elastic scattering amplitude $f_0(a)$ as previously described is adequate, $S_0(s)$ possesses enough structure[28] to be a useful explicit approximation to the partial-wave amplitude $T_0(s)$ in the physical region.

[28] The so-called triangle amplitude for $T_0(s)$, not $T(s_1,s_2,s_3)$ [cf. Refs. 10 and 11 and I. J. R. Aitchison and C. Kacser, Phys. Rev. **173**, 1700 (1968)], is partially represented by $S_0(s)$, since its δ-function part has been added to $T_{0,1}(s)$: $c_0T_{0,0}(s)\phi(k^2)=1+if_0(s)$. Its principal-value part is the third term in Eq. (8) with $Y_0(s)$ set equal to $W_0{}^I(s)$. Now $\phi(k^2)$ vanishes on the right-hand cut when the phase shift δ goes through $\frac{1}{2}\pi$, but it does not vanish at the exact (complex) position of the resonance, $s=\alpha_0$. Hence it is not obvious how much interference there is between the integrals of the direct and crossed Watson terms in $T(s_1,s_2,s_3)$ when s_1, say, is near α_0. As far as the logarithmic singularity of $T_0(s)$ at $a_1\in W_-$ is concerned, the discontinuity across the cut which it generates is proportional to $c_0T_{0,0}(s)\phi(k^2)$. Addition of a further $if_0(s)$ to $c_0T_{0,0}(s)\phi(k^2)$, to represent the contribution from the principal part of the triangle amplitude near $s=\alpha_1$, would result in a factor $\exp(2i\delta)$. This may be more appropriate than $c_0T_{0,0}(s)\times\phi(k^2)$ for s near α_1, as shown in Ref. 11 and the paper by Aitchison and Kacser cited above, which study in detail the triangle singularity at $s=\alpha_1$ and its effect on the singularity of the crossed Watson term at the same point. However, $f_0{}^*(s)\times\exp(2i\delta)/k$ does have an undesirable right-hand cut starting at $k=0$. It is not quite obvious which of the two effects, the one at $s=4$ or the one at $s=\alpha_1$, will be more important. $s=4$ is the physical threshold. α_1 is also close to the physical region, since one meets it by crossing from above the right-hand cut of $S_0(s)$ [remaining in the same sheet of $W_0(s)$] or the right-hand cut of $T_{0,1}(s)+$triangle amplitude, as the case may be. However, for a resonance of finite width, α_1 will not be exactly in the physical region but some distance away. It is for this reason and the fact that, even though α_1 is a fairly strong (logarithmic) singularity, no spectacular effects seem to be associated with it, whether the contribution of the triangle amplitude is considered or not, (cf., e.g., Refs. 2, 4, 10, and 11) that the approximation to $T_0(s)$ has been chosen so as to satisfy property (1) of the Introduction. As a compromise one could, of course, replace $\phi(k^2)$ in Eq. (9) by some polynomial in k^2 to be determined from experiment. This is a more realistic procedure in any case, but it has the disadvantage of introducing additional unknown parameters into $S_0(s)$.

PHYSICAL REVIEW D VOLUME 2, NUMBER 8 15 OCTOBER 1970

Analytic Hard-Pion Methods: The $A_1\rho\pi$ System*

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(Received 5 December 1969)

We use current algebra and analyticity to study vertex functions occurring in the $A_1\rho\pi$ system. Employing the conserved vector current and partially conserved axial-vector current relations, and the $SU(2)\times SU(2)$ algebra of currents, we generate Ward identities which relate two- and three-point functions of vector and axial-vector currents. Extracting the pion poles from these vertex functions and exposing their isospin content, we define form factors whose analytic properties may readily be studied and, in particular, deduce from the Ward identities a relation involving the pion form factor. With suitable low-energy approximations which maintain the correct cut structure, we use this relation to calculate an effective-range formula for the pion form factor, and consequently from unitarity, the p-wave $\pi\pi$ phase shift. Our results are generally in agreement with experiment. Using A₁ dominance, we are able to obtain analytic effective-range formulas for the form factors appearing in the vector-current matrix element of π, A_1 mesons. From these form factors, measurable in the reaction $e^+e^-\to\pi A_1$, we calculate the $A_1\to\rho\pi$ width and the A_1-ρ spin correlation. Finally, we extend our methods to encompass both $\pi\pi$ and πA_1 cut contributions, and derive a set of coupled integral equations which we solve approximately for the $\pi\pi$ and πA_1 form factors. We conclude with a general observation on the complementary roles played by current algebra and by unitarity.

INTRODUCTION

HARD-PION methods refer to the procedure whereby hadronic matrix elements of physical interest can be extrapolated to off-mass-shell values of the particle momenta. The foundations[1] of the procedure originate in the conserved vector-current (CVC) theory, the partially conserved axial-vector current (PCAC) hypothesis, and current algebra. Earlier approaches involved zero four-momentum limits and provided such exact statements about extrapolated amplitudes as the Adler consistency relation[2] and the soft-pion theorems.[3] The hard-pion techniques[4] pertain to arbitrary four-momenta, thus extending the range of utility of these off-shell methods. Physical mesonic matrix elements are expressed, in their extrapolated form, in terms of vacuum expectation values of products of local operators which can be identified with hadronic vector and axial-vector currents satisfying the algebra of currents. As Schnitzer and Weinberg[4] have shown, the content of this method is summarized in a set of Ward identities, which give constraints among the N-point functions of the theory. Further dynamical structure must be added to this system of constraints in order to obtain detailed knowledge of the matrix elements in question. In this

* Supported in part by the National Science Foundation.

[1] For general background, see S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968).

[2] S. L. Adler, Phys. Rev. **139**, B1638 (1965).

[3] See, e.g., S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[4] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967), hereafter referred to as SW. See also S. G. Brown and G. B. West, *ibid.* **168**, 1605 (1968); T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 1067 (1967), and Refs. 9 and 12 below.

paper, we show how the constraints of current algebra may be complemented with constraints of a basically different, but equally general, nature. In particular, we suggest that the concepts of analyticity and unitarity can be incorporated advantageously into the Ward identity structure and that the simultaneous implications of current algebra and of unitarity provide more predictive power than either scheme is capable of giving when applied disjointly. We will discuss the extent to which we are presently able to implement the complementary constraints of current algebra and of analyticity and unitarity. These considerations have already been applied in a limited way to the determination of the pion form factor and the $T=J=1$ $\pi\pi$ phase shift.[5] Some extensions of this work and some more general conclusions will be given below.

I. FORM FACTORS AND WARD IDENTITIES

In this paper we shall confine our attention to the three-point functions of the vector and axial-vector currents of $SU(2)\times SU(2)$:

$$\begin{aligned} W_\lambda{}^{abc}(q,p) &= \int dxdy\, e^{-iqx}e^{ipy}\langle 0|T\partial_\mu A_\mu{}^a(x)\partial_\nu A_\nu{}^b(y)V_\lambda{}^c(0)|0\rangle\,, \\ W_{\nu\lambda}{}^{abc}(q,p) &= \int dxdy\, e^{-iqx}e^{ipy}\langle 0|T\partial_\mu A_\mu{}^a(x)A_\nu{}^b(y)V_\lambda{}^c(0)|0\rangle\,, \\ W_{\mu\nu\lambda}{}^{abc}(q,p) &= \int dxdy\, e^{-iqx}e^{ipy}\langle 0|TA_\mu{}^a(x)A_\nu{}^b(y)V_\lambda{}^c(0)|0\rangle\,, \end{aligned} \tag{1}$$

in which a, b, and c are isospin indices. To illustrate the relation of the above quantities to matrix elements of direct physical interest, we show how the first of them, for $c=3$, gives the off-shell electromagnetic form factor of the pion, extrapolated in the momenta q, p, and $k=p-q$. On shell, the pion form factor $F(t)$ is defined by

$$\langle\pi(qa)|V_\lambda{}^3(0)|\pi(pb)\rangle = -\frac{i\epsilon_{ab3}}{(4\omega_q\omega_p)^{1/2}}F(t)Q_\lambda\,, \tag{2}$$

where $Q=p+q$ and $t=-k^2$. Straightforward reduction and application of the PCAC relation[6]

$$\partial_\mu A_\mu{}^a = F_\pi m_\pi{}^2\pi^a \tag{3}$$

yields

$$\langle\pi(qa)|V_\lambda{}^3(0)|\pi(pb)\rangle = -\frac{[(m_\pi{}^2+q^2)(m_\pi{}^2+p^2)W_\lambda{}^{ab3}(q,p)]_{p^2=q^2=-m_\pi{}^2}}{(4\omega_q\omega_p)^{1/2}F_\pi{}^2m_\pi{}^4}\,. \tag{4}$$

[5] J. J. Brehm, E. Golowich, and S. C. Prasad, Phys. Rev. Letters **23**, 666 (1969). Further work along these lines appears in R. Rockmore, *ibid.* **24**, 541 (1970).

