol{\pi}_0{}^2)\,, \quad (2.1)$$

[10] J. Schwinger, Ann. Phys. (N. Y.) **2**, 407 (1957).

where σ_0 and $\boldsymbol{\pi}_0$ are the isoscalar—scalar and isovector—pseudoscalar meson fields, respectively.

Except for the term $c_0\sigma_0$, the Lagrangian (2.1) is invariant under $SU(2)\times SU(2)$ transformations, whose infinitesimal forms are given by

$$\boldsymbol{\pi}_0\to\boldsymbol{\pi}_0+\boldsymbol{\alpha}\times\boldsymbol{\pi}_0+\boldsymbol{\beta}\sigma_0\,,$$
$$\sigma_0\to\sigma_0-\boldsymbol{\beta}\cdot\boldsymbol{\pi}_0\,.$$

The axial-vector currents, which are the responses of the Lagrangian density to the variation in $\partial_\mu\boldsymbol{\beta}(x)$, are given by

$$\mathbf{A}_\mu(x)=[\boldsymbol{\pi}_0\partial_\mu\sigma_0-\sigma_0\partial_\mu\boldsymbol{\pi}_0](x)\,; \quad (2.2)$$

they satisfy the PCAC condition,

$$\partial_\mu\mathbf{A}^\mu(x)=c_0\boldsymbol{\pi}_0(x)\,. \quad (2.3)$$

Because of the Lagrangian possesses the chiral-transformation properties assumed in current algebra, the tree diagrams of the Lagrangian (2.1) give precisely the results of current algebra in the soft-pion limit.[11] In Refs. 1 and 12 it was shown to be possible to renormalize the theory in such a way as to preserve the current-algebra constraints and the PCAC condition in each order of the perturbation expansion.

For $\mu^2>0$, if we renormalize the fields and coupling constants according to

$$(\sigma_0,\boldsymbol{\pi}_0)=\hat{Z}^{1/2}(\hat{\sigma},\hat{\boldsymbol{\pi}})\,, \quad (2.4a)$$
$$g_0=gZ_g/\hat{Z}^2\,, \quad (2.4b)$$
$$c_0=\hat{\gamma}\hat{Z}^{-1/2}\,, \quad (2.4c)$$

and if we choose $\delta\mu^2$, $\hat{Z}$, and Z_g to be the ones that make the symmetric theory (i.e., $c_0=0$, $\mu^2>0$) finite, then all the Green's functions of the σ model become finite functions of finite parameters μ^2, g, and $\hat{\gamma}$, and external momenta. After this intermediate renormalization, all physical quantities may be expressed in terms of the physical pion mass $m_\pi{}^2$, the renormalized coupling constant g, and the vacuum expectation value $\hat{v}$ of the σ field:

$$\hat{v}=\langle 0|\hat{\sigma}(0)|0\rangle\,, \quad (2.5)$$

where $\hat{v}$ is related to $\hat{\gamma}$, as we shall see presently. Once the renormalized theory is expressed in terms of these parameters, the theory contains no reference to the unphysical parameter μ^2, and may be extended to the domain which corresponds to $\mu^2<0$, in particular, to the Goldstone case $\mu^2<0$, $c_0=0$ (see Appendix A for a discussion of this case). The intermediate renormalization renders the theory finite, but the intermediate renormalized field $\hat{\boldsymbol{\pi}}$ is not asymptotically normalized to the unit amplitude:

$$\langle 0|\hat{\pi}^\alpha(x)|\pi(p,\beta)\rangle=[e^{-ipx}/(2\pi)^{3/2}(2p_0)^{1/2}]\delta_{\alpha\beta}C_\pi{}^{1/2}\,, \quad (2.6)$$

[11] W. Bardeen and B. W. Lee, Phys. Rev. **177**, 2389 (1969).

[12] K. Symanzik, Nuovo Cimento Letters **11**, 1 (1969); Commun. Math. Phys. **16**, 48 (1970).

where $C_\pi \neq 1$ and the final renormalization of the π is necessary:

$$\hat{\boldsymbol{\pi}}(x)=C_\pi^{1/2}\boldsymbol{\pi}(x)\,, \tag{2.7a}$$

with C_π now finite. Since the σ particle will be unstable and therefore unobservable asymptotically, there is no need for a final renormalization of the σ field. It is sometimes convenient, however, to further renormalize $\hat{v}$ and $\hat{\gamma}$ according to

$$\hat{v}=vC_\pi^{1/2}\,,\quad \hat{\gamma}=\gamma C_\pi^{-1/2}\,. \tag{2.7b}$$

The Feynman rules for the σ model can be derived most succinctly if we translate the σ field by its vacuum expectation value in Eq. (2.1):

$$\sigma_0=s_0+v_0\,,\quad v_0=\hat{Z}^{1/2}\hat{v}\,, \tag{2.8a}$$

$$\langle s_0\rangle_0=0\,. \tag{2.8b}$$

The resulting Lagrangian is

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\text{int}}\,, \tag{2.9}$$

where

$$\mathcal{L}_0=\tfrac{1}{2}[(\partial_\mu s_0)^2-\mu_\sigma^2 s_0^2]+\tfrac{1}{2}[(\partial_\mu\boldsymbol{\pi}_0)^2-\mu_\pi^2\boldsymbol{\pi}_0^2]\,, \tag{2.10a}$$

$$\mu_\sigma^2=\mu^2+3g_0v_0^2\,, \tag{2.10b}$$

$$\mu_\pi^2=\mu^2+g_0v_0^2\,, \tag{2.10c}$$

and

$$\mathcal{L}_{\text{int}}=-\tfrac{1}{4}g_0(s_0^2+\boldsymbol{\pi}_0^2)^2-(g_0v_0)s_0(s_0^2+\boldsymbol{\pi}_0^2)$$
$$-\tfrac{1}{2}\delta\mu^2(s_0^2+\boldsymbol{\pi}_0^2)+[c_0-v_0(\mu_\pi^2+\delta\mu^2)]s_0\,. \tag{2.11}$$

The (unrenormalized) perturbation series is defined as the expansion of Green's functions in g_0 with $g_0v_0^2$ fixed. It was shown in Ref. 1 that the nth term in the expansion corresponds to diagrams with $n-1$ loops. The explicit Feynman rules have been written down in Ref. 1; we shall summarize them in Fig. 1.

The value of v_0 cannot be given *a priori*; it must be determined by the condition (2.8b). In lowest order it is

$$c_0=v_0\mu_\pi^2\,. \tag{2.12}$$

In higher orders the value of v_0 is determined by the requirement that the sum of the so-called s tadpole diagrams must be zero:

$$c_0=v_0[\mu_\pi^2+\delta\mu^2+\mathfrak{F}(g_0,v_0)]\,, \tag{2.13a}$$

where $v_0\mathfrak{F}(g_0,v_0)$ is the sum of s tadpole diagrams with one or more internal loops. The precise form of Eq. (2.13a) is determined from the relation

$$\partial^\mu\langle 0|T(A_\mu{}^\alpha(x),\pi_0{}^\beta(0))|0\rangle$$
$$=i\delta^{\alpha\beta}\langle\sigma_0\rangle_0\delta^4(x)+c_0\langle T(\pi_0{}^\alpha(x)\pi_0{}^\beta(0))\rangle\,, \tag{2.13b}$$

which gives

$$v_0=-c_0\Delta_\pi(0)\,, \tag{2.13c}$$

where $\Delta_\pi(k^2)$ is the unrenormalized full-pion propagator. Therefore, renormalizing the pion propagator c_0 and v_0

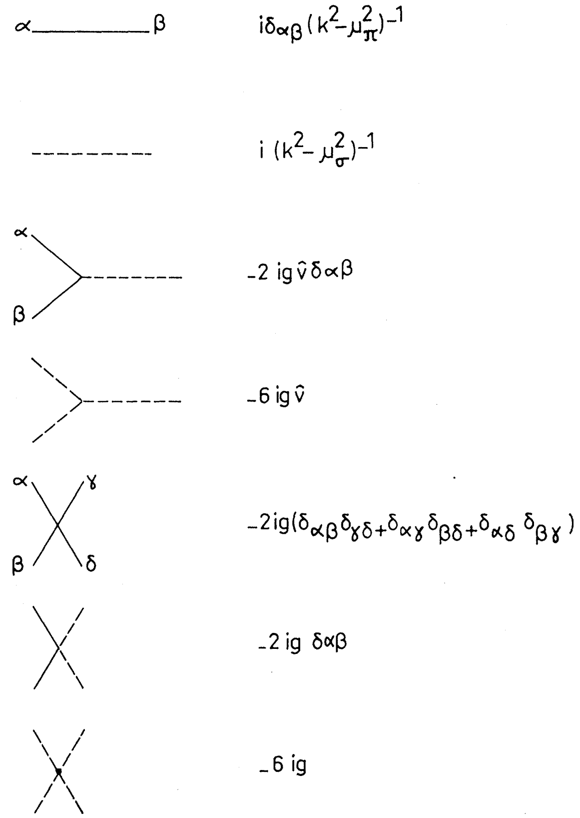

FIG. 1. Feynman rules for the renormalized Lagrangian. Straight lines represent pions, dashed lines represent σ particles.

by

$$\Delta_\pi=Z_\pi\Delta_\pi'\,,\qquad v_0=Z_\pi^{1/2}v\,,\qquad c_0=Z_\pi^{-1/2}\gamma\,,$$

where $Z_\pi=\hat{Z}C_\pi$, and Δ_π' is the renormalized full-pion propagator, we obtain the renormalized form of Eq. (2.13c):

$$\gamma=v[-\Delta_\pi'(0)]^{-1}\,, \tag{2.14}$$

which is a nonlinear equation for v in terms of γ, g, and the physical mass of the pion, m_π^2. Note also that Eq. (2.14) is the formal statement of the Goldstone theorem.

The renormalized parameter γ has a direct physical meaning. From Eq. (2.3) we see that

$$\partial_\mu\mathbf{A}^\mu(x)=c_0\boldsymbol{\pi}_0(x)$$
$$=(c_0Z_\pi^{1/2})[Z_\pi^{-1/2}\boldsymbol{\pi}(x)]=\gamma\boldsymbol{\pi}(x)\,, \tag{2.15}$$

so that

$$\gamma=f_\pi m_\pi^2\,, \tag{2.16}$$

f_π being the physical pion decay constant. We may therefore write Eq. (2.14) in the form

$$f_\pi=v[-\Delta_\pi'^{-1}(0)/m_\pi^2]\,. \tag{2.17}$$

For the ensuing discussion, the following point should be kept in mind: The Lagrangian (2.1) contains three parameters μ_0^2, g_0, and c_0. The perturbation series T_G of the Green's function G renormalized according to the

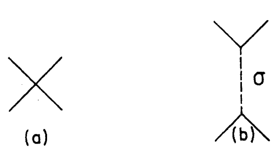

FIG. 2. First-order Feynman graphs; one has to take the contribution of each graph in all channels (s,t,u).

prescription just discussed is a power series in g with $g\hat{v}^2$ and $m_\pi{}^2$ considered as fixed parameters:

$$T_G(g;g\hat{v}^2,m_\pi{}^2)=\sum_n g^n(T_G)_n(g\hat{v}^2,m_\pi{}^2). \quad (2.18)$$

The parameter $\hat{v}$ is determined from Eq. (2.17), which is of the form

$$f_\pi=\hat{v}F(g;g\hat{v}^2,m_\pi{}^2)$$
$$=\hat{v}\sum_n g^nF_n(g\hat{v}^2,m_\pi{}^2), \quad (2.19)$$

once f_π is given. Thus, in principle, there is only one unknown parameter in the theory, which may be taken to be g, since $m_\pi{}^2$ and f_π are known.

III. RENORMALIZATION TO SECOND ORDER

As discussed in Ref. 1, in order to perform the renormalization of the σ model up to second order (i.e., in the one-loop approximation), we need to know the counter terms $\delta\mu^2$, $\hat{Z}$, and Z_g in the one-loop approximation in the symmetric theory. Since there is no one-loop self-energy insertion in the symmetric theory, we take

$$\delta\mu^2=0, \quad (3.1a)$$

$$\hat{Z}=0. \quad (3.1b)$$

The renormalization constant Z_g is, to this order,

$$Z_g\equiv 1-12gB_0, \quad (3.2)$$

where the infinite part of B_0 must be of the form

$$B_0\simeq-\frac{1}{4\pi^2}\ln\left(\frac{\Lambda}{\mu}\right)\simeq i\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2},$$

Λ being a cutoff parameter. By convention we choose B_0 to be

$$B_0=i\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2-m_\pi{}^2}\right)^2. \quad (3.3)$$

Let us consider the pion self-energy. As shown in Eq. (28′) of Ref. 1, the inverse pion propagator is

$$[\hat{\Delta}_\pi(k^2)]^{-1}=k^2-m_\pi{}^2+4(g\hat{v})^2[B_{\sigma\pi}(m_\pi{}^2)-B_{\sigma\pi}(k^2)] \quad (3.4)$$

after eliminating μ^2 in favor of $m_\pi{}^2$. The quantity $B_{xy}(k^2)$ is defined by

$$B_{xy}(k^2)=i\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-\mu_x{}^2}\frac{1}{(k-p)^2-\mu_y{}^2}, \quad (3.5)$$

where, to this order, we may take

$$\mu_\pi{}^2=m_\pi{}^2, \quad \mu_\sigma{}^2=m_\pi{}^2+2g\hat{v}^2. \quad (3.5')$$

The final renormalization constant C_π may be computed from Eq. (3.4):

$$C_\pi=1+4(g\hat{v})^2B_{\sigma\pi}{}'(m_\pi{}^2), \quad (3.6)$$

where $B_{\sigma\pi}{}'(s)=[dB_{\sigma\pi}(s)/ds]$.