[6] The pion decay constant is $F_\pi=94$ MeV. It is defined by $\langle 0|\partial_\mu A_\mu{}^a(0)|\pi(pb)\rangle=(2\omega_p)^{-1/2}\delta_{ab}F_\pi$.

The second and third of Eqs. (1) possess physical interpretations which will be indicated later in this section.

Along with the three-point functions of Eqs. (1) we shall have need of the following spectral representations for the propagators:

$$\begin{aligned} &\int dy\, e^{iky}\langle 0|TV_\nu{}^b(y)V_\lambda{}^c(0)|0\rangle \\ &\qquad = -i\delta_{bc}[\Delta_{\nu\lambda}{}^V(k)-C_V\delta_{\nu4}\delta_{\lambda4}]\,, \\ &\int dx\, e^{-iqx}\langle 0|TA_\mu{}^a(x)A_\nu{}^b(0)|0\rangle \\ &\qquad = -i\delta_{ab}\left[\Delta_{\mu\nu}{}^A(q)+F_\pi{}^2\frac{q_\mu q_\nu}{q^2+m_\pi{}^2}-(C_A+F_\pi{}^2)\delta_{\mu4}\delta_{\nu4}\right]\,, \end{aligned} \tag{5}$$

where

$$\Delta_{\mu\nu}{}^{V,A}(k) = \int\frac{dx}{x+k^2}\rho_{V,A}(x)\left(\delta_{\mu\nu}+\frac{k_\mu k_\nu}{x}\right) \tag{6}$$

and

$$C_{V,A} = \int\frac{\rho_{V,A}(x)}{x}dx\,. \tag{7}$$

The spin-zero part of the axial-vector current spectrum has been saturated with the pion state. The spectral functions ρ_V and ρ_A are related to $\langle 0|V_\mu{}^a(x)V_\nu{}^b(0)|0\rangle$ and $\langle 0|A_\mu{}^a(x)A_\nu{}^b(0)|0\rangle$ as usual. In addition, we have

$$\begin{aligned} &\int dx\, e^{-iqx}\langle 0|T\partial_\mu A_\mu{}^a(x)A_\nu{}^b(0)|0\rangle = -\delta_{ab}\frac{m_\pi{}^2F_\pi{}^2}{q^2+m_\pi{}^2}q_\nu \\ &\text{and} \\ &\int dx\, e^{-iqx}\langle 0|T\partial_\mu A_\mu{}^a(x)\partial_\nu A_\nu{}^b(0)|0\rangle = -i\delta_{ab}\frac{m_\pi{}^4F_\pi{}^2}{q^2+m_\pi{}^2}\,. \end{aligned} \tag{8}$$

The chiral commutation relations[7] we use are

$$\begin{aligned} \delta(x_0-y_0)[V_4{}^a(x),V_\nu{}^b(y)] &= -\delta(x-y)\epsilon_{abc}V_\nu{}^c(y)+\text{ST}\,, \\ \delta(x_0-y_0)[V_4{}^a(x),A_\nu{}^b(y)] &= -\delta(x-y)\epsilon_{abc}A_\nu{}^c(y)+\text{ST}\,, \\ \delta(x_0-y_0)[A_4{}^a(x),A_\nu{}^b(y)] &= -\delta(x-y)\epsilon_{abc}V_\nu{}^c(y)+\text{ST}\,, \\ \delta(x_0-y_0)[A_4{}^a(x),V_\nu{}^b(y)] &= -\delta(x-y)\epsilon_{abc}A_\nu{}^c(y)+\text{ST}\,, \end{aligned} \tag{9}$$

where ST refers to the Schwinger terms which, following Weinberg,[8] we assume are not isovector operators. Equations (9) provide relations among the quantities in Eqs. (1) and (5)–(8); the results[4] are the Ward identities

$$k_\lambda W_\lambda{}^{abc}(q,p) = i\epsilon_{abc}m_\pi{}^4F_\pi{}^2\left(\frac{1}{q^2+m_\pi{}^2}-\frac{1}{p^2+m_\pi{}^2}\right)\,, \tag{10}$$

$$k_\lambda W_{\nu\lambda}{}^{abc}(q,p) = \epsilon_{abc}m_\pi{}^2F_\pi{}^2\left(\frac{q_\nu}{q^2+m_\pi{}^2}-\frac{p_\nu}{p^2+m_\pi{}^2}\right)\,, \tag{11}$$

[7] We differ from SW in convention by a factor of 2.

[8] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

$$k_\lambda W_{\mu\nu\lambda}{}^{abc}(q,p)=i\epsilon_{abc}\left[\Delta_{\mu\nu}{}^A(q)+F_\pi{}^2\frac{q_\mu q_\nu}{q^2+m_\pi{}^2}\right.$$
$$\left.-\Delta_{\mu\nu}{}^A(p)-F_\pi{}^2\frac{p_\mu p_\nu}{p^2+m_\pi{}^2}\right], \quad (12)$$

$$p_\nu W_{\nu\lambda}{}^{abc}(q,p)=iW_\lambda{}^{abc}(q,p)$$
$$+\epsilon_{abc}m_\pi{}^2F_\pi{}^2q_\lambda/(q^2+m_\pi{}^2), \quad (13)$$

$$q_\mu W_{\mu\nu\lambda}{}^{abc}(q,p)=-iW_{\nu\lambda}{}^{abc}(q,p)$$
$$+i\epsilon_{abc}[\Delta_{\nu\lambda}{}^A(p)+F_\pi{}^2p_\nu p_\lambda/(p^2+m_\pi{}^2)$$
$$-\Delta_{\nu\lambda}{}^V(k)-(C_A+F_\pi{}^2-C_V)\delta_{\nu4}\delta_{\lambda4}]. \quad (14)$$

By comparing q_μ contracted into (12) with k_λ contracted into (14), and using (11), we obtain

$$(C_A+F_\pi{}^2-C_V)(k_\nu-k_4\delta_{\nu4})=0,$$

whence

$$C_V=C_A+F_\pi{}^2. \quad (15)$$

Thus Weinberg's first sum rule[8] depends in no way on the assumption of a conserved axial-vector current. The above derivation of (15) is identical with that of Weinberg except that (3) has been invoked rather than $\partial_\mu A_\mu{}^a=0$.

Equations (10)–(14) exhaust the content of the current commutators and must now be supplemented with further structure. For Schnitzer and Weinberg, the next step ultimately took the form of pole dominance. Each of the channels (qa), (pb), and (kc) was saturated with the appropriate π, ρ, or A_1 meson poles, and the residual vertex factors were assumed to be as smooth in momenta q, p, and k as possible. The procedure of meson-pole saturation of the vacuum expectation values and imposition of chiral symmetry has been usefully cast in an equivalent Lagrangian scheme by Arnowitt and co-workers.[9] It is the implementation of the Ward identities with pole dominance in all channels that we specifically wish to avoid. In particular, we shall not employ ρ-pole dominance of the vector current (kc) channel but rather incorporate the $\pi\pi$ branch cut, and, in principle, higher-mass cuts as well. In this way, the instability of the ρ meson is properly treated and, more importantly, the foundations are laid for introducing analyticity and unitarity as added ingredients to the theory.

Pion-pole dominance will continue to be used; the stability of the pion and the fundamental role of PCAC place this assumption on a different footing than obtains for ρ dominance and A_1 dominance. Indeed, pion dominance of the spin-zero axial-vector spectral function has already been used in (5) and (8), and, from (4), is needed to secure the interpretation of $W_\lambda{}^{ab3}(q,p)$ as the off-shell pion form factor. Accordingly, we factor the pion poles from the vacuum expectation values and write

$$W_\lambda{}^{abc}(q,p)=i\epsilon_{abc}\frac{F_\pi{}^2m_\pi{}^4}{(m_\pi{}^2+p^2)(m_\pi{}^2+q^2)}F_\lambda(q,p), \quad (16)$$

$$W_{\nu\lambda}{}^{abc}(q,p)=\epsilon_{abc}\frac{F_\pi m_\pi{}^2}{m_\pi{}^2+q^2}\left[F_{\nu\lambda}(q,p)\right.$$
$$\left.+\frac{F_\pi}{m_\pi{}^2+p^2}p_\nu F_\lambda(q,p)\right], \quad (17)$$

$$W_{\mu\nu\lambda}{}^{abc}(q,p)=i\epsilon_{abc}\left\{F_{\mu\nu\lambda}(q,p)\right.$$
$$+F_\pi\left[\frac{p_\nu}{m_\pi{}^2+p^2}F_{\mu\lambda}(p,q)+\frac{q_\mu}{m_\pi{}^2+q^2}F_{\nu\lambda}(q,p)\right]$$
$$\left.+\frac{F_\pi{}^2}{(m_\pi{}^2+p^2)(m_\pi{}^2+q^2)}q_\mu p_\nu F_\lambda(q,p)\right\}. \quad (18)$$

Before embarking on a theoretical analysis of Eqs. (16)–(18), we digress briefly to exhibit part of the physical content of the functions F_λ and $F_{\nu\lambda}$ and also to review some pertinent results of pole-dominated hard-pion calculations.[4] It is convenient for this purpose to express F_λ and $F_{\nu\lambda}$ in terms of matrix elements of the vector current. From (4) and (16), one can identify F_λ as

$$\langle\pi(qa)|V_\lambda{}^c(0)|\pi(pb)\rangle$$
$$=\frac{-i\epsilon_{abc}}{(4\omega_p\omega_q)^{1/2}}[F_\lambda(q,p)]_{q^2=p^2=-m_\pi{}^2}. \quad (19a)$$

Similarly, $F_{\nu\lambda}$ can be interpreted in terms of the matrix element for the experimentally remote process, A_{l3} decay:

$$\langle\pi(qa)|V_\lambda{}^c(0)|A_1(pbi)\rangle$$
$$=\frac{-\epsilon_{abc}}{(4\omega_q\omega_p)^{1/2}}\epsilon_\nu{}^{(i)}(p)$$
$$\times\left[\frac{m_A{}^2+p^2}{g_A}F_{\nu\lambda}(q,p)\right]_{p^2=-m_A{}^2,\,q^2=-m_\pi{}^2}, \quad (19b)$$

where $\epsilon_\nu{}^{(i)}(p)$ is the polarization vector of the A_1 meson of momentum p, helicity i, and g_A is defined by

$$\langle 0|A_\mu{}^a(0)|A_1(pbi)\rangle=(2\omega_p)^{-\frac{1}{2}}\delta_{ab}g_A\epsilon_\mu{}^{(i)}(p). \quad (20)$$

In Eq. (19b) we have ignored A_1 instability in constructing the asymptotic state $|A_1(pbi)\rangle$. The matrix elements in (19a) and (19b) may be expressed in terms of invariant functions, or form factors, with appropriate

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[9] R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters 19, 1085 (1967), and Ref. 17 below. For a complete listing of the effective Lagrangian literature, see S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. 41, 531 (1969).

kinematical factors. This has been done already for Eq. (19a) in Eq. (2). For Eq. (19b) we have, taking $t=-k^2$,

$$\langle\pi(qa)|V_\lambda{}^c(0)|A_1(pbi)\rangle = -\epsilon_{abc}\epsilon_\nu{}^{(i)}(p)\frac{A(t)\delta_{\nu\lambda}+B(t)k_\nu p_\lambda+C(t)k_\nu k_\lambda}{(4\omega_p\omega_q)^{1/2}}, \quad (21)$$

in which A, B, and C are constrained by vector current conservation to satisfy

$$A+p\cdot kB+k^2C=0. \quad (22)$$

If the A_{l3} form factors are continued to the region $t\geq(m_A+m_\pi)^2$, then A, B, and C describe that part of the process $e^+e^-\to\pi A_1$ which proceeds through the isovector component of the electromagnetic current, a process measurable in colliding-beam experiments. We note that F_λ can be expanded off shell as

$$F_\lambda(q,p)=FQ_\lambda+Gk_\lambda, \quad (23)$$

where F and G are functions of q^2, p^2, and k^2 with F symmetric and G antisymmetric in q^2 and p^2. The A_1-dominated off-shell form of $F_{\nu\lambda}$ can be expressed as

$$F_{\nu\lambda}(q,p)=g_A\frac{\delta_{\nu\sigma}+p_\nu p_\sigma/m_A{}^2}{p^2+m_A{}^2} \times(A\,\delta_{\sigma\lambda}+Bk_\sigma p_\lambda+Ck_\sigma k_\lambda+Dp_\sigma p_\lambda+Ep_\sigma k_\lambda), \quad (24)$$

where A, etc., now depend on q^2, p^2, and k^2.

Each of the functions F_λ and $F_{\nu\lambda}$ contains a ρ-meson pole in the variable k^2. We can determine the following coupling constants from the pole residues:

$$\begin{aligned} g_{\rho\pi\pi}&=(1/g_\rho)[(m_\rho{}^2+k^2)F]_{q^2=p^2=-m_\pi{}^2,\ k^2=-m_\rho{}^2},\\ g_{A\rho\pi}&=(1/g_\rho)[(m_\rho{}^2+k^2)A]_{q^2=-m_\pi{}^2,\ p^2=-m_A{}^2,\ k^2=-m_\rho{}^2}, \quad (25)\\ h_{A\rho\pi}&=(1/g_\rho)[(m_\rho{}^2+k^2)B]_{q^2=-m_\pi{}^2,\ p^2=-m_A{}^2,\ k^2=-m_\rho{}^2},\end{aligned}$$

where we have defined g_ρ by

$$\langle 0|V_\lambda{}^c(0)|\rho(kjd)\rangle=(2\omega_k)^{-\frac{1}{2}}\delta_{cd}g_\rho\epsilon_\lambda{}^{(j)}(k). \quad (26)$$

The expressions in (25) presuppose the treatment of decay matrix elements $\langle\pi\pi|\rho\rangle$ and $\langle\pi\rho|A_1\rangle$ in which the unstable mesons ρ and A_1 are assigned real masses in accordance with the asymptotic condition. The coupling constants $g_{\rho\pi\pi}$, $g_{A\rho\pi}$, and $h_{A\rho\pi}$ determine the decay rates

$$\Gamma(\rho\to\pi\pi)=\frac{2}{3}\frac{g_{\rho\pi\pi}{}^2}{4\pi}\frac{|\mathbf{q}|^3}{m_\rho{}^2}, \quad (27a)$$

$$\Gamma(A_1\to\rho\pi)=\frac{|\mathbf{q}|}{12\pi m_A{}^2}\left\{3g_{A\rho\pi}{}^2 +\frac{|\mathbf{q}|^2}{m_\rho{}^2}[(g_{A\rho\pi}-m_A\omega_k h_{A\rho\pi})^2-m_A{}^2m_\rho{}^2h_{A\rho\pi}{}^2]\right\}, \quad (27b)$$

where $|\mathbf{q}|$ is the magnitude of the decay momentum in the parent rest frame. The hard-pion calculation of Schnitzer and Weinberg gives the coupling constants of Eq. (25) in terms of a single parameter δ,

$$\begin{aligned} g_{\rho\pi\pi}{}^{(\mathrm{SW})}&=(3-\delta)m_\rho{}^2/4g_\rho,\\ g_{A\rho\pi}{}^{(\mathrm{SW})}&=[\delta(m_\pi{}^2-m_\rho{}^2)-m_A{}^2]/4F_\pi, \quad (28)\\ h_{A\rho\pi}{}^{(\mathrm{SW})}&=-\delta/2F_\pi,\end{aligned}$$

as well as the pion form factor

$$F^{(\mathrm{SW})}(t)=\frac{1+\delta}{4}+\frac{3-\delta}{4}\frac{m_\rho{}^2}{m_\rho{}^2-t}. \quad (29)$$

The value $\delta\simeq-\frac{1}{2}$ gives $\Gamma(A_1\to\rho\pi)$ and $\Gamma(\rho\to\pi\pi)$, in reasonable agreement with experiment. We note in passing that some results of ρ dominance in its simplest form (unadorned by chiral dynamics) are reproduced by $\delta=-1$,

$$g_{\rho\pi\pi}g_\rho=m_\rho{}^2,\quad F(t)=m_\rho{}^2/(m_\rho{}^2-t), \quad (30)$$

but for which $\Gamma(A_1\to\rho\pi)$ is too small. The SW results of Eqs. (28)–(30), provided here for future reference, are based on meson-pole dominance in all channels. If we avoid the use of ρ-pole dominance and instead incorporate t-plane analyticity, we can supply a further test for the value of the SW parameter δ. In addition, a crucial test of the parametrization is the A_1-ρ spin correlation, to which we shall return later.

In Eqs. (16)–(18) we have exposed the isospin and pion-pole structure of the vacuum expectation values given in Eq. (1). The quantities thus defined, F_λ, $F_{\nu\lambda}$, and $F_{\mu\nu\lambda}$, are the fundamental amplitudes in our approach. The Ward identities (10)–(14) impose constraints on them as follows:

$$k_\lambda F_\lambda(q,p)=p^2-q^2, \quad (31)$$

$$k_\lambda F_{\nu\lambda}(q,p)=-F_\pi k_\nu, \quad (32)$$

$$k_\lambda F_{\mu\nu\lambda}(q,p)=\Delta_{\mu\nu}{}^A(q)-\Delta_{\mu\nu}{}^A(p), \quad (33)$$

$$p_\nu F_{\nu\lambda}(q,p)=F_\pi[q_\lambda-F_\lambda(q,p)], \quad (34)$$

$$q_\mu F_{\mu\nu\lambda}(q,p)=\Delta_{\nu\lambda}{}^A(p)-\Delta_{\nu\lambda}{}^V(k)-F_\pi F_{\nu\lambda}(q,p). \quad (35)$$

If we contract p_ν into (35), and use (34) and (15) we get

$$q_\mu p_\nu F_{\mu\nu\lambda}(q,p)=p_\lambda C_V-p_\nu\Delta_{\nu\lambda}{}^V(k) +F_\pi{}^2[F_\lambda(q,p)-Q_\lambda]. \quad (36)$$