The σ propagator is, to second order,

$$\hat{\Delta}_\sigma(k^2)=\frac{1}{k^2-m_\pi{}^2-2g\hat{v}^2+i\epsilon}$$
$$+(g\hat{v})^2\left(\frac{1}{k^2-m_\pi{}^2-2g\hat{v}^2+i\epsilon}\right)^2$$
$$\times[18\bar{B}_{\sigma\sigma}(k^2)-4\bar{B}_{\sigma\pi}(m_\pi{}^2)+6\bar{B}_{\pi\pi}(k^2)], \quad (3.7)$$

where

$$\bar{B}_{xy}(s)=B_{xy}(s)-B_0. \quad (3.7')$$

The connection between $\hat{v}=C_\pi{}^{1/2}v$ and f_π is given by Eq. (2.17), which, in this order, reads as

$$f_\pi=C_\pi{}^{1/2}\hat{v}\{1+4g(g\hat{v}^2/m_\pi{}^2)[B_{\sigma\pi}(0)-B_{\sigma\pi}(m_\pi{}^2)]\}. \quad (3.8)$$

This equation, together with the expression for the real part $M_\sigma{}^2$ of the pole of the σ propagator,

$$M_\sigma{}^2=m_\pi{}^2+2g\hat{v}^2+g(g\hat{v}^2)$$
$$\times[18\bar{B}_{\sigma\sigma}(M_\sigma{}^2)-4\bar{B}_{\sigma\pi}(m_\pi{}^2)+6\,\mathrm{Re}\bar{B}_{\pi\pi}(M_\sigma{}^2)], \quad (3.9)$$

connects the values of g and $\hat{v}$ in terms of f_π and the physical σ-resonance mass M_σ ($\simeq$700 MeV).

The proper π^4 vertex, given by Eqs. (39) and (40) of Ref. 1, is then

$$V_{\alpha\beta\gamma\delta}(p_1,p_2,p_3,p_4)$$
$$=-2igC_\pi{}^2[\delta_{\alpha\beta}\delta_{\gamma\delta}V(p_1,p_2;p_3,p_4)+\delta_{\alpha\gamma}\delta_{\beta\delta}V(p_1,p_3;p_2,p_4)$$
$$+\delta_{\alpha\delta}\delta_{\beta\gamma}V(p_1,p_4;p_2,p_3)]. \quad (3.10)$$

The function V is given by

$$V(p_1,p_2;p_3,p_4)$$
$$=1+g[7\bar{B}_{\pi\pi}(s)+2\bar{B}_{\pi\pi}(t)+2\bar{B}_{\pi\pi}(u)+\bar{B}_{\sigma\sigma}(s)]$$
$$+4g(g\hat{v}^2)[E_\pi(p_1,p_2)+E_\pi(p_3,p_4)+E_\sigma(p_1,p_2)$$
$$+E_\sigma(p_3,p_4)+E_\sigma(p_2,p_4)+E_\sigma(p_1,p_3)+E_\sigma(p_2,p_3)$$
$$+E_\sigma(p_1,p_4)]+8g(g\hat{v}^2)^2[D(p_1,p_2;p_3,p_4)$$
$$+D(p_1,p_2;p_4,p_3)], \quad (3.11)$$

where, as usual, $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$, and

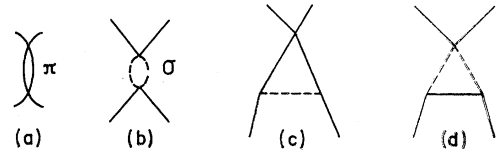

FIG. 3. Second-order diagrams with no poles and and a single spectral function.

$u=(p_1+p_4)^2$. The contribution of triangle graphs, for instance $E_\sigma(p,q)$, is given by

$$E_\sigma(p,q)=i\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p+k)^2-m_\pi^2}\times\frac{1}{(q-k)^2-m_\pi^2}\frac{1}{k^2-m_\sigma^2},\quad(3.12)$$

and $E_\pi(p,q)$ is obtained from E_σ by interchanging m_π and m_σ in the integrand. The box-diagram contributions D are given by

$$D(p_1,p_2;p_3,p_4)=i\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k+p_1)^2-m_\sigma^2}\frac{1}{(p_2-k)^2-m_\sigma^2}\times\frac{1}{k^2-m_\pi^2}\frac{1}{(p_1+p_3-k)^2-m_\pi^2}.\quad(3.13)$$

The $\sigma\pi^2$ vertex is given by Eqs. (41) and (42) of Ref. 1 as

$$\Gamma_{\alpha\beta}(p_1,p_2;k)=-2ig\hat{v}C_\pi\Gamma(p_1,p_2;k)\delta_{\alpha\beta},\quad(3.14)$$

where Γ is given by

$$\Gamma(p_1,p_2;k)=1+g\{2\bar{B}_{\sigma\pi}(p_1^2)+2\bar{B}_{\sigma\pi}(p_2^2)+3\bar{B}_{\sigma\sigma}(k^2)+5\bar{B}_{\pi\pi}(k^2)-4(g\hat{v}^2)[E_\sigma(p_1,p_2)+3E_\pi(p_1,p_2)]\}.\quad(3.15)$$

It is now straightforward to write the $\pi\pi$ scattering amplitude up to second order. The first- and second-order Feynman diagrams are represented in Figs. 2–5. It is also easy to check, from the preceding equations, that Adler's self-consistency condition[13] and Weinberg's relation[8] are true in the second-order $\pi\pi$ amplitude. In Appendix B we give the analytic expressions for the partial-wave projections of the first- and second-order amplitudes, together with their analytic continuation in the complex angular momentum plane.

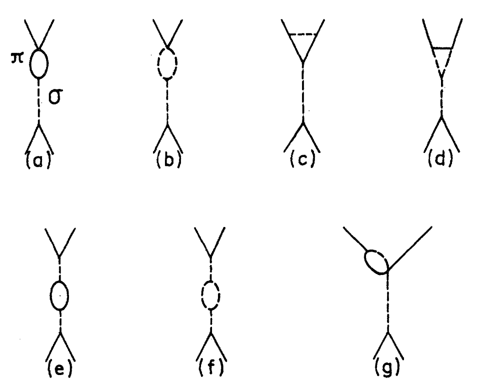

FIG. 4. Second-order diagrams, pole terms.

[13] S. L. Adler, Phys. Rev. **137**, B1022 (1965).

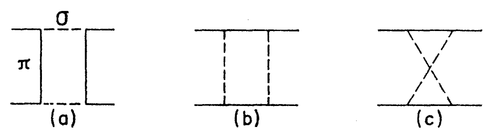

FIG. 5. Second-order box diagrams.

IV. PADÉ METHOD

A. Principle of Method

The Padé approximation has been used extensively in recent years in order to compute scattering amplitudes of strongly interacting systems.[3–5] For a general review of the properties of the approximation, we refer the reader to Ref. 14.

The method consists in approximating a function (in our case, the scattering amplitude considered as a function of the coupling constant g) by a rational function of g instead of a polynomial (like the truncated perturbation series). Rational functions are almost as easy to deal with as polynomials, but they are better suited to simulate singularities of the original function and, therefore, give meaningful approximations in much larger regions. Given the formal power-series expansion in g of the S matrix,

$$S(g)=1+gS_1+g^2S_2+\cdots g^nS_n+\cdots,\quad(4.1)$$

where the dependence of the coefficients S_n on dynamical variables is implicit, the $[N,M]$ Padé approximant to $S(g)$ is defined as the ratio of two polynomials in g, $P_N(g)$, and $Q_M(g)$ of degrees N and M, respectively, which has the same Taylor-series expansion around $g=0$ as $S(g)$ of Eq. (4.1) up to order $N+M$:

$$S^{[N,M]}(g)\equiv P_N(g)/Q_M(g)=S(g)+O(g^{N+M+1}).\quad(4.2)$$

In potential scattering, it has been proved that the diagonal Padé approximants ($N=M$) converge to the true solution as $N\to\infty$,[15] and therefore this method is capable of handling the usual problem of nonconvergence of the perturbation series associated with strong coupling. In quantum field theory no such proof exists at the moment. We accept on faith that the Padé approximants in field theory converge to the exact solution, or failing that, is a better asymptotic expansion of the exact solution than a truncated perturbation series.[16] It is of some interest to note that experience

[14] G. A. Baker, J. Advan. Theoret. Phys. 1, 1 (1965); J. L. Basdevant, in *Methods of Subnuclear Physics*, edited by M. Nikolic (Gordon and Breach, London, 1969), Vol. IV; see also the mathematical appendices of Ref. 4 and further references therein.

[15] R. Chisholm, J. Math. Phys. 4, 12 (1963).

[16] A proof of convergence of Padé approximants in field theory must come from the derivation of some analytic properties of Green's functions in the coupling constant. In this respect, the recent result of J. J. Loeffel, A. Martin, B. Simon, and A. S. Wightman [Phys. Letters **30B**, 656 (1969)], which shows that the Padé approximants converge for the energy levels of the anharmonic oscillator because of the analytic properties they possess in the coupling constant, is encouraging.

with some models of field theory (such as the $\lambda\Phi^4$ theory) has shown a quite remarkable stability of the approximations as the order is increased, and this may be taken as an indication of the reliability of the scheme.

Independently of its original motivation, which is to sum divergent series, the method has the important feature of yielding an amplitude which is exactly unitary in the elastic region. More precisely, if we consider the perturbation series for a partial-wave amplitude

$$t^l(s)=gt_1{}^l(s)+g^2t_2{}^l(s)+\cdots \quad (4.3)$$

for which the elastic unitary is an *algebraic* relation, for example,

$$\mathrm{Im}t^l(s)=[t^l(s)]^*\rho(s)t^l(s)\,, \quad (4.4)$$

[the relation (4.4) may be a matrix equation for a coupled-channel problem; the important criterion is that Eq. (4.4) is an algebraic, and not an integral equation], then the Padé approximants $t^{l[N,M]}(s)$ with $N\leq M$ are unitary, i.e., satisfy Eq. (4.4) exactly[17]:

$$\mathrm{Im}t^{l[N,M]}(s)=[t^{l[N,M]}(s)]^\dagger\rho(s)t^{l[N,M]}(s)\,. \quad (4.5)$$

This fact is in particular true for the diagonal Padé approximants $N=M$.[18]

Thus the application of the Padé method to the partial-wave amplitude will yield a unitary amplitude. On the other hand, the crossing symmetry present in the perturbation series is thereby lost. That is to say, the amplitude $\tilde{T}(s,t)$, defined by

$$\tilde{T}(s,t)=\sum_l(2l+1)t^{l[N,M]}(s)P_l(Z_s)\,,\quad Z_s=1+t/2s \quad (4.6)$$

does not enjoy the crossing symmetry (or relations) that Eq. (4.3) does. Of course, $T(s,t)-\tilde{T}(s,t)$, where

$$T(s,t)=\sum_l(2l+1)t^l(s)P_l(Z_s)\,,$$

is of order g^{N+M+1}, so that for weak coupling the violation of crossing is only of this order. The construction of the amplitude (4.6) would not be practical since it involves an infinite summation.[19] A practical way of testing the crossing relation in terms only of partial-wave amplitudes is afforded by the works of Balachandran and Nuyts,[20] and Roskies,[9] who derive a complete set of relations among partial-wave amplitudes which follow from the crossing symmetry of the full amplitude [complete in the sense that if the partial-wave amplitudes satisfy these relations, then the full amplitude constructed by the device of Eq. (4.6) is necessarily crossing symmetric]. In particular, Roskies relations for $\pi\pi$ scattering are an infinite set, each one of which however includes only a finite number of partial waves. Another test of crossing is the Martin inequalities,[21] which follow from the crossing symmetry and the positivity of the absorptive part of $\pi^0\pi^0$ amplitude. These tests have been applied to the Padé amplitudes for $\pi\pi$ scattering in the $\lambda\Phi^4$ theory,[22] and have shown that the violation of crossing is very small for the Padé amplitudes in the range of the coupling constant of interest.

Alternatively, one may apply the Padé method to the full amplitude:

$$T(s,t)=\sum_{n=1}^{\infty}g^nT_n(s,t)\,, \quad (4.7)$$

thereby obtaining a Padé approximant $T^{[N,M]}(s,t)$. The Padé approximant $T^{[N,M]}(s,t)$ is manifestly crossing-symmetric, but no longer satisfies the elastic unitarity relation, since it is an integral relation for the full amplitude. A consequence of this is that the pole of $T^{[N,M]}(s,t)$ corresponding to a dynamical bound state depends, in general, on both variables s and t, and the trajectory of the pole is a curve in the (s,t,u) plane (the so-called poloid; the trajectory of a bound-state pole of the T matrix must be a straight line parallel to the s, t, or u axis, depending on the channel in which it appears). Obvious tests of convergence of the method when applied to the full amplitude consist in checking the degree to which $T^{[N,M]}(s,t)$ violates elastic unitarity, and checking the flatness of the poloids. Again these tests have been performed for $\pi\pi$ scattering in the Φ^4 theory with satisfactory results.[4,23]

B. Application to σ Model

We now turn to the application of the method to the construction of a unitary $\pi\pi$ scattering amplitude from the σ model. The Feynman diagrams which contribute to the T matrix for $\pi\pi$ scattering up to second order in g are shown in Figs. 2–5 and the corresponding amplitudes are evaluated in Appendix B. It is important to recall that for the use in the Padé method we need a genuine expansion of the T matrix in the coupling constant g with $g\bar{v}^2$ and $m_\pi{}^2$ treated as fixed parameters. This

[17] S. Caser, C. Piquet, and J. L. Vermeulen, Nucl. Phys. **B14**, 119 (1969); D. Masson, J. Math. Phys. **8**, 512 (1967).

[18] As pointed out in Ref. 4, this is due to the fact that there exist groups of transformations on the function and its Padé approximants, and that homographic functional relations such as $S(s)S^*(s)=1$ are preserved under the mapping from the function to its Padé approximant.

[19] Since Padé approximants can be defined throughout the complex angular momentum plane (Ref. 3) for $\mathrm{Re}l>l_{\min}$ ($l_{\min}=0$ or 1 according to the renormalizable Lagrangian considered), one can convert the infinite summation into an integral by the Sommerfeld-Watson transformation. However, there are technical problems in calculating the residues of the Regge poles and this method has not been used up to now.

[20] A. P. Balachandran and J. Nuyts, Phys. Rev. **172**, 1821 (1968); A. P. Balachandran, W. J. Meggs, J. Nuyts, and P. Ramond, Phys. Rev. **187**, 2080 (1969), and further references therein.

[21] A. Martin, Nuovo Cimento, **47A**, 265 (1967); Nuovo Cimento Letters **58A**, 303 (1968); Nuovo Cimento **63A**, 167 (1969). The method has been extended to the general $\pi\pi$ case with isospin by G. Auberson, O. Brander, G. Mahoux, and A. Martin, *ibid.* **65A**, 743 (1970).