Since $W_{\mu\nu\lambda}{}^{abc}(q,p)$ possesses the crossing property

$$W_{\mu\nu\lambda}{}^{abc}(q,p)=-W_{\nu\mu\lambda}{}^{abc}(-p,-q), \quad (37)$$

it follows that

$$F_{\mu\nu\lambda}(q,p)=-F_{\nu\mu\lambda}(-p,-q). \quad (38)$$

Similarly,

$$F_\lambda(q,p)=-F_\lambda(-p,-q). \quad (39)$$

If we replace $q\to-p$, $p\to-q$ in (36) and use (38) and (39), we obtain

$$q_\mu p_\nu F_{\mu\nu\lambda}(q,p)=q_\lambda C_V-q_\nu\Delta_{\nu\lambda}{}^V(k)+F_\pi{}^2[F_\lambda(q,p)-Q_\lambda]. \quad (40)$$

By adding (36) and (40), we get

$$F_\pi{}^2(F_\lambda(q,p)-Q_\lambda)=q_\mu p_\nu F_{\mu\nu\lambda}(q,p)+\frac{1}{2}\int\frac{dx\,\rho_V(x)}{x}\frac{(p^2-q^2)k_\lambda-k^2Q_\lambda}{x+k^2}. \quad (41)$$

Given the assumptions of CVC, PCAC, and current algebra, Eq. (41) is *exact.* The contributions to the Ward identities from incalculable Schwinger terms and σ terms are rigorously absent by virtue of the isospin structure of the three-point functions for chiral $SU(2)$. Divergences appearing in certain models of the three-point functions do not affect the validity of Eq. (41) because regularization procedures can be shown to leave the Ward identities, Eqs. (10)–(14), intact.[10] In addition, only pion poles have been extracted from the matrix elements of currents in Eq. (1). No assumptions about vector or axial-vector current dominance by, e.g., the ρ or A_1 mesons have been made in obtaining Eq. (41). Thus, Eq. (41) is a sound starting point for confronting the basic assumptions of hard-pion theory with experiment, and it is to this task that we now turn our attention.

II. EFFECTIVE-RANGE APPROXIMATION

The attitude we shall take in working with Eq. (41) is motivated by considerations of t-plane analyticity. The last term in (41) is mainfestly an analytic function of $t=-k^2$ with a right-hand cut starting at the $\pi\pi$ threshold, the lowest mass hadronic contribution to ρ_V. The three-point functions F_λ and $F_{\mu\nu\lambda}$ contain form factors which are real analytic functions in the same cut t plane, provided we fix p^2 and q^2 at small enough real values, e.g., $p^2=q^2=-m_\pi{}^2$. Examination of the discontinuity of (41) across the t-plane cut reveals an interesting relation between analytic functions of t, a point we discuss further in the concluding section of the paper. In this section we confine our attention to performing a calculation of the pion form factor. In doing so, we must treat the function $F_{\mu\nu\lambda}$ approximately because our present knowledge of the form factors associated with this term is limited. We can use the SW approximation for $F_{\mu\nu\lambda}$ to obtain an expression having the correct t-plane cut, thus leaving (41) in the form of an integral equation whose solution involves the SW parameter δ.

The SW construction of $F_{\mu\nu\lambda}(q,p)$ presumes that the q^2, p^2, and k^2 dependence of the form factors resides in propagation functions for each of the three currents,

$$F_{\mu\nu\lambda}{}^{\mathrm{SW}}(q,p)=(1/g_A{}^2g_\rho)\Delta_{\mu\tau}{}^A(q)\Delta_{\nu\sigma}{}^A(p)\times\Delta_{\lambda\eta}{}^V(k)\Gamma_{\tau\sigma\eta}(q,p), \quad (42)$$

and that the proper vertex factor $\Gamma_{\tau\sigma\eta}(q,p)$ is at most linear in q,p,

$$\Gamma_{\tau\sigma\eta}(q,p)=\Gamma_1\delta_{\tau\sigma}Q_\eta+\Gamma_2(\delta_{\tau\eta}k_\sigma-\delta_{\sigma\eta}k_\tau)+\Gamma_3(\delta_{\tau\eta}p_\sigma+\delta_{\sigma\eta}q_\tau), \quad (43)$$

in which Γ_1, Γ_2, and Γ_3 are constants. Given this hypothesis, we recover the SW result that Δ^A is pole dominated. To demonstrate this, it suffices to consider $p^2=q^2$, in which case

$$k_\lambda F_{\mu\nu\lambda}{}^{\mathrm{SW}}(q,p)\,|_{p^2=q^2}=(C_VC_A\Gamma_3/g_A{}^2g_\rho)\times[p_\nu p_\tau\Delta_{\mu\tau}{}^A(q)-q_\mu q_\sigma\Delta_{\nu\sigma}{}^A(p)]_{p^2=q^2}, \quad (44)$$

to be compared with the Ward identity (33), which takes the form

$$k_\lambda F_{\mu\nu\lambda}\,|_{p^2=q^2}=\int\frac{dx}{x+p^2}\frac{\rho_A(x)}{x}(q_\mu q_\nu-p_\mu p_\nu). \quad (45)$$

Equations (44) and (45) are mutually consistent only if

$$\int\frac{dx}{x+p^2}\rho_A(x)\left(\frac{1}{x}+\frac{C_VC_A\Gamma_3}{g_A{}^2g_\rho}\right)=0, \quad (46)$$

which implies that $\rho_A(x)$ is localized at some mass $m_A{}^2$, with strength $g_A{}^2$:

$$\rho_A(x)=g_A{}^2\delta(x-m_A{}^2),\quad C_A=g_A{}^2/m_A{}^2, \quad (47)$$

and simultaneously

$$\Gamma_3=-g_\rho/C_V. \quad (48)$$

Insertion of these results into (42) for $p^2\neq q^2$, and comparison with the Ward identity (33), yields

$$\Gamma_1=-\Gamma_3, \quad (49a)$$

with no condition on Γ_2. We adopt the SW parametrization

$$\Gamma_2=\Gamma_1(2+\delta). \quad (49b)$$

We can now solve for $F_{\nu\lambda}$ directly from Eq. (35); we get

$$F_\pi F_{\nu\lambda}=\frac{g_A{}^2}{p^2+m_A{}^2}\left(\delta_{\nu\lambda}+\frac{p_\nu p_\lambda}{m_A{}^2}\right)-\int dx\frac{\rho_V(x)}{k^2+x}\left(\delta_{\nu\lambda}+\frac{k_\nu k_\lambda}{x}\right)-\frac{g_A{}^2}{m_A{}^2C_V}\frac{\Delta_{\lambda\eta}{}^V(k)}{p^2+m_A{}^2}\left(\delta_{\nu\sigma}+\frac{p_\nu p_\sigma}{m_A{}^2}\right)\times[q_\sigma Q_\eta+(2+\delta)(q_\eta k_\sigma-q\cdot k\delta_{\sigma\eta})-q_\eta p_\sigma-\delta_{\sigma\eta}q^2]. \quad (50)$$

This expression for $F_{\nu\lambda}$ satisfies the Ward identity (32). We note that pole dominance of Δ^V is not necessary in this approach. Thus a hybrid treatment of the $A_1\rho\pi$ system is permitted, in which we use pole dominance in the axial-vector current variables p^2 and q^2, but use

[10] K. Wilson, Phys. Rev. 181, 1909 (1969).

t-plane analyticity in the vector current channel. An analogous treatment of $\langle 0|TV_\mu{}^aV_\nu{}^bV_\lambda{}^c|0\rangle$ is not valid because it would violate crossing symmetry. From Eqs. (42)–(43) and (47)–(49) we obtain the following expression to be used in Eq. (41):

$$q_\mu p_\nu F_{\mu\nu\lambda}(q,p)$$
$$=\frac{g_A{}^2}{m_A{}^4C_V}\frac{1+\delta}{2}[Q_\lambda k^2-k_\lambda(p^2-q^2)]\int\frac{dx\,\rho_V(x)}{x+k^2}. \quad (51)$$

We shall take the point of view that the $k^2=0$ value of $\int dx\,\rho_V(x+k^2)^{-1}$ is adequately given by the SW result

$$C_V=g_\rho{}^2/m_\rho{}^2. \quad (52a)$$

If we also use[8]

$$g_A=g_\rho \quad\text{and}\quad m_A{}^2=2m_\rho{}^2, \quad (52b)$$

Eq. (41) becomes

$$F_\pi{}^2[F_\lambda(q,p)-Q_\lambda]=\frac{1}{2}\int\frac{dx\,\rho_V(x)}{x(x+k^2)}$$
$$\times\left(1-\frac{1+\delta}{4}\frac{x}{m_\rho{}^2}\right)[k_\lambda(p^2-q^2)-Q_\lambda k^2]. \quad (53)$$

For $p^2=q^2=-m_\pi{}^2$ we get, for the pion form factor,

$$F(t)=1+\frac{t}{2F_\pi{}^2}\int\frac{dx}{x}\frac{\rho_V(x)}{x-t}\left(1-\frac{1+\delta}{4}\frac{x}{m_\rho{}^2}\right). \quad (54)$$

Our first step in solving Eq. (54) is the specification of ρ_V, the vector spectral function. Since (51) is a low-energy approximation, it is consistent for us to consider only the $\pi\pi$ contribution,

$$\rho_V{}^{\pi\pi}(t)=\frac{1}{6\pi^2}|F(t)|^2\frac{P^3}{\sqrt{t}},\quad t>4m_\pi{}^2 \quad (55)$$

where $P^2=\frac{1}{4}(t-4m_\pi{}^2)$. Equations (54) and (55) imply that, on the $\pi\pi$ cut,

$$\mathrm{Im}F=\frac{1}{a_{11}}|F|^2\frac{P^3}{\sqrt{t}}\left(1-\frac{1+\delta}{4}\frac{t}{m_\rho{}^2}\right), \quad (56)$$

where $a_{11}=12\pi F_\pi{}^2$. Given (56), we can use the inverse amplitude method to solve (54) as follows. Defining $G(t)=F^{-1}(t)$, with $G(0)=F(0)=1$, we have

$$\mathrm{Im}G=-\frac{\mathrm{Im}F}{|F|^2}=-\frac{1}{a_{11}}\frac{P^3}{\sqrt{t}}\left(1-\frac{1+\delta}{4}\frac{t}{m_\rho{}^2}\right) \quad (57)$$

and therefore

$$G(t)=1-\frac{t}{8\pi a_{11}}\int_{4m_\pi{}^2}^{\Lambda}\left(\frac{x-4m_\pi{}^2}{x}\right)^{3/2}$$
$$\times\left(1-\frac{1+\delta}{4}\frac{x}{m_\rho{}^2}\right)\frac{dx}{x-t}, \quad (58)$$

where we use a cutoff Λ, in accordance with our low-energy approximation. Equation (58) is readily integrated, giving the effective-range formula

$$F(t)=a_{11}\left[a_{11}+bt+g(t)\left(1-\frac{1+\delta}{4}\frac{t}{m_\rho{}^2}\right)-g(0)\right]^{-1}, \quad (59)$$

where $8\pi b=-\ln(\Lambda/m_\pi{}^2)$, for large Λ, and

$$g(t)=\frac{2}{\pi}\frac{P^3}{\sqrt{t}}\ln\frac{(\sqrt{t})+2P}{2m_\pi}-\frac{iP^3}{\sqrt{t}},\quad g(0)=-\frac{m_\pi{}^2}{\pi}. \quad (60)$$

Equation (59) contains an effective-range parameter b, which we fix by requiring that the ρ-pole appear at the correct mass,

$$\mathrm{Re}F^{-1}(m_\rho{}^2)=0, \quad (61)$$

from which

$$b=\frac{-1}{m_\rho{}^2}\left(a_{11}+\frac{m_\pi{}^2}{\pi}+\frac{3-\delta}{4}\frac{2}{\pi}\frac{P_\rho{}^3}{m_\rho}L_\rho\right),\qquad L_\rho=\ln\left(\frac{m_\rho+2P_\rho}{2m_\pi}\right). \quad (62)$$

For $\delta=-\frac{1}{2}$, this corresponds to $\Lambda/m_\pi{}^2=e^{17}$. Such a large cutoff, in a model which, as we shall show, is confirmed by every direct experimental test, would imply that the elastic unitarity approximation is a better one physically than one might *a priori* have expected.

We can determine the ρ resonance parameters by examining $F(t)$ near $t=m_\rho{}^2$, where Eq. (59) becomes

$$F(t)=a_{11}/\{-\lambda[m_\rho{}^2-t-i\Gamma m_\rho(P/P_\rho)^3m_\rho/\sqrt{t}]\}, \quad (63)$$

implying a ρ width

$$\Gamma_{\rho\pi\pi}=-\frac{1}{\lambda}\frac{3-\delta}{4}\frac{P_\rho{}^3}{m_\rho{}^2}, \quad (64)$$

with

$$\lambda=a_{11}\left(\frac{d}{dt}\,\mathrm{Re}F^{-1}\right)_{t=m_\rho{}^2}. \quad (65)$$

Numerical values for the ρ width as a function of the parameter δ are given in Table I. Extracting the $\rho\pi\pi$ coupling constant from the ρ width,

$$\Gamma=\frac{2}{3}\frac{g_{\rho\pi\pi}{}^2}{4\pi}\frac{P_\rho{}^3}{m_\rho{}^2} \quad (66)$$

and comparing with (64), we obtain, within 5%, the KSRF relation,[11–13] modified by the factor $\frac{1}{4}(3-\delta)$,

$$2F_\pi{}^2=\tfrac{1}{4}(3-\delta)m_\rho{}^2/g_{\rho\pi\pi}{}^2. \quad (67)$$

[11] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

[12] The KSRF relation cannot be proven from current algebra alone: D. Geffen, Phys. Rev. Letters **19**, 770 (1967); S. G. Brown and G. B. West, *ibid.* **19**, 812 (1967).

[13] The KSRF relation has been obtained using current algebra and an effective-range formula for the $T=J=1$ $\pi\pi$ phase shift: L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 346 (1968).

TABLE I. Parameters of the $A_1\rho\pi$ system as a function of δ. Entry (b), $|g_1/g_0|$, is the ratio of helicity coupling constants, defined in Sec. IV, for the decay $A_1 \to \rho\pi$. Other notation is self-explanatory.

	δ = 0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$
(a) $\Gamma_{A_1\rho\pi}$ (MeV)	250.0	116.0	48.0	16.0
(b) $\|g_1/g_0\|$	0.95	0.87	0.76	0.55
(c) $\Gamma_{\rho\pi\pi}$ (MeV)	102.0	124.0	146.0	170.0
(d) r_π(F)	0.627	0.635	0.643	0.651
(e) $\|F(m_\rho^2)\|^2$	57.2	42.0	32.2	25.4
(f) $\Gamma(\rho\to e^+e^-)/\Gamma(\rho\to\pi^+\pi^-)$	6.8×10^{-5}	5.0×10^{-5}	3.8×10^{-5}	3.0×10^{-5}
(g) $[(P^3/\sqrt{t})\cot\delta_{11}]_{P=0}(m_\pi^2)$	15.2	14.9	14.6	14.3

In Fig. 1, we plot the colliding-beam data[14] for $|F(t)|^2$ versus t. The maximum value that the quantity $|F(t)|^2$ attains near the ρ pole depends sensitively upon the SW parameter δ. Comparison with theoretical values given in Table I tends to favor values of $\delta \geq -\frac{1}{2}$. Since the approximations made in obtaining Eq. (59) become less valid for $t > m_\rho^2$, we shall not look for any significance of the results in this region.

There exist data in the spacelike region from electroproduction experiments[15]; in Ref. 5 we have compared our results with these data. The pion charge radius r_π is determined from the form factor derivative at $t=0$,

$$r_\pi^2 = 6\left.\frac{dF}{dt}\right|_{t=0} = \frac{6}{m_\rho^2}\left\{1 - \frac{m_\rho^2}{\pi a_{11}} \times\left[\frac{1}{3} - \frac{3-\delta}{4m_\rho^2}\left(m_\pi^2 + \frac{2P_\rho^3}{m_\rho}L_\rho\right)\right]\right\}. \quad (68)$$

Numerical results are given in Table I, and are comparable in all cases to the ρ-dominance value $r_\pi = (6/m_\rho^2)^{1/2} = 0.64$ F.

A representation for the pion form factor with the correct $\pi\pi$ branch cut has previously been given by Gounaris and Sakurai,[16] who write

$$F(t) = f(0)/f(t)\,, \quad f(t) = [\cot\delta_{11}(t) - i]P^3/\sqrt{t}\,, \quad (69)$$

where δ_{11}, the $T=J=1$ $\pi\pi$ phase shift, is given by an effective-range formula, and serves as input to their calculation. In contrast to this, our method of obtaining $F(t)$ avoids the use of δ_{11} as input. In fact, we can *predict* δ_{11} by using unitarity. In the $\pi\pi$ region, we have

$$\operatorname{Im}F = F^* e^{i\delta_{11}}\sin\delta_{11}\,, \quad (70)$$

and comparing this with Eq. (56), we conclude that

$$\cot\delta_{11} = \frac{\sqrt{t}}{P^3}\left(1 - \frac{1+\delta}{4}\frac{t}{m_\rho^2}\right)^{-1} a_{11}\operatorname{Re}F^{-1}. \quad (71)$$

A comparison of our expression for δ_{11} with that of Arnowitt *et al.*[17] and of Brown and Goble[13] has been presented in Ref. 5. The p-wave scattering length is obtained from

$$\left(\frac{P^3}{\sqrt{t}}\cot\delta_{11}\right)_{t=4m_\pi^2} = \left[a_{11} + \frac{m_\pi^2}{\pi}\left(1 - \frac{3-\delta}{2}\frac{P_\rho}{m_\rho}L_\rho\right)\right] \times\left(1 + \frac{3-\delta}{4}\frac{m_\pi^2}{P_\rho^2}\right)^{-1}. \quad (72)$$