[22] J. L. Basdevant, G. Cohen-Tannoudji, and A. Morel, Nuovo Cimento **64**, 585 (1969).

[23] J. L. Basdevant and J. Zinn-Justin (unpublished).

means that no partial summation of the self-energy parts, for example, is allowed.

In order to obtain a unitary $\pi\pi$ amplitude, we shall apply the Padé method to the partial-wave amplitude $t(s; I,J)$ for definite isospin I and angular momentum J. For brevity, we shall often suppress the indices I and J. Let

$$t(s)=gt_1(s)+g^2t_2(s)+\cdots.$$

The [1,1] Padé approximant is built up of t_1 and t_2 and is

$$t^{[1,1]}=gt_1{}^2/(t_1-gt_2). \quad (4.8)$$

This approximant is exactly unitary in the elastic region $4m_\pi{}^2\leq s\leq 4m_\sigma{}^2$,

$$m_\sigma{}^2=m_\pi{}^2+2g\hat{v}^2, \quad (4.9)$$

in the $I=0$ states, and for all values of s in the $I=1$ and 2 states.

The $I=0$ s-wave amplitude deserves a special comment. This channel is unique in that it contains a pole corresponding to the elementary σ particle. To lowest order in g, the σ mass is given by Eq. (4.9). That is, the first-order term $t_1(s)$ has a pole at $s=m_\sigma{}^2$. The second-order term $t_2(s)$ has a double pole, as well as a simple pole at $s=m_\sigma{}^2$. In the neighborhood of $s=m_\sigma{}^2$, t_1 and t_2 behave as

$$\begin{aligned} t_1&=a+b/(s-m_\sigma{}^2), \\ t_2&=A(s-m_\sigma{}^2)^{-2}+B(s-m_\sigma{}^2)^{-1}+C, \end{aligned} \quad (4.10)$$

where A, B, and C are functions of s with elastic cuts along $s\geq 4m_\pi{}^2$. Hence the Padé approximant (4.8) behaves as

$$t^{[1,1]}\sim\frac{gb^2(b-gB)^{-1}}{s-m_\sigma{}^2-gA(b-gB)^{-1}}. \quad (4.11)$$

Let $M_\sigma{}^2$ be the value of s at which the real part of the Padé denominator vanishes. $M_\sigma{}^2$ so defined agrees with the expression (3.9) up to (and including) the first order in g. Owing to the cut structure of $A(s)$ and $B(s)$, the pole of $t^{[1,1]}(s)$ lies in the second sheet of the complex s plane if $M_\sigma{}^2>4m_\pi{}^2$. Clearly, this feature of the Padé approximant will persist in high orders as long as the approximant satisfies the elastic unitarity exactly, i.e., for $N\leq M$.

For higher partial waves, we encounter a different problem associated with the validity of the [1,1] Padé approximation itself. In many cases it has been shown that the [1,1] which is the simplest approximant one can build, can only yield qualitative results and that higher orders must be computed in order to obtain accurate quantitative results.[5,17,24,25] What happens here is that for $l>1$, the [1,1] approximation is not defined in the limit $m_\sigma{}^2\to\infty$. In fact, in first order of the perturbation series, the only term which contributes to higher waves ($l>0$) is Fig. 2(b), which corresponds to σ exchange between two pions. In the limit of large $m_\sigma{}^2$, this Born term behaves as

$$t_1\simeq g(m_\sigma{}^2)^{-l}\simeq(1/2\hat{v}^2)(m_\sigma{}^2)^{-l+1} \quad (4.12)$$

and vanishes for $m_\sigma{}^2\to\infty$ for $l>1$ [see, for instance, Eqs. (B4), (B6), and (B7) of Appendix B]. On the other hand, the second- and higher-order terms contain two-pion-exchange diagrams [e.g., Figs. 3(a), 3(c), and 6] which are finite in this limit. In our case, for a relatively large $(m_\sigma{}^2/m_\pi{}^2)$, the knowledge of t_1 and t_2 may be insufficient to generate a reliable approximant for $l>1$, since t_1 is abnormally small compared to other terms in the series. For instance, if we try to fix the coupling constant g by requiring that a given resonance occurs at its physical mass $m_R{}^2$ in Eq. (4.8), we have clearly

$$g_R=\mathrm{Re}[t_1(m_R{}^2)/t_2(m_R{}^2)]. \quad (4.13)$$

But since t_1 is too small, g_R will be smaller than the value obtained in higher calculations, the "true" value of g. In turn, if we compute Eq. (4.8) with the "true" g, the resonance mass will be too small and this effect will increase with the angular momentum, as is clear from Eq. (4.12). As will be seen in the next section, there is a tendency for higher partial-wave resonances to occur at low energies for the value of g which gives a good fit to the s-wave phase shift.

However, in this calculation $m_\sigma{}^2$ and g are not independent parameters once $\hat{v}$ is fixed and the value of $(m_\sigma{}^2/m_\pi{}^2)$ is not too large; also, the effective expansion parameter $g/8\pi^2$ is small in the cases discussed below, so that we expect the [1,1] Padé approximant to be a reasonable approximation to the true amplitude in the low-energy region for s, p, and perhaps d waves, although higher-order calculations are certainly necessary to verify these assertions.

The Padé approximation is basically a low-energy approximation. As the order of the approximation increases, the domain in the energy plane in which the approximation is valid becomes enlarged. This statement is based on the observation that the successive terms in the perturbation series include the effects of higher and higher mass intermediate states. Thus, the second-order terms in the σ model in Eq. (4.3) contains only 2π and 2σ intermediate states. Since, in the range of parameters of interest, the σ meson is unstable, the cut corresponding to the 2σ intermediate state must be

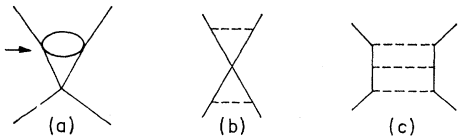

FIG. 6. Some third-order graphs which are dominant in higher partial waves; (a) contributes to all isospin channels, (b) contributes only to $I=0$ and $I=2$, (c) only to $I=0$.

[24] J. L. Basdevant and B. W. Lee, Nucl. Phys. **B13**, 182 (1969).
[25] H. M. Nieland and J. A. Tjon, Phys. Letters **27B**, 5 (1968).

TABLE I. Some values of the computed $\pi\pi$ phase shifts and resonance masses for several values of f_π. We first give the value of g, then the difference $\delta_0-\delta_2$ at the K mass (495 MeV) [experimentally, $(\delta_0-\delta_2)(m_K)=58°\pm17°$; see, for instance, C. D. Buchanan and K. Lande, Phys. Rev. Letters 21, 169 (1968)]; next δ_0 and δ_2 at the ρ mass; then the computed ρ and f_0 masses, and finally the mass and width of the σ resonance defined through the position of the second-sheet pole.

f_π (MeV)	g	$(\delta_0-\delta_2)$ (495 MeV)	δ_0 (760 MeV)	δ_2 (760 MeV)	m_ρ (MeV)	m_{f_0} (MeV)	M_σ (MeV)	Γ_σ (MeV)
95	6.96	81°	89°	−32°	600	870	425	220
110	6.20	65°	92°	−22°	680	970	470	260
120	5.72	56°	95°	−17°	750	1080	510	290
125	5.63	45°	90°	−13°	780	1115	530	310

displaced into some unphysical sheet in the exact amplitude. However, the lowest Padé approximant (4.8) approximates this cut by a cut $s\geq 4m_\sigma^2$ on the physical sheet. Presumably, higher-order Padé approximations will remedy this deficiency, but the approximant (4.8) is in any case unreliable for $s\gtrsim 4m_\sigma^2$.

To conclude this section, let us point out that there are other tests of convergence besides increasing the order and checking the crossing properties. In fact, one can also think of computing two- and three-point functions besides the scattering amplitude. For instance, one can directly compute the σ propagator itself:

$$D_\sigma(s)=[k^2-m_\sigma^2-\Sigma(k^2)]^{-1}, \qquad (4.14)$$

where the mass operator $\Sigma(k^2)$ is given as a series in the coupling constant

$$\Sigma(k^2)=g\Sigma_1(k^2)+g^2\Sigma_2(k^2)+\cdots. \qquad (4.15)$$

Again, all the coefficients of this series are finite after renormalization and one can sum the series with Padé approximants. It is then possible to compare, at a given finite order, the position and width of the physical σ resonance obtained in the two-point function and in the four-point function. One can also compute the electromagnetic form factor of the pion,

$$F(q^2)=1+gF_1(q^2)+g^2F_2(q^2)+\cdots. \qquad (4.16)$$

In the exact solution we know that the phase of the form factor in the region $16m_\pi^2\geq q^2\geq 4m_\pi^2$ is equal to the $\pi\pi$ p-wave phase shift, and, at finite order, one can compare the ρ resonance obtained in Padé approximations to the form factor and in the scattering amplitude. However, for two- and three-point functions we have no convergence theorems, and we do not know which Padé approximants to use preferably, whereas for the scattering amplitude the unitary properties indicate that we should use $N\leq M$ approximants to the T matrix. Further investigations are needed about these points.

V. NUMERICAL RESULTS

In the ensuing discussion, the physical pion mass m_π^2 is chosen as unit mass scale of the theory. We are then left with two parameters: the vacuum expectation value of the σ field $\hat{v}$, and the dimensionless coupling constant g. The values of these parameters can be fixed in terms of the pion decay constant f_π and the physical σ-resonance mass.

In Sec. III we saw that in second order the pion decay constant f_π is related to $\hat{v}$ and g by

$$f_\pi=C_\pi^{1/2}\hat{v}\{1+4g(g\hat{v}^2/m_\pi^2)[B_{\sigma\pi}(0)-B_{\sigma\pi}(m_\pi^2)]\}, \qquad (5.1)$$

where $B_{\sigma\pi}(k^2)$ and C_π are defined by Eqs. (3.5) and (3.6). In the lowest order we have, of course, $f_\pi=\hat{v}$.

Similarly, in order to fix the value of the coupling constant g, we could also use Eq. (3.9), knowing the physical mass of the σ resonance ($M_\sigma\sim 700$ MeV). However, there is a large uncertainty on the exact value of this mass; the σ resonance is very broad ($\Gamma_\sigma\sim$ 200–600 MeV) and its position is not well determined. In fact, since the width is large, the exact position of the second-sheet pole is difficult to determine accurately in terms of physical-region data. Therefore, in the present calculation we have preferred to determine g by imposing directly on the physical s-wave amplitude that the isospin zero phase shift δ_0 be close to 90° around 700 MeV, which seems a more reasonable assumption. Once g is fixed by this procedure, and $\hat{v}$ determined in terms of f_π by Eq. (5.1), we can compute all other characteristics of low-energy $\pi\pi$ scattering such as scattering lengths, isospin-two s-wave phase shift δ_2, higher partial-wave resonances such as ρ and f_0, and so on, with no additional assumption or parameter, and all our partial-wave amplitudes satisfy unitarity.

As we shall see further on, the important features of the results are (i) that the s-wave phase shifts agree well with experimental data and (ii) that the theory also predicts higher partial-wave resonances which can be identified with the ρ and f_0 mesons. However, the computed masses of the ρ and f_0 are generally smaller than the experimental values. This is a consequence of the poor convergence of the [1,1] Padé approximant, as explained in Sec. IV. We shall explain later on how this deficiency may be remedied in higher-order calculations. It is quite clear that in the present second-order calculation, we cannot reasonably expect the numerical results to deviate less than 20% from the experimental figures. In order to obtain a good over-all numerical representation of low-energy $\pi\pi$ scattering, we have considered the pion decay constant f_π as a parameter of the theory, and we have allowed its variation from the experimental value $f_\pi=95$ MeV up to $f_\pi=125$ MeV. As we explained in Sec. II, the T matrix is expanded in powers of g while $g\hat{v}^2$ is kept constant [see Eq. (2.7)]. The value of $\hat{v}$ is to be determined in principle from Eq. (2.18) in terms of f_π. Equations (5.1) is an approximation of Eq. (2.18). Therefore, we regard the variation of f_π as resulting from the approximate nature of Eq. (5.1). In other words, we shall vary $\hat{v}$ within certain range, so that *if* Eq. (5.1) *were exact* (which it is not), f_π would range from 95 to 125 MeV.

In Table I, we give for each value of f_π, the value of g which is used in the calculation, and which is determined by the $I=0$ s wave, together with some values of s-wave phase shifts, the ρ and f_0 masses, and the mass and width of the output σ resonance defined through the computed position of the corresponding second-sheet pole.

From the requirements that the f_π computed from the right-hand side of Eq. (5.1) does not deviate too much from the experimental value, and that the σ-resonance occurs near 700 MeV, we obtain the coupling constant g around 6 (see Table I). An inspection of the order of magnitude of various second-order terms shows (see Appendix B) that they are of order $g/8\pi^2$ compared to the first-order terms. The effective expansion parameter therefore appears to be $g/8\pi^2\simeq 0.1$. A notable exception to this "rule of thumb" is the σ mass squared, $M_\sigma^2\simeq m_\sigma^2$, Eq. (3.4), where the second-order term has a large ($\simeq 20$) numerical factor. In general, however, the the [1,1] Padé approximant deviates noticeably from the first-order approximation only near the singularities of the former (resonances and left-hand cut), and in some sense, the Padé approximant is a unitarization of the current-algebra amplitude which is "as smooth as possible" consistent with unitarity.

A. Scattering Lengths

Because the model satisfies the PCAC condition (2.15), the first- and second-order amplitudes satisfy Adler's self-consistency condition and Weinberg's relation

$$\lim_{p_1\to 0} T_{\alpha\beta,\gamma\delta}(p_1,p_2;p_3,p_4)=0\,,\quad p_2^2=p_3^2=p_4^2=m_\pi^2\,,$$

$$\lim_{p_1,p_3\to 0} T_{\alpha\beta,\gamma\delta}(p_1,p_2;p_3,p_4) = T^0\delta_{\alpha\gamma}\delta_{\beta\delta}+(i/f_\pi^2)(\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})p_1\cdot p_2\,.$$

Since our matrix element for $\pi\pi$ scattering is not linear in s, t, and u, we expect that the scattering lengths will differ somewhat from those of Weinberg. Since the effective expansion parameter $g/8\pi^2$ is small, we may estimate the scattering lengths from the renormalized first-order amplitude. To lowest order in g, we have

$$\hat{v}=f_\pi\,,\quad m_\sigma^2=m_\pi^2+2gf_\pi^2\,.$$

TABLE II. Scattering lengths obtained in this model. We give the value of the coupling constant g, the first-order vacuum expectation value of the σ field $\hat{v}$ which is very close to f_π, and for a_0 and a_2 our values together with those obtained through the Weinberg relations for each value of f_π. The coupling constant g is adjusted so that the resonance position of the σ meson lies between 700 and 760 MeV.

f_π (MeV)	g	$\hat{v}$ (MeV)	a_0 calc	a_0 Weinberg	a_2 calc	a_2 Weinberg
95	7	90	0.24	0.16	−0.043	−0.046
110	6.2	105	0.17	0.12	−0.033	−0.034
125	5.6	120	0.12	0.09	−0.025	−0.026

In this order, the $\pi\pi$ scattering amplitude is

$$T_{\alpha\beta,\gamma\delta}{}^1(s,t,u)=-2g\left[\frac{s-m_\pi^2}{s-m_\sigma^2}\delta_{\alpha\beta}\delta_{\gamma\delta}+\frac{t-m_\pi^2}{t-m_\sigma^2}\delta_{\alpha\gamma}\delta_{\beta\delta}+\frac{u-m_\pi^2}{u-m_\sigma^2}\delta_{\alpha\delta}\delta_{\beta\gamma}\right]. \quad (5.2)$$

As $g\to\infty$ ($m_\sigma^2\to\infty$, while $m_\sigma^2/g\to 2f_\pi^2$), Eq. (5.2) reduces to the Weinberg amplitude (linear current-algebra amplitude)

$$T_{\alpha\beta,\gamma\delta}{}^W(s,t,u)=(1/f_\pi^2)[(s-m_\pi^2)\delta_{\alpha\beta}\delta_{\gamma\delta}+(t-m_\pi^2)\times\delta_{\alpha\gamma}\delta_{\beta\delta}+(u-m_\pi^2)\delta_{\alpha\delta}\delta_{\beta\gamma}]. \quad (5.3)$$

The scattering lengths that follow from Eq. (5.2) are

$$\lim_{s\to 4m_\pi^2} 32\pi\left(\frac{s}{s-4m_\pi^2}\right)^{1/2} e^{i\delta_I}\sin\delta_I$$

$$\simeq 7\left(\frac{m_\pi}{f_\pi}\right)^2\left[1+\frac{29}{7}\left(\frac{m_\pi}{m_\sigma}\right)^2+\cdots\right],\quad I=0,\ l=0$$

$$\simeq -2\left(\frac{m_\pi}{f_\pi}\right)^2\left[1-\left(\frac{m_\pi}{m_\sigma}\right)^2+\cdots\right],\quad I=2,\ l=0$$

$$\simeq \frac{4}{3}\left(\frac{m_\pi}{f_\pi}\right)^2\left(\frac{s-4m_\pi^2}{4m_\pi^2}\right)\left[1-2\left(\frac{m_\pi}{m_\sigma}\right)^2+\cdots\right],\quad I=l=1\,. \quad (5.4)$$

As $m_\sigma^2\to\infty$, these values of course reduce to the Weinberg values. We expect from Eq. (5.4) that in the σ model the s-wave scattering length in the $I=0$ (2) state is somewhat larger (smaller) in magnitude than those of Weinberg (for the same value of f_π). This is borne out in the Padé-approximant calculation (see Table II). Note that in higher partial waves ($l=2$, 3), we have

$$\lim_{s\to 4m_\pi^2} 32\pi\left(\frac{s}{s-4m_\pi^2}\right)^{1/2} e^{i\delta_l}\sin\delta_l$$

$$\simeq \frac{m_\pi^2}{f_\pi^2}\,\frac{\pi^{1/2}\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}\,\frac{(m_\sigma^2-m_\pi^2)^2}{m_\sigma^2 m_\pi^2}\left(\frac{k^2}{m_\sigma^2}\right)^l,\quad l=2,3,\ldots \quad (5.5)$$

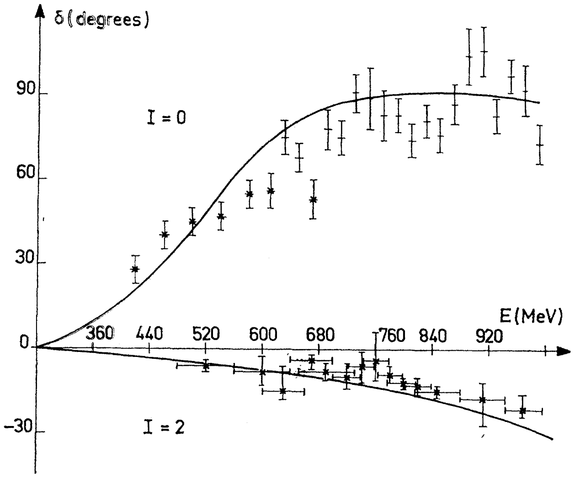

FIG. 7. The s-wave phase shifts obtained in the σ model, as a function of the total c.m. energy in MeV. This corresponds to $f_\pi=125$ MeV, $g=5.63$. Upper curve: isospin zero; experimental data from Ref. 27. Lower curve: isospin two, experimental data from Ref. 26.

where $k^2=\frac{1}{4}(s-4m_\pi^2)$, valid for *all* isospins. In the limit $m_\sigma\to\infty$, the "scattering lengths" vanish, as do the partial-wave projections of the Weinberg amplitude (5.3) which contains only s and p waves [note that from (5.5) we recover the results of the preceding section, and in particular Eq. (4.12)]. Note that since Eq. (5.5) is valid for all isospins, the Born term of the σ model tends to give a degeneracy between $I=0$ and $I=2$ states in higher partial waves. Actually it is only higher-order corrections which will suppress this degeneracy. We will come back to this point later on.

B. s-Wave Phase Shifts

In the previous section we have explained the mechanism which makes the σ meson unstable. For each value of the pion decay constant f_π we can compute the s-wave phase shifts. We have plotted them in Fig. 7 for $f_\pi=125$ MeV, together with the experimental values of Baton, Laurens, and Reignier[26] for isospin two, and the up-down solution of Malamud and Schlein[27] for isospin zero.

On these results the following features appear clearly.

1. The $I=2$, s-wave phase shift δ_2 is small and negative, quite compatible with experimental data. It agrees much better with experiment than the δ_2 phase shift of the Brown-Goble model.[7] Its major contribution comes from the Φ^4 part of the interaction [Figs. 2(a) and 3(a)], and this agrees with previous Φ^4 calculations.[3,4] As a consequence, we can presumably trust the result, since Φ^4 calculations have shown to be very stable as the order of approximation is increased.

2. In the $I=0$ channel, when we constrain the phase shift δ_0 to be 90° near 700 MeV, the σ resonance comes out very broad. In fact, it can be seen from Table I that the renormalization effects [see Eq. (4.11)] have put the σ pole far away in the complex plane. As a consequence, δ_0 has a maximum and stays close to 90° in a large region around the ρ mass. This is in good agreement with the up-down solution of Malamud and Schlein,[27] which this calculation therefore favors. It is clear that such a behavior of the phase shift does not lead to an observable clear-cut σ resonance, although the $I=0$ s-wave interaction is very strong. This possible stability of δ_0 around 90° may be interesting in explaining experimental histograms; in particular, our results agree very well with the recent measurement of the $\pi^+\pi^-\to\pi^0\pi^0$ cross section in the region 600–900 MeV by Deinet *et al.*,[28] as shown in Fig. 8. We should not trust this result blindly, however, since the [1,1] approximation may be insufficient in this region, higher-order terms being increasingly singular at $s=m_\sigma^2$; but the agreement is striking and higher-order corrections may leave this situation unchanged. Nevertheless, as a countercheck we have also computed the renormalized σ propagator itself up to first order in g [see Eqs. (4.14) and (4.15)]. This procedure yields a rather sharp resonance; with $f_\pi=125$ MeV and $g=5.6$, we get $M_\sigma=640$ MeV, $\Gamma_\sigma=45$ MeV. However, we prefer the solution obtained in the four-point function rather than in the two-point function since it contains more physical information from the perturbation series (order g^2 instead of g), and since it yields the physically observable phase shift.

C. Higher Partial Waves

We find resonances as poles of the corresponding amplitudes (zeros of the Padé denominators). Higher

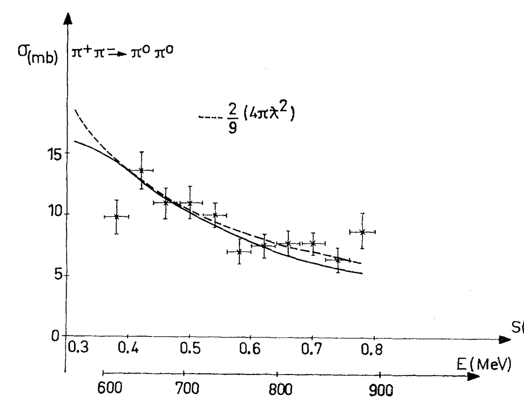

FIG. 8. Cross section for the reaction $\pi^+\pi^-\to\pi^0\pi^0$. Dashed line: unitarity limit. Continuous curve: prediction of the σ model with $f_\pi=125$ MeV, $g=5.63$. Experimental data from Deinet *et al.*, Ref. 28.

[26] J. P. Baton, G. Laurens, and J. Reignier, Nucl. Phys. **B3**, 349 (1967).

[27] E. Malamud and P. E. Schlein, in Proceedings of Argonne National Laboratory Conference, 1969, p. 107 (unpublished).

[28] W. Deinet, A. Menzione, H. Müller, H. M. Staudenmaier, S. Buniatov, and D. Schmitt, Phys. Letters **30B**, 359 (1969); see also P. Sonderegger and P. Bonamy, Contribution to the 1969 Lund Conference (unpublished).

partial-wave resonances lie on Regge trajectories. It can be seen from Table I that ρ and f_0 are always generated by the interaction, but that they are too strongly bound, as was explained previously, owing presumably to a defect of the [1,1] Padé approximation. In Sec. VI we will explain why the good crossing properties of the approximation indicate that these resonances can be considered as true dynamical effects of the theory and not as artifacts due to the type of approximation employed.

The best fit compatible with reasonable s wave is obtained with $f_\pi=125$ MeV and $g=5.6$ (this corresponds to the s waves of Fig. 7), yielding

$$\begin{aligned} J=1\,, &\quad m_\rho=780\text{ MeV}\ , \quad \Gamma_\rho=35\ \text{ MeV}\,,\\ J=1\,, &\quad m_{f_0}=1115\text{ MeV}, \quad \Gamma_{f_0}=180\text{ MeV}, \\ &\quad m_{I=2}=1335\text{ MeV}. \end{aligned} \tag{5.6}$$

where $m_{I=2}$ is the mass of an isospin-two d-wave partner of the f_0. Of course we know of no such exotic $I=2$ object, and this is probably the only unpleasant product of the theory. Notice, however, that the splitting of the f_0 from its $I=2$ partner is by more than 200 MeV, whereas in a pure Φ^4 calculation when only pions are considered, these two states are nearly *degenerate*.[3]

The exact or nearly exact degeneracies which occur in Φ^4 theories are due to the fact that the Φ^4 Lagrangian is purely s wave and as a consequence the dominant forces are also purely s wave.[29] From the structure of Feynman graphs, the σ model appears as a combination of Φ^4 and Φ^3 s-wave interactions. However, as a consequence of the Adler self-consistency condition, these two contributions have to cancel each other exactly for $s=t=u=m_\pi^2$, leaving us with important p-wave forces (this is reflected by the Weinberg condition) which are known to be attractive in $I=0$ and repulsive in $I=2$. We have noted in Eq. (5.5) that the first-order terms are the same for both isospin channels for $l\geq 2$. The degeneracy is removed here since the second-order amplitudes are different in two isospin channels (for example, the 2σ intermediate states are allowed in the $I=0$ states, but not in the $I=2$ states). In second order, there is more attraction in $I=0$ than in $I=2$, actually for the same reason that makes s-wave phase shifts be positive in $I=0$ and negative in $I=2$, and one may expect that this effect will propagate at higher orders. Also, it has been shown in the Φ^4 theory[4] that as soon as one introduces the $K\bar{K}$ channel, the $I=2$ resonance is pushed up in energy about 250 MeV (in a lowest-order calculation, independently of the sign of the additional coupling constant that one needs to introduce). Therefore we believe that higher-order terms and the introduction of the $K\bar{K}$ channel [perhaps through an $SU(3)\times SU(3)$ scheme] may very well raise the unwanted $I=2$ resonance to the 2-GeV region (or even suppress it) thus giving rise to a perfectly acceptable situation.