Numerical values of this, given in Table I, are to be compared with the soft-pion value,[18] $a_{11} = 17m_\pi^2$,

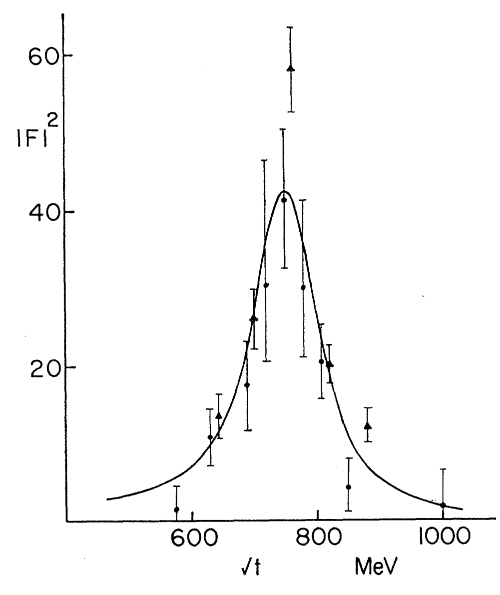

FIG. 1. Comparison of the pion form factor, Eq. (59), with experimental data taken from Ref. 14. Closed circles refer to the data of Auslander *et al.*; closed triangles to Augustin *et al.* The solid curve is calculated with $\delta = -\frac{1}{2}$.

[14] V. L. Auslander *et al.*, Phys. Letters **25B**, 433 (1967); and in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968); J. E. Augustin *et al.*, Phys. Letters **28B**, 508 (1969).

[15] C. W. Akerlof *et al.*, Phys. Rev. **163**, 1482 (1967); C. Mistretta *et al.*, Phys. Rev. Letters **20**, 1523 (1968).

[16] G. J. Gounaris and J. J. Sakurai, Phys. Rev. Letters **21**, 244 (1968); see also M. Parkinson, Phys. Rev. D **1**, 368 (1970).

[17] R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. **175**, 1820 (1968).

[18] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966); see also Ref. 13.

Olsson's result,[19] $(15.0\pm1.2)m_\pi^2$, deduced from a forward-dispersion sum rule, and with the hard-pion result of Ref. 17, $14.5m_\pi^2$.

The main result of this section is our effective-range formula for the pion form factor, Eq. (59). The calculation was based on the $SU(2)\times SU(2)$ algebra of currents and PCAC. It has recently been proposed[20] that our world is mathematically near one in which the pion mass vanishes, $m_\pi=0$, and $SU(2)\times SU(2)$ is an exact symmetry of the Hamiltonian. If so, our formula for $F(t)$ should be expected to make sense in the limit $m_\pi\to0$. However, because of the presence of terms proportional to $\ln m_\pi$ [which arise from our approximation of the exact relation (41)], the limit $m_\pi\to0$ is, in general, singular. The only exception to this is the case $\delta=-1$ for which the singular terms cancel, and we find

$$[F(t)]_{m_\pi=0}=a_{11}\left[a_{11}\left(1-\frac{t}{m_\rho^2}\right)+\frac{P^2}{\pi}\ln\frac{P}{P_\rho}-\tfrac{1}{2}iP^2\right]^{-1}. \quad (73)$$

The value $\delta=-1$ corresponds in the SW approximation to $q_\mu p_\nu F_{\mu\nu\lambda}=0$, which suggests that at least for $|t|\lesssim m_\rho^2$, $q_\mu p_\nu F_{\mu\nu\lambda}$ is in reality quite small. This may explain in part the success of our calculation, despite a limited knowledge of this term.

Having obtained a reasonable representation of the pion form factor for energies at and below the ρ pole, it is natural to consider an extension of the calculation to higher energies. In doing so, it is important to take into account explicitly additional intermediate states. This leads us into the multichannel approach discussed in Sec. III. Before turning to this, we note that, once $F(t)$ is determined then so are the A_{l3} form factors $A(t)$ and $B(t)$. To establish this we must obtain equations for the invariant functions associated with $F_{\nu\lambda}(q,p)$ in its A_1-dominated form (24). Inserting this relation into (50) and taking $p^2=-m_A^2$, $q^2=-m_\pi^2$, and $k^2=-t$, we find

$$A=\frac{g_\rho}{F_\pi}\left(1+\frac{1}{4g_\rho^2}\int\frac{dx}{x-t}\rho_V(x)\times[(2+\delta)(t-m_A^2)+\delta m_\pi^2]\right),$$

$$B=-\frac{\delta}{2F_\pi g_\rho}\int\frac{dx}{x-t}\rho_V(x), \quad (74)$$

$$C=\frac{1}{2F_\pi g_\rho}\int\frac{dx}{x-t}\rho_V(x)\left(1+\delta-\frac{m_A^2}{x}\right).$$

The functions D and E do not contribute on shell. In obtaining (74) we have used Eqs. (52). Equation (22) is satisfied by Eqs. (74), so that only A and B are independent on shell. The basic analytic function of t occurring in F, A, and B is the integral $\int dx\,\rho_V(x)/(x-t)$. We can show that, by virtue of (52a),

$$\int\frac{dx\,\rho_V}{x-t}=2F_\pi^2F(t)\left(1-\frac{1+\delta}{4}\frac{t}{m_\rho^2}\right)^{-1}, \quad (75)$$

so that formulas for A and B follow from the one for F, Eq. (59). For the multichannel problem, it is no longer the case that F can be obtained, and then A and B determined from it; there the on-shell form factors are coupled and must be solved for simultaneously.

III. COUPLED FORM-FACTOR PROBLEM

Two approximations have been employed so far. The first was the use of the SW form for $F_{\mu\nu\lambda}$, Eqs. (42) and (43); from this, Eqs. (54) and (74) followed. These amount to integral representations for the form factors with analytic properties dictated by the structure of the spectral function ρ_V. The contributions to ρ_V from the more massive intermediate states correspond to cuts whose branch points are further removed from the effective-range region. The second approximation of Sec. II was to calculate ρ_V using only the $\pi\pi$ contribution. We can easily do better without needing more equations than (54) and (74). In particular, if ρ_V contains the contributions from both $\pi\pi$ and πA_1 intermediate states, then (54) and (74), together with the truncated formula for ρ_V, provide a coupled system of equations for the form factors F, A, and B.

The vector spectral function is given by

$$\langle0|V_\mu^a(x)V_\nu^b(0)|0\rangle=\delta_{ab}(2\pi)^{-3}\int d^4k\,e^{ikx}\theta(k)\rho_V(-k^2)(\delta_{\mu\nu}-k_\mu k_\nu/k^2). \quad (76)$$

With just $\pi\pi$ intermediate states we obtained Eq. (55). If πA_1 states are also included, we find

$$\pi\rho_V=\frac{\theta_1}{6\pi}\frac{P_1^3}{\sqrt{t}}|F|^2+\frac{\theta_2}{4\pi}\frac{P_2}{\sqrt{t}}\times\left[|A|^2+\tfrac{1}{3}P_2^2\left(\frac{1}{m_A^2}|A+p\cdot kB|^2-t|B|^2\right)\right], \quad (77)$$

where θ_1 and θ_2 are θ functions referring to the $\pi\pi$ and πA_1 thresholds, respectively. The momenta are defined by

$$P_1^2=\tfrac{1}{4}(t-4m_\pi^2),$$
$$P_2^2=[t-(m_A+m_\pi)^2][t-(m_A-m_\pi)^2]/4t,$$

and

$$p\cdot k=-\tfrac{1}{2}(t+m_A^2-m_\pi^2).$$

[19] M. G. Olsson, Phys. Rev. 162, 1338 (1967), and unpublished. We have converted Olsson's number to conform to our Eq. (63).

[20] M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968); R. Dashen, *ibid.* 183, 1245 (1969), and references therein.

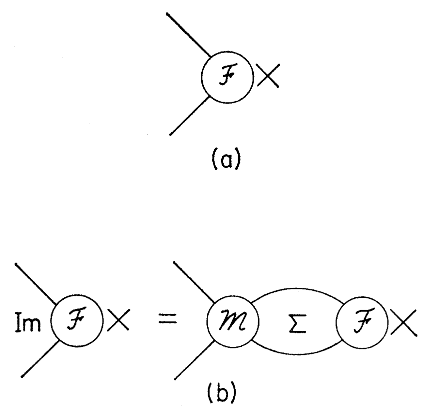

FIG. 2. Diagrammatic representations of (a) form factor $\mathfrak{F}$, (b) Im $\mathfrak{F}$, as calculated from the discontinuity formula implied by unitarity. In (b) $\sum$ represents a sum over allowed intermediate states, and $\mathfrak{M}$ denotes the set of relevant partial-wave scattering amplitudes.