Since we identify our $J=I=1$ and $J=2$, $I=0$ resonances with the ρ and f_0, we can make the following comments on their masses and widths. We explained in the previous section that since the long-range forces (i.e., the two-pion-exchange contributions) are absent from the Born term, and are present in all other terms of the perturbation series, the [1,1] Padé approximant may approximate these forces inaccurately. Since long-range contributions certainly dominate low-energy scattering in $l\neq 0$, we expect that the values obtained here for the ρ and f_0 masses will be somewhat modified by higher-order terms. We regard as a very positive aspect of the present calculation that the value of g, which is determined here by adjusting *s waves*, is very close to the value of the same parameter ($g\sim 6$) which in pure Φ^4 calculations generates *higher partial-wave resonances* such as the ρ and f_0 so well.[3,4] Since Φ^4 calculations have been performed up to fourth order showing a remarkable stability, and since the long-range forces which generate the ρ and f_0 are very similar in the σ model and in the Φ^4 model, we expect that higher-order corrections will improve the agreement between theoretical and experimental values of these masses.[30]

From Eq. (5.6) one notices that the widths obtained in this model are wrong. The ρ width is too small, the f_0 width somewhat too large; the reason for this may be traced to the fact that these widths come from the box diagrams with two σ's in the t channel for the former [Fig. 5(b)], and with two pions in the t channel for the latter [Fig. 5(a)]. Clearly the former is much smaller than the latter, and this is why the ρ width is anormalously small and the f_0 width large. Previous experiences[3–5] show that although the masses of resonances obtained from Padé approximants are relatively stable, the widths obtained at lowest order are unreliable in higher partial waves. Actually one may have

[29] The only renormalizable interaction involving *only* pseudoscalar mesons is the Φ^4 interaction. Hence, in building an "$SU(3)$-invariant" Lagrangian with an *octet* of pseudoscalar mesons, we can only use $(\Phi_\alpha\Phi^\alpha)^2$ if we require renormalizability. Clearly the symmetry group is then larger than $SU(3)$; it is $O(8)$. The P^8+P^8 amplitude can be decomposed in the $1_s\oplus 28_a\oplus 35_s$ representations of $O(8)$ [where s (a) means that the representation is symmetric (antisymmetric)]. Clearly, then, in this $SU(3)$-symmetric Φ^4 interaction, the vector octet is degenerate with a vector $10+\overline{10}$ representation, in order to form the antisymmetric **28** representation of $O(8)$, and the 2^+ octet is degenerate with a **27** representation to form the symmetric **35**. Hence, in such a model, the f_0 is degenerate with an $I=2$ resonance, the $K^*_{I=1/2}$ with a $K^*_{I=3/2}$ one, etc., i.e. the $O(8)$ symmetry generates exotic resonances. This was noticed by L. Copley, D. Elias, and D. Masson, Phys. Rev. **173**, 1552 (1968) and in Ref. 4. This kind of effect remains when one performs a Φ^4 calculation with only pions, but then the f_0 and its $I=2$ partner are no longer exactly degenerate. Notice, however, that the most general Φ^4 Lagrangian involving only pions and kaons is invariant under $O(4)$ transformations in kaon space, so that (a) KK amplitudes are exactly degenerate with $K\bar{K}$ amplitudes, and (b) $I=\frac{1}{2}$ πK amplitudes are degenerate with $I=\frac{3}{2}$ ones. This is the basic reason for the degeneracies obtained in Ref. 4; it had also been noticed in a different context by B. W. Lee, Phys. Rev. **120**, 325 (1960).

[30] In the final analysis, however, the proof of the pudding is in the eating: There is no substitute for direct demonstration of this conjectured improvement through the evaluation of higher-order corrections.

to go to much higher orders in order to obtain reasonable widths, since the widths will come from diagrams with double spectral functions (higher partial waves have no single spectral functions); such diagrams appear in only one term here (T_2). It may very well be that by studying functions other than scattering amplitudes (for instance, form factors), one might obtain a better approach to widths.

VI. CROSSING SYMMETRY AND DISCUSSION

We now turn to the question of determining to what extent crossing symmetry is violated by our unitary amplitudes. In fact, the perturbation series truncated at any order satisfies crossing exactly but violates unitary by a large amount in regions where the phase shifts are large, while the $[N,M]$ Padé approximant to the partial-wave amplitude satisfies crossing up to order g^{N+M}. The question is whether the violation of crossing by unitary Padé approximants is small in the low-energy region (which must be the case if the method actually converges) or large as is the violation of unitary by the perturbation series (in this case, the method would not be more reliable than the perturbation series). Naturally, since our amplitudes are exactly unitary, and since they have only a finite number of inelastic channels, they cannot satisfy crossing exactly.[31] The best we can hope is that they will satisfy crossing to the maximum extent compatible with unitarity.

As mentioned in Sec. IV, two methods are actually available for testing crossing. The first one, which we call the poloid method, consists in building crossing-symmetric Padé approximants (nonunitary) and comparing them with the unitary ones. The second one consists in testing crossing constraints (Roskies relations and Martin inequalities) directly on partial-wave amplitudes. In the Φ^4 theory, these tests have been quite positive.[4,22,23] In the present calculation, although we do not expect to have such good results since the $[1,1]$ approximant may be insufficient (see preceding sections), the crossing conditions are satisfied within a few percent and therefore we can trust the gross features of the numerical results. We shall also see that forward dispersion relations, which provide a test of crossing in terms of *physical-region* amplitudes, are well satisfied.

A. Poloid Method

For technical reasons, the poloid method[4] has not been used in the present calculation up to now. However, we shall point out some of its properties.

In order to build a crossing-symmetric Padé approximant to the $\pi\pi$ amplitude, we start with the usual definition

$$T_{\alpha\beta,\gamma\delta}(s,t,u)=\delta_{\alpha\beta}\delta_{\gamma\delta}A(s,t,u)+\delta_{\alpha\gamma}\delta_{\beta\delta}B(s,t,u)+\delta_{\alpha\delta}\delta_{\beta\gamma}C(s,t,u)\,,\quad(6.1)$$

[31] M. Froissart and A. Martin (private communication); D. W. Greenberg and A. L. Licht, J. Math. Phys. 4, 613 (1963).

with the well-known relations from crossing

$$A(s,t,u)=A(s,u,t)\,,\quad(6.2a)$$

$$B(s,t,u)=A(t,s,u)\,,\quad(6.2b)$$

$$C(s,t,u)=A(u,t,s)\,.\quad(6.2c)$$

We now consider the perturbation expansion in g of the function $A(s,t,u)$; it has the form

$$A(s,t,u)\simeq gA_1(s,t,u)+g^2A_2(s,t,u)\,,\quad(6.3)$$

where

$$A_1(s,t,u)=a+b/(s-m_\sigma^2)\,,\quad(6.4)$$

$$A_2(s,t,u)=\Sigma_1(s)/(s-m_\sigma^2)^2+\Sigma_2(s)/(s-m_\sigma^2)+R(s,t,u)\,,\quad(6.5)$$

where the function $R(s,t,u)$ is regular in the vicinity of $s\sim m_\sigma^2$ and has no polar singularities in t and u; $\Sigma_1(s)$ and $\Sigma_2(s)$ have the elastic cut in s and the unitarity property reflects itself in the relation

$$\mathrm{Im}\Sigma_1(s)=\frac{1}{32\pi}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}b^2\,,\quad s\geq 4m_\pi^2\,.\quad(6.6)$$

We now build the Padé approximant to the function $A(s,t,u)$:

$$A^{[1,1]}(s,t,u)=g\frac{A_1^2(s,t,u)}{A_1(s,t,u)-gA_2(s,t,u)}\,.\quad(6.7)$$

If we take into account Eqs. (6.2b) and (6.2c) and insert the values of $A^{[1,1]}$, $B^{[1,1]}$, and $C^{[1,1]}$ into Eq. (6.1), we obtain a $\pi\pi$ amplitude which is exactly crossing symmetric, but no longer unitary.

It is clear that the poles of $A^{[1,1]}$ in the various channels are no longer straight lines in the (s,t,u) plane, for instance, the poles in s generally depend on t and u (hence the name poloid). If we project Eq. (6.7) onto partial waves in a given channel, the partial-wave amplitudes will not have poles, but small cuts coming from the spreading of the poloids in the transfer variables.

However, this does not hold for the σ resonance itself. In fact, it is clear from Eqs. (6.4), (6.5), and (6.7) that the σ pole will appear as a *straight line* in the (s,t,u) plane (thus contributing to the s wave only). Furthermore, in the vicinity of this pole, the s-wave amplitude will satisfy *unitarity*, as a consequence of Eq. (6.6). In other words, elementary particles (i.e., those which have been inserted directly in the Lagrangian) will appear as true pole of the crossing-symmetric amplitudes (with accompanying unitarity properties in their vicinity) and dynamical resonances will appear as poloids. In the Φ^4 theory it has been found[4] that the poloids associated with the ρ and f_0 are quite flat, at least in the physical region ($s>4m_\pi^2$, $t<0$, $u<0$, for instance) and that their average positions are quite close to the masses found in unitary partial-wave amplitudes.

B. Roskies Relations

A first approach for writing crossing-symmetry constraints on partial-wave amplitudes has been developed by Balachandran and Nuyts.[20] This consists in performing a two-variable orthogonal-polynomial expansion of the amplitude in the Euclidean region $s\geq 0$, $t\geq 0$, $u\geq 0$ (note that all three scattering angles are physical in that region, so the expansion converges). If one then applies crossing symmetry on the expansion, one obtains a complete set of relations each of which involves a finite number of partial-wave amplitudes, expressing crossing symmetry. This approach has been extensively studied by Roskies[9] in the $\pi\pi$ case, and generalized and applied to the Φ^4 theory in Ref. 22. Roskies has shown that in the simple parametrization proposed by Brown and Goble,[7] in order to unitarize the amplitude given by current algebra, the presence of both the σ resonance in $I=J=0$ and the ρ in $I=J=1$ leads to a contradiction with crossing symmetry. Since in the present calculation we obtain both the σ and the ρ, it is of interest to see how our amplitudes fare in the Roskies relations.

We shall denote the $I=0$ and 2 s-wave amplitudes by $t_{(0)}$ and $t_{(2)}$ and the $I=1$ p-wave amplitude by $t_{(1)}$:

$$t_{(0)}=32\pi\left(\frac{s}{s-4m^2}\right)^{1/2}\sin\delta_0{}^0\exp i\delta_0{}^0\,,\quad\text{etc.}$$

Roskies relations which involve only the s and p waves are

$$I_0=I_2\,, \tag{6.8}$$

$$J_0=J_2=J_1\,, \tag{6.9}$$

where

$$\begin{pmatrix}I_0\\ I_2\end{pmatrix}=\begin{pmatrix}2\\ 5\end{pmatrix}\int_0^{4m_\pi^2}ds(s-4m_\pi^2)\begin{pmatrix}t_{(0)}(s)\\ t_{(2)}(s)\end{pmatrix}, \tag{6.10}$$

$$\begin{pmatrix}J_0\\ J_2\end{pmatrix}=\begin{pmatrix}1\\ -2\end{pmatrix}\int_0^{4m_\pi^2}ds(s-4m_\pi^2)\times(3s-4m_\pi^2)\begin{pmatrix}t_{(0)}(s)\\ t_{(2)}(s)\end{pmatrix}, \tag{6.11}$$

and

$$J_1=2\int_0^{4m_\pi^2}ds(s-4m_\pi^2)^2t_{(1)}(s)\,. \tag{6.12}$$

Note that Eq. (6.9) relates the s and p waves and it is there that Roskies found a conflict with crossing in the simple parametrization of Brown and Goble when both the σ and p resonances are imposed. On the other hand, the Padé treatment of the Φ^4 theory is known[22] to lead to a very good agreement with Eqs. (6.8) and (6.9).

In the present calculation of the σ model, we have computed Eqs. (6.10)–(6.12) with the amplitudes $t_I(s)$ given by the [1,1] Padé approximant, in three cases considered in Table I ($f_\pi=95$ MeV, $g=6.96$; $f_\pi=110$ MeV, $g=6.20$; $f_\pi=125$ MeV, $g=5.63$). The values we obtain are given in Table III where one can see that the Roskies relations are always satisfied to within a few percent (less than 5% in all cases). This holds actually for a wide range of the parameters g and f_π; as is to be expected, the smaller g is, the better the relations hold. What is important is that the relation between *s waves* and *p waves*, Eq. (6.9), is well satisfied although the σ and the ρ are both present.[32]

TABLE III. Test of the Roskies crossing relations for three solutions of Table I. The exact crossing relations are $I_0=I_2$, $J_0=J_2=J_1$.

f_π (MeV)	g	I_0	I_2	J_0	J_2	J_1
95	6.96	−72.25	−69.54	−95.06	−98.20	−99.33
110	6.20	−57.58	−56.05	−76.22	−78.78	−79.04
125	5.63	−41.68	−40.15	−56.02	−57.70	−58.48

The fact that the equalities cannot be satisfied *exactly* except for $g=0$ is a clear consequence of the incompatibility between exact crossing and exact unitarity at finite order. It is nevertheless possible to add extra parameters in the amplitudes in order to satisfy some crossing relation *exactly*, as is done in some theoretical models.[33] Instead of Eq. (4.8), one may parametrize the partial-wave amplitude in the channel I, J, as

$$\bar{t}=g\frac{t_1{}^2}{(1+\epsilon)t_1-gt_2}\,. \tag{6.13}$$

Here, the phenomenological quantities $\epsilon=\epsilon(I,J)$ must be real between $s=0$ and the first inelastic threshold in order to preserve elastic unitarity. In order to satisfy the three lowest-order Roskies relations exactly, one would have to adjust three numbers $\epsilon(I,J)$ whose orders of magnitude would be a few percent. (This would therefore not change appreciably the physical amplitudes.) The quantities $\epsilon(I,J)$ must be regarded as describing phenomenologically the contributions of inelastic channels in the true solution. In order to satisfy the higher-order Roskies relations, one would have to give them a dependence in energy; however, one cannot go too far in this direction, since the actual contributions of inelastic channels would appear consistently in higher orders of perturbation theory.

[32] For comparison, we give here the corresponding values for the Brown-Goble model. For $f_\pi=0.67m_\pi$, $m_\rho=5.5m_\pi$, $m_\sigma=5m_\pi$, $I_0=-68.3$, $I_2=-58.5$; $J_0=-112.6$, $J_2=-94.75$, $J_1=-84.9$. Roskies suggested that we should test whether $t^{[1,1]}-t^W$, where t^W is the projection of the Weinberg amplitude, satisfies Eqs. (4) and (5), since the first order term in $t^{[1,1]}$ is just t^W and t^W is manifestly crossing-symmetric. For the difference amplitudes, we find $I_0=-8.76, I_2=-7.23, J_0=-3.35, J_2=-5.03, J_1=-5.81$. For the Brown-Goble model, the corresponding values are $I_0=-8.3$, $I_2=+1.5$, $J_0=-14.5$, $J_2=+3.25$, $J_1=13.1$.

[33] G. Auberson, O. Piguet, and G. Wanders, Phys. Letters **28B**, 41 (1968).

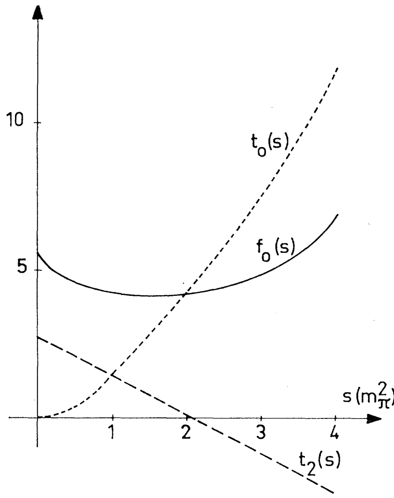

FIG. 9. Plot of the s-wave $\pi\pi$ amplitudes obtained in the [1,1] Padé approximation in the unphysical region $0\leq s\leq 4m_\pi^2$. $t_{(0)}$ and $t_{(2)}$ are the $I=0$ and 2 amplitudes; f_0 is the $\pi^0\pi^0\to\pi^0\pi^0$ amplitude $(=t_{(0)}+2t_{(2)})$, which in the Weinberg limit is a constant.