Assembling F, A, and B into a row matrix

$$\mathfrak{F}=(F\,A\,B)\,,$$

we see that

$$\rho_V=(1/\pi)\mathfrak{F}\rho\mathfrak{F}^\dagger\,, \tag{78}$$

where ρ is a square matrix with nonvanishing elements:

$$\rho_{FF}=\frac{\theta_1}{6\pi}\frac{P_1{}^3}{\sqrt{t}}\,,$$

$$\rho_{AA}=\frac{\theta_2}{4\pi}\frac{P_2}{\sqrt{t}}\left(1+\frac{P_2{}^2}{3m_A{}^2}\right),$$

$$\rho_{AB}=\rho_{BA}=\frac{\theta_2}{4\pi}\frac{P_2{}^3}{3(\sqrt{t})m_A{}^2}p\cdot k\,,$$

$$\rho_{BB}=\frac{\theta_2}{4\pi}\frac{P_2{}^3}{3(\sqrt{t})m_A{}^2}[(p\cdot k)^2-tm_A{}^2]=\frac{\theta_2}{4\pi}\frac{(\sqrt{t})P_2{}^5}{3m_A{}^2}\,.$$

From (54) and (74) we conclude that

$$\mathrm{Im}\mathfrak{F}=Z\pi\rho_V\,, \tag{79}$$

where Z is the row matrix

$$\begin{aligned} &Z=(Z_F\,Z_A\,Z_B)\,,\\ &Z_F=\frac{1}{2F_\pi{}^2}\left(1-\frac{(1+\delta)}{4}\frac{t}{m_\rho{}^2}\right),\\ &Z_A=(1/4F_\pi g_\rho)[(2+\delta)(t-m_A{}^2)+\delta m_\pi{}^2]\,,\\ &Z_B=-\delta/2F_\pi g_\rho\,. \end{aligned} \tag{80}$$

Our definition of the form factors A and B is such that the phase-space matrix ρ in (78) is not diagonal. While this is no disadvantage in seeking the solution of the coupled equations, it is worth seeing how the form factors may be transformed to diagonalize ρ. If we define new A_{l3} form factors $\bar{G}_T$ and $\bar{G}_L$ by

$$\begin{aligned} &\bar{G}_T=m_A{}^2A\,,\\ &\bar{G}_L=tP_2{}^2B+p\cdot kA\,, \end{aligned} \tag{81}$$

then we get

$$\pi\rho_V=\frac{\theta_1}{6\pi}\frac{P_1{}^3}{\sqrt{t}}|F|^2+\frac{\theta_2}{12\pi}\frac{P_2}{m_A{}^2t^{3/2}}\left(|\bar{G}_L|^2+2\frac{t}{m_A{}^2}|\bar{G}_T|^2\right). \tag{82}$$

To describe the decay $A_1\to\rho\pi$, we can define coupling constants[21]

$$\begin{aligned} &\bar{g}_T=\left(\frac{m_\rho{}^2-t}{g_\rho}\bar{G}_T\right)_{t=m_\rho{}^2},\\ &\bar{g}_L=\left(\frac{m_\rho{}^2-t}{g_\rho}\bar{G}_L\right)_{t=m_\rho{}^2} \end{aligned} \tag{81'}$$

in terms of which Eq. (27b) becomes

$$\Gamma(A_1\to\rho\pi)=\frac{|\mathbf{q}|}{12\pi m_A{}^4m_\rho{}^2}\left(2\frac{m_\rho{}^2}{m_A{}^2}|\bar{g}_T|^2+|\bar{g}_L|^2\right).$$

Equation (79) is the basic equation of our approximate method for calculating form factors analytic in the cut t plane. The equation reads

$$\mathrm{Im}\mathfrak{F}_i=Z_i(\mathfrak{F}\rho\mathfrak{F}^\dagger)\,. \tag{83}$$

In this paper we shall give only a brief discussion of our attempt to obtain its solution. Quite apart from the problem of how F, A, and B are to be determined from (83), we view the equation as being of some interest as the extension of Sec. II to the problem of coupled form factors. The novelty of this form-factor method is easily visualized in terms of diagrams. Conventional dispersion theory based on unitarity employs a discontinuity formula which can be illustrated in the familiar way as in Fig. 2. If Fig. 2(a) represents the matrix element $\langle\alpha|V_\nu{}^a(0)|\beta\rangle$, then Fig. 2(b) illustrates the prescription for the imaginary part of its analytic continuation to the region $t>4m_\pi{}^2$. Physical intermediate states contribute to the sum $\sum$ and a calculation of the form

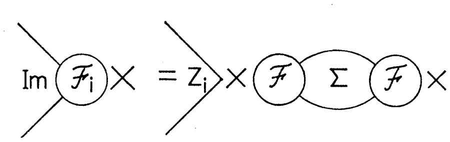

FIG. 3. Diagrammatic representation of Eq. (83), giving an approximate procedure for calculating Im $\mathfrak{F}_i$ in the general problem of coupled form factors, $\mathfrak{F}_i$. The sum $\sum$ denotes contributions to ρ_V from those two-particle states occurring in the coupled-channel problem, and Z_i are known polynomials.

[21] These are proportional to the $A_1\rho\pi$ couplings of F. J. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968); see also S. G. Brown and G. B. West, *ibid.* **180**, 1613 (1969).

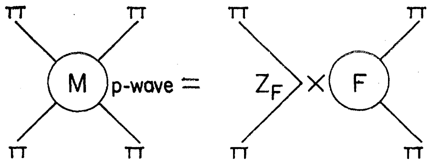

FIG. 4. Method for calculating the p-wave $\pi\pi$ scattering amplitude M in the effective-range approximation from knowledge of the pion form factor F and a known polynomial Z_F. See Eq. (71).

factors requires knowledge of the multichannel partial-wave scattering amplitudes. Diagrams of a different sort suggest themselves to represent Eq. (83); these are as shown in Fig. 3. This method also employs an intermediate-state sum, but one in which the states enter as they contribute to ρ_V. The factor Z_i appearing in Fig. 3 is a known polynomial in t. The method illustrated in Fig. 3 is self-contained, albeit approximate. Knowledge of the partial-wave amplitudes is not needed in solving for the form factors. If (83) is used to obtain the form factors, then the method shown in Fig. 2 may be used to solve for the partial waves. This procedure has already been employed by us to calculate the $\pi\pi$ p-wave amplitude, in effective-range approximation. The result (71) is graphically displayed in Fig. 4. The Z factors appearing in (83) are readily interpreted. If we use $\rho_V = g_\rho^2\delta(m_\rho^2 - t)$ in Eqs. (54) and (74), we obtain the SW form factors, which may be written

$$F^{\mathrm{SW}} = \frac{g_\rho^2}{m_\rho^2 - t} Z_F\,,$$

$$A^{\mathrm{SW}} = \frac{g_\rho}{F_\pi} + \frac{g_\rho^2}{m_\rho^2 - t} Z_A\,, \tag{84}$$

$$B^{\mathrm{SW}} = \frac{g_\rho^2}{m_\rho^2 - t} Z_B\,.$$

Thus we identify the Z factors as the polynomials in t which multiply $g_\rho^2/(m_\rho^2 - t)$ in the SW formulas for the form factors.

It is not difficult to find functions $\mathfrak{F}_i$ whose imaginary parts are given by (83). Let $G(t)$ have the t-plane cuts and set

$$\mathfrak{F}_i = Z_i/G\,. \tag{85}$$

Then

$$\mathrm{Im}\mathfrak{F}_i = -Z_i(\mathrm{Im}G)/|G|^2\,,$$

so that, with (79), we find

$$\begin{aligned}\mathrm{Im}G &= -\pi\rho_V|G|^2\\ &= -Z\rho Z^T \equiv -R\,,\end{aligned}$$

where

$$R = (2F_\pi^2 a_{11})^{-1}(\theta_1 P_1 t^{-\frac{1}{2}} f_1 + \tfrac{3}{2}\theta_2 P_2 t^{-3/2} f_2)$$

and f_1 and f_2 are the entire functions

$$\begin{aligned} f_1 &= P_1^2[1 - \tfrac{1}{4}(1+\delta)t/m_\rho^2]^2\,,\\ f_2 &= (t/m_A^2)[(1+\delta)t - m_A^2 - \tfrac{1}{2}\delta(t + m_A^2 - m_\pi^2)]^2\\ &\quad + (P_2^2 t/3m_A^4)\{[(1+\delta)t - m_A^2]^2 - \delta^2 t m_A^2\}\,.\end{aligned}$$