C. Martin Inequalities

Another approach for putting together crossing and unitarity has been initiated in recent years by Martin.[21] This method is based on the positivity condition of the absorptive part which comes from unitarity and crossing symmetry. In the $\pi^0\pi^0$ case it leads to various inequalities involving the values of partial-wave amplitudes in the unphysical region $0\leq s\leq 4m_\pi^2$. These inequalities are necessary conditions for crossing to be satisfied. Again, in the Φ^4 theory, the Martin inequalities have been shown to be well satisfied by unitary Padé approximants.[22]

We have tested these inequalities for the $\pi^0\pi^0\to\pi^0\pi^0$ s wave. Denoting this amplitude by $f_0(s)=t_0(s)+2t_2(s)$, and expressing s in units of m_π^2, we have the following inequalities together with our numerical results (with $g=5.63$, $f_\pi=125$ MeV, and $m_\rho=780$ MeV):

$$\begin{aligned} &f_0(3.205)>f_0(0.2134)>f_0(2.9863)\,,\\ &5.072> \quad 4.945 \quad >4.808\,, \end{aligned}\tag{6.14}$$

TABLE IV. The Martin inequalities for various values of the coupling constant g. Here the solution $f_\pi=125$ MeV, $g=5.63$, has been chosen for definiteness. We represent only inequality (6.14) since the other one is always satisfied. For each value of g we give the masses of ρ, f_0, and σ resonances. Clearly one inequality is violated for $g=7$. m_ρ and m_{f_0} are the ρ and f_0 masses for each value of g, and m_σ is the real part of the position of the second-sheet σ pole (all masses are in MeV).

g	m_ρ	m_{f_0}	m_σ	$f(3.205)>f(0.2134)>f(2.9863)$		
7	600	850	670	4.39	4.66	4.17
5.63	780	1115	530	5.07	4.95	4.81
4	1000	1400	390	6.72	6.37	6.07

and

$$\begin{aligned} &f_0(4)>f_0(0)>f_0(3.190)\,,\\ &6.5>5.44>5.05. \end{aligned}\tag{6.15}$$

The inequalities on the derivatives are also well satisfied. They are

$$\begin{aligned} \frac{d}{ds}f_0(s)&<0 \quad \text{for } 0\leq s\leq 1.05\\ &>0 \quad \text{for } 1.696\leq s\leq 4\,, \end{aligned}\tag{6.16}$$

and

$$\frac{d^2}{ds^2}f_0(s)>0 \quad \text{for } 0\leq s\leq 1.7\,. \tag{6.17}$$

The amplitude $f_0(s)$ is plotted on Fig. 9; the minimum of $f_0(s)$ occurs between $s=1.6$ and $s=1.68$ in all our calculations.

An interesting result appears when we vary the coupling constant g, keeping f_π fixed. As g increases, the masses of the ρ and f_0 decrease (more binding), whereas the σ mass increases [see Eq. (3.9)]. In Table IV, we show the situation in two rather extreme cases: (a) M_σ small, m_ρ and m_{f_0} large; (b) M_σ large, m_ρ and m_{σ_0} small, together with the intermediate case of our calculation. The Martin inequalities are well satisfied in the first case, but they are violated when the σ mass becomes too large. There are two reasons for this: First, when g is smaller, one expects a better convergence for the Padé approximation; secondly, in the [1,1] approximant, the contribution of σ exchange in crossed channels appears to be taken into account more accurately than that of two-pion exchange. Therefore, crossing is satisfied better when the nearby left-hand cut is dominated by the σ rather than by two-pion states. Hence, higher orders are needed in order to treat higher partial waves accurately, since two-pion forces are dominant in those waves.

The Martin inequalities are more subtle and perhaps more indicative than other tests of crossing. Consider for instance Weinberg's linear amplitudes, Eq. (5.3); in that approximation, the $\pi^0\pi^0$ amplitude is a constant and the Martin inequalities become *equalities*. Clearly, if a certain procedure alters Weinberg's amplitudes by a few percent, the Roskies relations will still hold within roughly the same relative amount, while the Martin inequalities can be violated no matter how small the variation is. Since these inequalities imply that the $\pi^0\pi^0$ amplitude is not monotonic in the interval $0\leq s\leq 4m_\pi^2$, it is not at all a trivial matter to preserve them in a unitarization procedure.[34,35]

[34] For comparison, we note that the Brown-Goble model gives $f_0(3.205)=9.34$, $f_0(0.2134)=6.57$, $f(2.9863)=8.85$, $f_0(0)=6.56$, $f_0(3.190)=9.31$.

[35] We shall come back to the question of the violation of crossing symmetry in various unitarizations of current algebra in a further paper.

D. Dispersion Relations

Since the previous tests of crossing were made on the values of amplitudes in the unphysical region $0 \leq s \leq 4m_\pi^2$, it is of interest to try and check crossing properties directly on physical region values. Such a test is provided by dispersion relations which are not satisfied exactly by our amplitudes.

Consider the usual antisymmetric combination of s-wave scattering lengths

$$L = \tfrac{1}{6}(2a_0 - 5a_2), \qquad (6.18)$$

where a_0 and a_2 are the $I=0$ and $I=2$, s-wave $\pi\pi$ scattering lengths. We have the following sum rule,[36] hereafter called the "L sum rule":

$$L = \frac{m_\pi}{8\pi^2} \int_{4m_\pi^2}^{\infty} \frac{ds}{[s(s-4m_\pi^2)]^{1/2}} [\sigma^{+-}(s) - \sigma^{++}(s)], \qquad (6.19)$$

where $\sigma^{+-}(s)$ and $\sigma^{++}(s)$ are the $\pi^+\pi^-$ and $\pi^+\pi^+$ total cross sections. Equation (6.19) is simply the unsubtracted forward dispersion relation for $\pi\pi$ scattering written for $s=4m_\pi^2$. In usual treatments, the L sum rule is used as follows. (a) One assumes that high-energy contributions can be neglected. (b) The contribution of low-energy resonances of spin $l>0$ (essentially ρ and f_0) to the right-hand side of Eq. (6.19) is positive; it can readily be evaluated in terms of physical masses and widths, for instance in the δ-function approximation. (c) This leaves us with a sum rule relating the s-wave amplitudes and the scattering lengths a_0 and a_2, which is of course the on-shell version of Adler's sum rule for $\pi\pi$ scattering.[37]

In our calculation, since higher partial-wave amplitudes are computed, we can use the L sum rule in order to check the internal consistency of the model. For convenience, we give the numerical results for Eq. (6.19) when both sides are multiplied by a factor of 100, and we express energies in units of m_π, namely,

$$L' \equiv \frac{100}{6}(2a_0 - 5a_2)$$
$$= \frac{100}{8\pi^2} \int_4^{\infty} [s(s-4)]^{-1/2} [\sigma^{+-}(s) - \sigma^{++}(s)] ds. \qquad (6.20)$$

The s-wave contributions A_I^0 ($I=0$, 2) to the right-hand side of Eq. (6.20) are

$$\begin{pmatrix} A_0^0 \\ A_2^0 \end{pmatrix} = \frac{100}{8\pi^2} \int_4^{\infty} [s(s-4)]^{-1/2} \begin{pmatrix} 2/3 \\ 5/3 \end{pmatrix} \begin{pmatrix} \sigma_0^0(s) \\ \sigma_2^0(s) \end{pmatrix} ds, \qquad (6.21)$$

where σ_I^0 is the isospin-I s-wave total cross section. Similarly, we define A_ρ to be the $I=J=1$ contribution to the right-hand side of Eq. (6.21), A_{f_0} the $I=0$, $J=2$ contribution, and A_2^2 the $I=J=2$ contribution which is non-negligible since we have an exotic resonance in our model. Neglecting high-energy contributions, the sum rule reads

$$L' \simeq A \equiv A_0^0 - A_2^0 + A_\rho + A_{f_0} - A_2^2. \qquad (6.22)$$

We give the corresponding numerical values for all these quantities in Table V. The sum rule is satisfied within 5% for a wide range of the parameters. This is in agreement with the check of crossing obtained by other methods (i.e., the Roskies relations), and shows the internal consistency of the model.

TABLE V. Test of the L sum rule for various values of f_π and g. A_0^0 and A_2^0 are the s-wave contributions in $I=0$ and 2 to the right-hand side of Eq. (6. 20). A_ρ, A_{f_0}, and A_2^2 are the $I=J=1$, $I=0$, $J=2$, and $I=J=2$ contributions. $A=A_0^0-A_2^0+A_\rho+A_{f_0}-A_2^2$ and the sum rule is $L'=A$, where $L'=100(2a_0-5a_2)/6$.

f_π (MeV)	g	A_0^0	A_2^0	A_ρ	A_{f_0}	A_2^2	A	L'
95	6.96	9.19	2.27	2.11	2.89	0.78	11.14	11.54
110	6.20	6.90	1.68	1.51	2.53	0.60	8.64	8.33
125	5.63	4.93	1.69	1.10	2.31	0.40	6.25	5.97

Since our s-wave amplitudes seem to agree with experiment, but the ρ and f_0 we obtain here do not have their experimental masses and widths, it is interesting to see how our s-wave amplitudes fare in the L sum rule when the *experimental* values of A_ρ and A_{f_0} are instead of ours. With the *physical* masses and widths of these resonances, $m_\rho = 765 \pm 10$ MeV, $\Gamma_\rho = 125 \pm 20$ MeV, $m_{f_0} = 1264 \pm 10$ MeV, and $\Gamma_{f_0} = 145 \pm 25$ MeV, one obtains

$$A_\rho = 3.8 \pm 0.8, \quad A_{f_0} = 1.0 \pm 0.2, \qquad (6.23)$$

and the relevant combination is

$$A_\rho + A_{f_0} = 4.8 \pm 1. \qquad (6.24)$$

From the values of Table V, we see that the combination $A_\rho + A_{f_0}$ in our calculation is close to the experimental value, and thus the sum rule will be reasonably well satisfied.[38] The reason for this is that, at this order, although the width of our ρ meson is too small, the fact that the ρ and f_0 are too strongly bound enhances their contributions to the right-hand side of Eq. (6.20). It is also clear that in this sense our s-wave amplitudes saturate Adler's sum rule; notice, however, that with $f_\pi = 125$ MeV, $g=5.6$, and with the experimental value on $\Delta_\rho + A_{f_0}$, we would have $L' > A$ (outside error bars). but this may simply mean that our scattering lengths are too small in that case and that higher order would increase them without changing the phase shifts appreciably at higher energies.

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[38] For comparison, we give the contributions of s waves to the L sum rule in the Brown-Goble model. With $M_\sigma \sim 700$ MeV, we have $A_0^0 = 7.1$, $A_2^0 = 4.61$, $L' = 11.0$ (the scattering lengths are $a_0 = 0.22$ and $a_2 = -0.045$), so $A = A_0^0 - A_2^0 + A_\rho^{\exp} + A_{f_0}^{\exp} = 7.3 \pm 1$ and the sum rule $L' = A$ is violated by more than 25%. The discrepancy can be seen to come from A_2^0, which is too large (δ_2 is too large in magnitude); see Ref. 35 for further comments.

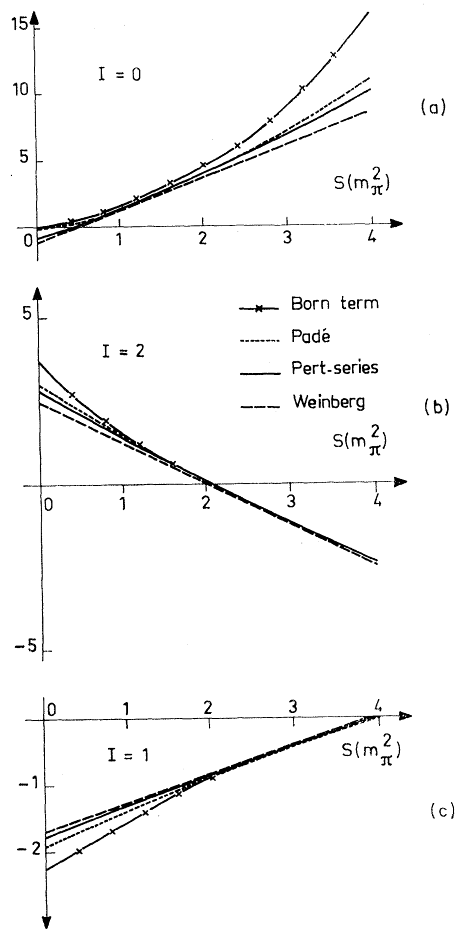

FIG. 10. Comparison of various amplitudes in the unphysical region $0 \leq s \leq 4m_\pi^2$. We have plotted the Born term of the σ model, the [1,1] Padé approximant, the perturbation series $gt_1+g^2t_2$, and the linear Weinberg amplitude. (a) $I=J=0$; (b) $I=2$, $J=0$; (c) $I=J=1$.

E. Discussion

In concluding this section, we shall make the following observations.

1. It is obvious that the Padé approximation to the amplitude cannot be valid throughout the entire complex-energy plane. For instance, it is certainly a bad approximation at high energy; in particular, it does not have Regge behavior at any finite order. The unitarity properties indicate that we should really use this approximation in low-energy regions where only a small number of inelastic channels contribute significantly to the amplitudes.

2. The Roskies relations express the physical content of crossing relations on the partial-wave amplitudes that the left-hand cut of a partial-wave amplitude is due to the physical processes in the crossed channels. Owing to the presence of the factor $(s-4m_\pi^2)$ in Eqs. (6.10)–(6.12), the contributions of the amplitudes to the integrals are enhanced near $s=0$ close to the left-hand cut. That the relation between the s and p waves is well satisfied is indicative of the ρ resonance we found to be a true dynamical effect of the theory, rather than an artifact of the approximation employed. Since crossing determines the left-hand cuts, ρ and σ must appear consistently as forces as well as direct-channel poles. The ρ width we obtained is much too small ($\simeq$35 MeV), but we have presented arguments that higher-order corrections would restore this to a value close to the experimental one.

3. The structure of the left-hand cut plays a crucial role also in the Martin inequalities. Our amplitudes satisfy these inequalities because the unitarization procedure is not arbitrary, it is based on a well-defined perturbation series. If, on the other hand, we test crossing directly on *physical-region* amplitudes, with dispersion relations, we see that we also have good agreement provided we consider only *low-energy* sum rules.

4. In our calculation, the effective expansion parameter $g/8\pi^2$ is small ($\sim$0.1). Therefore, the Padé amplitudes are expected to converge well and are not too different from the Born amplitudes except near singularities of the former (such as near resonances and left-hand cuts). The unphysical region $0 \leq s \leq 4m_\pi^2$ is of particular interest in connection with current algebra and zeros of amplitudes. In Fig. 10 we have plotted, in this region, the s- and p-wave amplitudes obtained from (a) the Weinberg amplitude [Eq. (5.3)], (b) the Born term of the σ model [Eq. (5.2)], (c) the perturbation series $gt_1+g^2t_2$, in the σ model, and (d) the [1,1] Padé approximant. All these curves are qualitatively similar. It is seen that the Padé amplitude is much closer to the linear Weinberg amplitude than the first-order term of the σ model is. This occurs because the σ mass is considerably raised in the Padé amplitude compared to the first-order σ mass, i.e., $m_\sigma^2 < M_\sigma^2$. Also, the perturbation series is not a worse approximation than the Padé amplitude in that region (it can actually be treated as an asymptotic series). The exact location of zeros of $t_0(s)$ at $s=\frac{1}{2}m_\pi^2$ for $I=0$, and of $t_2(s)$ at $s=2m_\pi^2$ for $I=2$ in the Weinberg amplitudes is a consequence of linearity; nevertheless, the presence of these zeros is a consequence of the Adler self-consistency condition, and we should observe them unless the smoothness assumption is wrong. In our case, the zero of $t_2(s)$ is always very close to $s=2m_\pi^2$, but the zero of $\mathbf{Re}(t_0(s))$ is shifted and occurs for $s<0$ in the Padé amplitude, and in the Born term, due to the presence of the $I=0$ σ pole. Notice, however, that in that channel, the zero of the perturbation series occurs at $s=0.4m_\pi^2$, close to Weinberg's value. It is also interesting to see that the $I=J=1$ amplitude is close to Weinberg's amplitude for $0 \leq s \leq 4m_\pi^2$ despite the presence of the ρ pole in the physical region.

5. The fact that the σ-model amplitude is close to the Born amplitude except near singularities is a clear advantage of the model. Here is a model in which the smoothness assumption made in current-algebra anal-

yses is very well justified, while the theory has the potentiality for accounting for the complex spectrum of the low-energy $\pi\pi$ system.

6. Our results for the s waves can be favorably compared with those of Auberson, Piguet, and Wanders,[33] who have exact unitarity and strict crossing relations, and who add some information about zeros of amplitudes, and also with those of Iliopoulos,[39] who extends the Weinberg calculation by keeping exact crossing and adding some contributions of elastic cuts. They are also in agreement with the solution of Morgan and Shaw.[40]

VII. CONCLUDING REMARKS

Let us briefly summarize what we have achieved. We began with a Lagrangian which possesses all the necessary ingredients to yield the current-algebra constraints in $\pi\pi$ scattering. The approximate amplitude constructed by the use of the Padé method is found to be in good agreement with experiment with regards to the s waves below, say 800 MeV. Aside from this numerical agreement, we consider the following result significant: Even when the phase shift in the $I=J=1$ is large in the physical region, the continuation of the amplitude below the threshold, in particular, in the range $0\leq s\leq 4m_\pi^2$, is "smooth," confirming the assumption usually made in conjunction with PCAC. We further find an indication that a model of this kind will correctly produce all the essential features of meson spectroscopy in the $\pi\pi$ system. The order of approximation is too low for us to be positive, but in the lowest-order approximation the ρ and f_0 resonances appear, albeit with "wrong" widths. We are comforted to know that the approximate amplitudes we constructed are manifestly unitary and, as the various tests have shown, satisfy various constraints imposed by crossing symmetry extremely well. The construction of higher-order approximate amplitudes is clearly desirable to test our conjectures, but has not been carried out.

An advantage of our approach was that we began with a Lagrangian rich in content. The σ model satisfies the conditions of current algebra, and in addition, in the version we considered (i.e., the neglect of nucleon fields) the effective expansion parameter of the perturbation series turns out to be reasonably small. Thus the Padé approximants are expected to converge well, and are not too different from the Born amplitudes except near singularities of the former, such as near resonances and left-hand cuts.

Our past and present experiences show that the Padé approximation is capable of eliciting the complex spectrum that a Hamiltonian possesses. Especially, the poloid method yields an approximate amplitude which is in many ways reminiscent of the Veneziano model. Higher-order Padé approximants to the full amplitude would presumably yield an amplitude which is very nearly unitary, exactly crossing symmetric, and exhibits a spectrum of resonances. Presumably, the poloids would become flatter with the increasing order of approximation. Whereas the Veneziano model is phenomenological, such an approximant to the full amplitude is a deduction from Lagrangian field theory. At any finite order, the Padé approximant will probably not have an infinite sequence of crossing-symmetric excitations. It will not exhibit a genuine Reggeistic high-energy behavior. Nonetheless it is possible that the Veneziano-like formula is an abstraction of the limit of the poloid amplitude as the order of approximation tends to infinity.

Also, it must be stressed that, even in relatively lower-order calculations, one should include more physical contents in the Lagrangian itself. We have neglected the baryon part of the $SU(2)\times SU(2)$ σ model. The inclusion of the nucleons in the scheme will affect low isospin states considerably. But, more importantly a complete description of all pseudoscalar-meson–pseudoscalar-meson interactions should be considered within the chiral $SU(3)\times SU(3)$ scheme.[41] The present calculation is based on the belief that, in some approximate sense, one can treat separately $\pi\pi$ dynamics from the rest. However, previous calculations indicate that the inclusion of the $K\bar{K}$ channel influences $\pi\pi$ scattering in a non-negligible way, and in fact improves the agreement with experiment. An $SU(3)$ version of the σ model exists[42,43] and shows many desirable features.[42] It is encouraging that the scalar mesons predicted by such a scheme do seem to appear experimentally (such as the S^* meson at 1080 MeV).

ACKNOWLEDGMENTS

We wish to thank Dr. D. Bessis, Professor M. Froissart, Professor R. Stora, and Dr. J. Zinn-Justin for many enlightening discussions. We are grateful to Dr. G. Cohen-Tannoudji, Professor A. Martin, Dr. A. Morel, Professor R. Roskies, and Dr. J. Yellin for their comments and interest. One of the authors (B.W.L.) thanks Professor Ph. Meyer and Professor C. Bouchiat for the hospitality at the Laboratorie de Physique Théorique et Hautes Energies at Orsay where this work was initiated and Professor C. Bloch for the hospitality at the Service de Physique Théorique of Saclay. Both authors thank Professor J. Prentki for enabling them to work at the Theory Division of CERN during the summer of 1969.

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APPENDIX A: GOLDSTONE PHASE FOR PION

We shall discuss the stability of the σ model for various ranges of parameters g, v, and γ. The discussion will be based on the classical approximation (or the tree approximation) to the σ model, but a similar conclusion also holds for the full theory at least asymptotically for small g with some modifications which we will outline below.

We recall Eqs. (2.12) and (2.10c):

$$\gamma = v\mu_\pi{}^2, \quad \text{(A1)}$$

$$\mu_\pi{}^2 = \mu^2 + gv^2. \quad \text{(A2)}$$

The coupling constant g is positive for the Hamiltonian to be non-negative:

$$g > 0. \quad \text{(A3)}$$

Equations (A1) and (A2) are invariant under $\gamma \to -\gamma$, $v \to -v$, so that we need only consider the case γ, $v \geq 0$.

As $\gamma \to 0$, either $v=0$ or $\mu_\pi{}^2=0$, according to Eq. (A1). Whether the first or second possibility obtains depends on the sign of μ^2: (1) If $\mu^2>0$, then $v=0$, since $\mu_\pi{}^2>0$, according to (A2) and (A3). (2) If $\mu^2<0$, then $\mu_\pi{}^2=0$ (and $\mu^2=-gv^2$), since if v were zero, then the physical mass of the pion would be negative and such a solution is not acceptable. (3) If $\mu^2=0$, then $\mu_\pi{}^2=gv^2=0$. We shall call cases (1) and (3) normal, while we shall call case (2) the Goldstone limit. For case (2), the stable solution of the model corresponds to spontaneously broken symmetry (Goldstone solution). The relationships among v, $\mu_\pi{}^2$, γ, and μ^2 can be seen readily in Fig. 11. In the $\mu_\pi{}^2-v$ plot, the lines of constant γ are hyperbolas; the lines of constant μ^2 are parabolas. We shall call the region below the line characterized by $\mu^2=0$ the normal phase, the region above, the Goldstone phase. If a solution in the normal phase (say at A) is continued to $\gamma=0$, while μ^2 is held fixed, we will obtain the normal solution (i.e., the symmetric solution: $\mu_\pi{}^2=\mu_\sigma{}^2$, $v=0$), while a solution in the Goldstone phase (say at B) will be reduced, under the same process, to the spontaneously broken symmetry solution (i.e., $\mu_\pi{}^2<\mu_\sigma{}^2$, $v\neq 0$).

The above discussion holds also for the full renormalized theory, by virtue of Eq. (2.14), for sufficiently small g. It is only necessary to replace $\mu_\pi{}^2$ and μ^2 by

$$\mu_\pi{}^2 \to [-\Delta_\pi{}^2(0)]^{-1},$$
$$\mu^2 \to [-\Delta_s{}'(0)]^{-1},$$

where $\Delta_\pi{}'$ is defined in Sec. II and $\Delta_s{}'$ is the renormalized full-pion propagator of the symmetric theory. Equation (A2) is replaced by

$$[-\Delta_\pi{}'(0)]^{-1} = [-\Delta_s{}'(0)]^{-1} + gv^2 + O(g^2).$$

Actually, each term in the perturbation series for the Green's function of the σ model constructed according to the discussion of Sec. II is continuous across the line $\mu^2=0$. The significance of the line $\mu^2=0$ is that the expansion of Green's function in v (with μ^2 fixed) becomes singular at $\mu^2=0$.

Since our renormalization is based on the comparison with the symmetric theory (i.e., the case $v=0$), it can be carried out only in the normal region. Each term in the renormalized perturbation series in g, with gv^2 fixed (i.e., summed to all orders in gv^2), is, however, continuous across the line $\mu^2=0$. Thus our prescription consists in renormalizing the theory at A, expressing the perturbative Green's function in terms of $m_\pi{}^2$, gv^2, and g, and continuing to the point B.

An example may clarify the point. Consider the inverse pion progapator (3.4),

$$[\hat{\Delta}_\pi(k^2)]^{-1} = k^2 - m_\pi{}^2 + 4g(g\hat{v})^2[B_{\sigma\pi}(m_\pi{}^2) - B_{\sigma\pi}(k^2)],$$

where

$$B_{\sigma\pi}(k^2) = i\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m_\sigma{}^2} \frac{1}{(k+p)^2 - m_\pi{}^2}.$$

This expression is meaningful for all values of $m_\pi{}^2 \geq 0$. However, if we write $B_{\sigma\pi}(m_\pi{}^2) - B_{\sigma\pi}(k^2)$ as a function μ^2 and gv^2 and expand it in gv^2, the expansion coefficients are undefined for $\mu^2 \leq 0$.

With $g \simeq 6$, $v \simeq f_\pi \simeq \frac{2}{3}m_\pi$, the system appears to be in the Goldstone phase.

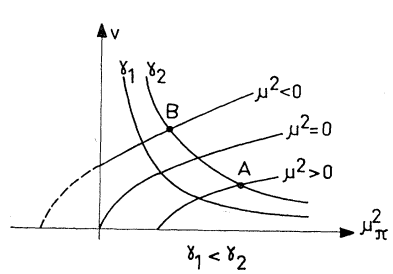

FIG. 11. Relationship among v, $\mu_\pi{}^2$, γ, and μ^2. $\gamma_1 < \gamma_2$. A is a pion in the normal phase, B in the Goldstone phase.

APPENDIX B: FIRST- AND SECOND-ORDER PARTIAL-WAVE $\pi\pi$ AMPLITUDES

We take the pion mass to be unity, $m_\pi{}^2=1$. The $\pi\pi$ amplitude is written

$$T_{\alpha\beta\gamma\delta} = C_\pi{}^2(A\delta_{\alpha\beta}\delta_{\gamma\delta} + B\delta_{\alpha\gamma}\delta_{\beta\delta} + C\delta_{\alpha\delta}\delta_{\gamma\beta}), \quad \text{(B1)}$$

so that the isospin amplitudes are

$$T^0 = C_\pi{}^2(3A+B+C), \quad \text{(B2a)}$$

$$T^1 = C_\pi{}^2(B-C), \quad \text{(B2b)}$$

$$T^2 = C_\pi{}^2(B+C). \quad \text{(B2c)}$$

Let l denote the angular momentum of the channel considered. The c.m. momentum q is given by

$$q^2 = \tfrac{1}{4}(s-4), \quad \text{(B3)}$$

and M is the input σ mass ($M^2 \equiv m_\sigma^2 = m_\pi^2 + 2g\hat{v}^2$). We denote by $T_n^{I,l}$ the nth-order amplitude in isospin I and angular momentum l, and by A_n^l, B_n^l, and C_n^l the lth partial-wave projections of A, B, and C at order n.

Our convention for unitarity is

$$\mathrm{Im}T^l(s) = \frac{1}{32\pi}\left(\frac{s-4}{s}\right)^{1/2} |T^l(s)|^2 .$$

1. First-Order Amplitudes

First-order amplitudes correspond to the Feynman graphs of Fig. 2. The partial-wave projection of the exchanged σ pole is

$$\Sigma_l = 8(g\hat{v}^2)Q_l\left(1+\frac{M^2}{2q^2}\right)\Big/ 2q^2 , \tag{B4}$$

where Q_l is the Legendre function of second kind, so we have

$$A_1{}^l = \delta_{l,0}[-2-4(g\hat{v}^2)/(s-M^2)] , \tag{B5}$$

$$(B_1{}^l + C_1{}^l) = -4\delta_{l,0} + \Sigma_l \text{ (even waves)}, \tag{B6}$$

$$(B_1{}^l - C_1{}^l) = \Sigma_l \text{ (odd waves)}. \tag{B7}$$

2. Second-Order Amplitudes

Let us define

$$\bar{B}_{xy}(s) = \frac{1}{8\pi^2}\Bigg\{-\frac{1}{s}[(m_x+m_y)^2-s]^{1/2}[(m_x-m_y)^2-s]^{1/2} \times \ln\left[\frac{[(m_x+m_y)^2-s]^{1/2}+[(m_x-m_y)^2-s]^{1/2}}{2(m_xm_y)^{1/2}}\right] + \frac{(m_x{}^2-m_y{}^2)}{2s}\ln\left(\frac{m_x}{m_y}\right) - 1 + \ln[(m_xm_y)^{1/2}]\Bigg\} \tag{B8}$$

and the constants

$$B_1 = \bar{B}_{\sigma\pi}(0) - \bar{B}_{\pi\pi}(0) = \frac{1}{8\pi^2}\left(\frac{M^2+1}{M^2-1}\ln M + \ln M - 1\right), \tag{B9}$$

$$B_2 = \bar{B}_{\sigma\sigma}(0) - \bar{B}_{\pi\pi}(0) = (1/8\pi^2)\ln M , \tag{B10}$$

$$B_3 = \bar{B}_{\sigma\pi}(1) . \tag{B11}$$

A. Pole Terms

The pole terms correspond to the Feynman graphs of Fig. 4. Poles in the s-channel are given by

$$A^{\mathrm{pole},l} = \delta_{l,0}\Bigg[-4(g\hat{v}^2)\Bigg(\frac{2}{s-M^2}[4B_3+3\bar{B}_{\sigma\sigma}(s)+5\bar{B}_{\pi\pi}(s)] + 2(g\hat{v}^2)\frac{1}{s-M^2}[4C_\sigma(s)+12C_\pi(s)] + (g\hat{v}^2)^2\frac{1}{(s-M^2)^2} \times [18\bar{B}_{\sigma\sigma}(s)+6\bar{B}_{\pi\pi}(s)-4(B_3-B_1)]\Bigg)\Bigg] , \tag{B12}$$

where the function C is defined as

$$C_\sigma(s) = -\frac{1}{8\pi^2}\int_4^\infty \frac{ds'}{s'-s}\frac{1}{2[s'(s'-4)]^{1/2}} \times \ln\frac{M^2}{s'-4+M^2} \tag{B13}$$

and

$$C_\pi(s) = -\frac{1}{8\pi^2}\int_{4M^2}^\infty \frac{ds'}{s'-s}\frac{1}{2[s'(s'-4)]^{1/2}} \times \ln\frac{s'-2M^2-[(s'-4)(s'-4M^2)]^{1/2}}{s'-2M^2+[(s'-4)(s'-4M^2)]^{1/2}} . \tag{B14}$$

For the pole terms in the t channel, we must define the analytic continuation in l of the projection since $M^2 > 4$, so that one cannot apply blindly the Froissart-Fribov integral. This is done in the following way. Consider an expression of the form

$$f(t) = \Phi(t)/(t-M^2) , \tag{B15}$$

where $\Phi(t)$ satisfies a dispersion relation

$$\Phi(t) = \frac{1}{\pi}\int_4^\infty \rho(t')\frac{dt'}{t'-t} . \tag{B16}$$

The angular momentum projection of $f(t)$ is

$$f_l = \int_{-1}^1 dz\, P_l(z) f(t) , \tag{B17}$$

which we can write formally as

$$f_l = \int_{-1}^1 dz \times P_l(z)\left[\frac{\Phi(M^2)}{t-M^2} + \frac{1}{\pi}\int_4^\infty \frac{\rho(t')dt'}{(t'-t)(t'-M^2)}\right] . \tag{B18}$$

Thus we can write

$$f_l = \frac{1}{2\pi q^2}\int_4^\infty \rho(t') \times \frac{Q_l(1+t'/2q^2) - Q_l(1+M^2/2q^2)}{t'-M^2} dt' , \tag{B19}$$

which is well defined and is analytic in l. A very similar procedure can be applied to the case of a double pole.

Let us now define

$$U_\sigma(s)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_4^\infty dt\,\frac{Q_l(1+t/2q^2)-Q_l(1+M^2/2q^2)}{t-M^2}\,\frac{1}{2[t(t-4)]^{1/2}}\ln\frac{M^2}{t-4+M^2}\,, \tag{B20}$$

$$U_\pi(s)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_{4M^2}^\infty dt\,\frac{Q_l(1+t/2q^2)-Q_l(1+M^2/2q^2)}{t-M^2}\,\frac{1}{2[t(t-4)]^{1/2}}\ln\frac{t-2M^2-[(t-4)(t-4M^2)]^{1/2}}{t-2M^2+[(t-4)(t-4M^2)]^{1/2}}\,, \tag{B21}$$

$$V_\pi(s)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_4^\infty dt\,\frac{1}{2}\left(\frac{t-4}{t}\right)^{1/2}\frac{Q_l(1+t/2q^2)-(M^2/t)Q_l(1+M^2/2q^2)}{t-M^2}\,, \tag{B22}$$

$$V_\sigma(s)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_{4M^2}^\infty dt\,\frac{1}{2}\left(\frac{t-4M^2}{t}\right)^{1/2}\frac{Q_l(1+t/2q^2)-(M^2/t)Q_l(1+M^2/2q^2)}{t-M^2}\,, \tag{B23}$$

$$W_\pi(s)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_4^\infty dt\,\frac{1}{2}\left(\frac{t-4}{t}\right)^{1/2}\frac{Q_l(1+t/2q^2)-Q_l(1+M^2/2q^2)-[(t-M^2)/2q^2](M^2/t)Q_l'(1+M^2/2q^2)}{(t-M^2)^2}\,, \tag{B24}$$

$$W_\sigma(s)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_{4M^2}^\infty dt\,\frac{1}{2}\left(\frac{t-4M^2}{t}\right)^{1/2}$$
$$\times\frac{Q_l(1+t/2q^2)-Q_l(1+M^2/2q^2)-[(t-M^2)/2q^2](M^2/t)Q_l'(1+M^2/2q^2)}{(t-M^2)^2}\,. \tag{B25}$$

The contribution of the exchanged poles are

$$B^{\text{pole},l}=-4(g\hat{v}^2)\left\{-\frac{1}{q^2}Q_l\left(1+\frac{M^2}{2q^2}\right)(4B_3+3B_2)\right.$$
$$-(g\hat{v}^2)\left(\frac{1}{2q^2}\right)^2Q_l'\left(1+\frac{M^2}{2q^2}\right)[18B_2-4(B_3-B_1)]$$
$$+2[3V_\sigma(s)+5V_\pi(s)]+2(g\hat{v}^2)$$
$$\left.\times[4U_\sigma(s)+12U_\pi(s)+3W_\pi(s)+9W_\sigma(s)]\right\}\,, \tag{B26}$$

with

$$C^{\text{pole},l}=(-)^lB^{\text{pole},l}\,, \tag{B27}$$

so that for the pole terms we have

$$(B\pm C)^{\text{pole},l}=2B^{\text{pole},l}\binom{\text{even waves}}{\text{odd waves}}\,. \tag{B28}$$

B. Contact Terms

The contact terms correspond to Feynman graphs of Fig. (3). Let us define

$$\beta_\pi^l(s)=\frac{1}{2}\int_{-1}^1 dz\,P_l(z)\bar{B}_{\pi\pi}(t)\,, \tag{B29}$$

$$\beta_\sigma^l(s)=\frac{1}{2}\int_{-1}^1 dz\,P_l(z)\bar{B}_{\sigma\sigma}(t)\,. \tag{B30}$$

For $l\neq0$ the analytic continuation in l is defined as

$$\beta_\pi^l(s)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_4^\infty dt$$
$$\times\frac{1}{2}\left(\frac{t-4}{t}\right)^{1/2}Q_l\left(1+\frac{t}{2q^2}\right)\,, \tag{B31}$$

$$\beta_\sigma^l(s)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_{4M^2}^\infty dt$$
$$\times\frac{1}{2}\left(\frac{t-4M^2}{t}\right)^{1/2}Q_l\left(1+\frac{t}{2q^2}\right)\,. \tag{B32}$$

We also define

$$\gamma_\sigma^l(s)=\frac{1}{2}\int_{-1}^1 dz\,P_l(z)C_\sigma(t)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_4^\infty dt$$
$$\times Q_l\left(1+\frac{t}{2q^2}\right)\frac{1}{2[t(t-4)]^{1/2}}\ln\frac{M^2}{t-4+M^2} \tag{B33}$$

and

$$\gamma_\pi^l(s)=-\frac{1}{8\pi^2}\frac{1}{2q^2}\int_{4M^2}^\infty dt\,\frac{Q_l(1+t/2q^2)}{2[t(t-4)]^{1/2}}$$
$$\times\ln\frac{t-2M^2-[(t-4)(t-4M^2)]^{1/2}}{t-2M^2+[(t-4)(t-4M^2)]^{1/2}}\,. \tag{B34}$$

We thus have

$$A^{\text{contact},l} = -2\{\delta_{l,0}[7\bar{B}_{\pi\pi}(s)+\bar{B}_{\sigma\sigma}(s) +4(g\hat{v}^2)[2C_\pi(s)+2C_\sigma(s)]] +4\beta_\pi{}^l(s)+16(g\hat{v}^2)\gamma_0{}^l(s)\}, \quad (B35)$$

$$(B+C)^{\text{contact},l} = -4\{\delta_{l,0}[2\bar{B}_{\pi\pi}(s)+8(g\hat{v}^2)C_\sigma(s)] +9\beta_\pi{}^l(s)+\beta_\sigma{}^l(s)+4(g\hat{v}^2) \times[4\gamma_\sigma{}^l(s)+2\gamma_\pi{}^l(s)]\}, \quad (B36)$$

$$(B-C)^{\text{contact},l} = -4[5\beta_\pi{}^l(s)+\beta_\sigma{}^l(s)+8(g\hat{v}^2)\gamma_\pi{}^l(s)]. \quad (B37)$$

C. Box Diagrams

The box diagrams correspond to the Feynman graphs of Fig. 5. Let us define

$$E(s,t) = \frac{1}{\pi^2}\int_4^\infty \frac{ds'}{s'-s}\int_{4M^2}^\infty \frac{dt'}{t'-t} \times \frac{\theta[(t'-4M^2)(s'-4)-4M^4]}{(s't')^{1/2}[(t'-4M^2)(s'-4)-4M^4]^{1/2}}. \quad (B38)$$

We have, recalling Eq. (3.13),

$$E(s,t) \equiv -8D(p_1p_2;\,p_3p_4). \quad (B39)$$

The function $E(s,t)$ has two pions in the s channel and two σ's in the t channel.

Let us define the partial-wave projections of $E(s,t)$ in the following way:

$[E(s,t)]_l{}^s$: two σ's in the crossed channel,

$[E(t,s)]_l{}^s$: two pions in the crossed channel.

We then have

$$[E(s,t)]_l{}^s = \frac{1}{\pi^2}\frac{1}{2q^2}\int_{4M^2}^\infty dt\,\frac{Q_l(1+t/2q^2)}{[t(t-4M^2)]^{1/2}} \times\frac{1}{[s(s-s_0)]^{1/2}}\left[i\pi+\ln\frac{s^{1/2}-(s-s_0)^{1/2}}{s^{1/2}+(s-s_0)^{1/2}}\right], \quad (B40)$$

with

$$s_0 = 4+4M^4/(t-4M^2), \quad (B41)$$

and

$$[E(t,s)]_l{}^s = \frac{1}{\pi^2}\frac{1}{2q^2}\int_4^\infty dt\,\frac{Q_l(1+t/2q^2)}{[t(t-4)]^{1/2}} \times\left[\frac{1}{[-s(s_0-s)]^{1/2}}\ln\frac{(-s)^{1/2}+(s_0-s)^{1/2}}{-(-s)^{1/2}+(s_0-s)^{1/2}}\right], \quad (B42)$$

with

$$s_0 = 4M^2+4M^4/(t-4). \quad (B43)$$

We then have

$$A^{\text{box},l} = 2(g\hat{v}^2)[E(t,s)+E(u,s)]_l{}^s, \quad (B44)$$

$$B^{\text{box},l} = 2(g\hat{v}^2)[E(s,t)+E(u,t)]_l{}^s, \quad (B45)$$

$$C^{\text{box},l} = 2(g\hat{v}^2)[E(s,u)+E(t,u)]_l{}^s. \quad (B46)$$

D. Second-Order Contributions

We have thus

$$A_2{}^l = A^{\text{pole},l}+A^{\text{contact},l}+A^{\text{box},l}, \quad (B47)$$

$$(B_2{}^l\pm C_2{}^l) = (B\pm C)^{\text{pole},l}+(B\pm C)^{\text{contact},l} +(B\pm C)^{\text{box},l}, \quad (B48)$$

If we write the isospin amplitudes $T^{I,l}$ as

$$T^{I,l} = C_\pi{}^2\mathcal{T}^{I,l}, \quad (B49)$$

we have

$$\mathcal{T}^{0,l} = g[3A_1{}^l+(B+C)_1{}^l]+g^2[3A_2{}^l+(B+C)_2{}^l], \quad (B50)$$

$$\begin{pmatrix}\mathcal{T}^{1,l}\\ \mathcal{T}^{2,l}\end{pmatrix} = g[B\mp C]_1{}^l+g^2[B\mp C]_2{}^l. \quad (B51)$$

Now we remember that we have [Eq. (3.6)]

$$C_\pi = 1+4g(g\hat{v}^2)\bar{B}_{\sigma\pi}{}'(m_\pi{}^2) = 1+g\xi. \quad (B52)$$

If we rewrite Eqs. (B50) and (B51), dropping the index l, as

$$\mathcal{T}^0 = gt_1{}^0+g^2t_2{}^0, \quad (B53a)$$

$$\mathcal{T}^1 = gt_1{}^1+g^2t_2{}^1, \quad (B53b)$$

$$\mathcal{T}^2 = gt_1{}^2+g^2t_2{}^2, \quad (B53c)$$

then the perturbation expansion in g of the partial-wave T matrix up to second order is, for each isospin I,

$$T^{I,l} = gt_1{}^I+g^2(t_2{}^I+2\xi t_1{}^I). \quad (B54)$$