Because $F(0) = 1$, we must impose the condition

$$G(0) = Z_F(0) = 1/2F_\pi^2\,.$$

A function $G(t)$ with these properties and one effective-range parameter c may be written

$$G = \frac{1}{2F_\pi^2}\left[1 + \frac{c}{a_{11}}t + \frac{1}{a_{11}}(f_1 g_1 + \tfrac{3}{2} f_2 g_2)\right],$$

where

$$g_1 = \frac{P_1}{\sqrt{t}}\left[\frac{2}{\pi}\ln\frac{(\sqrt{t}) + 2P_1}{2m_\pi} - i\right] - \frac{1}{\pi}\,,$$

$$\begin{aligned} g_2 &= \frac{(P_2^2 t)^{1/2}}{t^2}\left[\frac{1}{\pi}\ln\frac{t - m_A^2 - m_\pi^2 + 2(P_2^2 t)^{1/2}}{2m_A m_\pi} - i\right]\\ &\quad + \frac{1}{2\pi t^2}\left[\left(m_A^2 - m_\pi^2 - t\frac{m_A^2 + m_\pi^2}{m_A^2 - m_\pi^2}\right)\ln\left(\frac{m_A}{m_\pi}\right) - t\right]\\ &\quad + \frac{m_A^4 - m_\pi^4 - 4m_A^2 m_\pi^2 \ln(m_A/m_\pi)}{4\pi(m_A^2 - m_\pi^2)^3}\,,\end{aligned}$$

with $g_1(0) = g_2(0) = 0$, and the appropriate continuation to $t < (m_A + m_\pi)^2$ being understood. From (85), we have

$$F = a_{11}\frac{1 - \frac{1}{4}(1+\delta)t/m_\rho^2}{a_{11} + ct + f_1 g_1 + \frac{3}{2} f_2 g_2}\,, \tag{86}$$

$$A = \frac{g_\rho}{F_\pi}\frac{a_{11}}{2m_A^2}\frac{(2+\delta)(t - m_A^2) + \delta m_\pi^2}{a_{11} + ct + f_1 g_1 + \frac{3}{2} f_2 g_2}\,, \tag{87}$$

$$B = -\frac{g_\rho}{F_\pi}\frac{a_{11}}{m_A^2}\frac{\delta}{a_{11} + ct + f_1 g_1 + \frac{3}{2} f_2 g_2}\,. \tag{88}$$

The procedure leading to relations (86)–(88) is unfortunately inadequate because it does not give $A(0)$ correctly. The quantities $A(0)$, $B(0)$ are already implicitly contained in (74), since from Eqs. (15) and (52) we infer $\int dx\,\rho_V(x)/x = 2F_\pi^2$, and thus from (74),

$$\begin{aligned} A(0) &= -(\delta g_\rho/2F_\pi m_A^2)(m_A^2 - m_\pi^2)\,,\\ B(0) &= -\delta g_\rho/F_\pi m_A^2\,,\end{aligned}$$

clearly inconsistent with (87). We can improve on this by writing

$$A = \frac{g_\rho}{F_\pi}\frac{a_{11}}{2m_A^2}\left(\frac{(2+\delta)(t - m_A^2) + \delta m_\pi^2}{a_{11} + ct + f_1 g_1 + \frac{3}{2} f_2 g_2} + \frac{2m_A^2}{a_{11}}\right). \tag{87$'$}$$

The form factors F, A, and B given in Eqs. (86), (87′), and (88) then have the correct values at $t = 0$ and appropriate branch points at $t = 4m_\pi^2$, $(m_A + m_\pi)^2$, but with the correct imaginary part only for $t<(m_A+m_\pi)^2$. In view of the latter, it is not clear that, although Eq. (83) describes the pion form factor with more of its dynamics than does Eq. (56), the provisional coupled-channel result (86) is any better than the single-channel result (59). Because of this, we shall eschew giving any further numerical results.

CONCLUSION

The basic aim of this paper has been to perform a hard-pion study of the $A_1\rho\pi$ system which adheres to the requirements of t-plane analyticity and unitarity. Our most important quantitative approximation was to adopt the SW construction of the pion-pole-free part of $W_{\mu\nu\lambda}{}^{abc}(q,p)$. As a consequence, we have been able to obtain effective-range formulas for form factors with the correct cut structure, but in terms of the SW parameter, δ. We devote the first part of this section to a comparison between our results and experiment and, in particular, assess the extent to which the δ parametrization is successful. Table I gives the relevant physical quantities for several values of δ.

(a) $\Gamma(A_1\to\rho\pi)$ is evaluated from Eqs. (25), (27b), (74), and (75). Our $A_1\rho\pi$ coupling constants are numerically related to those of SW, Eq. (28), by a δ-dependent factor which differs little from unity:

$$g,\, h=g^{\mathrm{SW}},\, h^{\mathrm{SW}}\left(\frac{4}{3-\delta}\frac{1.07}{1.16+0.07\delta}\right).$$

The widths in Table I are to be compared with the current experimental value 80 ± 35 MeV.[22]

(b) $|g_1/g_0|$ pertains to the spin structure of the decay $A_1\to\rho\pi$. The transition matrix element may be written in general as

$$\langle\pi(qa)\rho(kcj)|A_1(pbi)\rangle = \frac{-i(2\pi)^4\delta(p-k-q)}{(8\omega_k\omega_p\omega_q)^{1/2}}\epsilon_{abc}g_jD_{ij}{}^{(1)*}(\phi,\theta,0)\,,$$

where (θ,ϕ) is the direction of $\mathbf{k}$, the ρ-meson momentum in the A_1 rest frame. In terms of quantities already defined in (81′), the helicity coupling constants are

$$g_1=g_{-1}=\bar{g}_T/m_A{}^2 \quad\text{and}\quad g_0=-\bar{g}_L/m_Am_\rho.$$

The ratio $|g_1/g_0|$ can be measured from the decay distribution of a polarized A_1. The two recent determinations[23] of $|g_1/g_0|$ are in conflict. Our formula, identical to the SW result in this case, takes a particularly simple form if we make the approximation $m_\pi=0$; then

$$|g_1/g_0|=\sqrt{2}(2+\delta)/(3+\delta)\,,$$

which clearly shows how the Gilman-Harari prediction[21] $(g_1\simeq0)$ corresponds to $\delta\simeq-2$.

(c) $\Gamma(\rho\to\pi\pi)$ is calculated from Eq. (64), which we derived from our effective-range formula for the pion form factor, Eq. (59), and is to be compared with the measurements in the colliding-beam experiments.[14] Roos and Pisut[24] have analyzed and parametrized these data and give

$$\Gamma(\rho\to\pi\pi)=122_{-6}^{+7}\text{ MeV}.$$

(d) We calculate the pion charge radius from (68). Electroproduction experiments[15] imply $r_\pi=0.86\pm0.14$ F (Harvard) and 0.80 ± 0.10 F (Cornell). The analysis of Ref. 24 gives $r_\pi=0.7$ F. Our calculations differ but little from the ρ-dominance value over a wide range of choices for δ.

(e) The peak value of $|F(t)|^2$ may be compared with the data[14] shown in Fig. 1. The sensitivity of this parameter to δ is evident from Table I. The data from Orsay and from Novosibirsk imply $\delta\gtrsim-\frac{1}{2}$; the peak values in the two experiments differ appreciably. A determination of δ from $|F(m_\rho{}^2)|^2$ cannot be made more precise, given this discrepancy, and we urge further intensive experimental investigation of the ρ region.

(f) Closely related to entry (e) is the determination of the branching ratio $\Gamma(\rho\to e^+e^-)/\Gamma(\rho\to\pi^+\pi^-)$. The colliding-beam cross section at the ρ mass is

$$\sigma(e^+e^-\to\pi^+\pi^-)|_\rho=\frac{8\pi\alpha^2}{3}\frac{P_\rho{}^2}{m_\rho{}^5}|F(m_\rho{}^2)|^2.$$

The branching ratio is given by[16]

$$\mathrm{BR}=m_\rho{}^2\sigma/12\pi\,,$$

so that our form factor leads to the expression

$$\mathrm{BR}=\frac{2}{9}\alpha^2\frac{a_{11}{}^2}{P_\rho{}^3m_\rho}\left(\frac{4}{3-\delta}\right)^2.$$

The tabulation may be compared with the Orsay result[14] $\mathrm{BR}=(6.56\pm0.72)\times10^{-5}$; as is the case with the peak value, the Novosibirsk result is some 25% smaller. Reference 22 quotes an average value $\mathrm{BR}=6.0\times10^{-5}$.

(g) The $\pi\pi$ p-wave scattering length is obtained from Eq. (72) and tabulated in the form of $(P^3/\sqrt{t})\cot\delta_{11}$. The most one can do by way of comparison with data is to refer to Olsson's result,[19] $(15\pm1.2)m_\pi{}^2$, which is deduced from a forward dispersion sum rule having the ρ parameters as input.

The original choice of $\delta\simeq-\frac{1}{2}$ used by SW was made to fit entries (a) and (c). Our tabulation indicates that this value of δ provides a reasonable picture of all listed parameters of the $A_1\rho\pi$ system, with the possible exception of entry (b). Here, the result of Ballam *et al.*[23] demands $\delta\simeq-\frac{3}{2}$, while that of Crennell *et al.*[23] is in excellent agreement with $\delta\simeq-\frac{1}{2}$. The ρ branching

[22] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **41**, 109 (1969).

[23] J. Ballam *et al.*, Phys. Rev. Letters **21**, 934 (1968); Phys. Rev. D **1**, 94 (1970); their result is $|g_1/g_0|=0.48\pm0.13$. D. J. Crennell *et al.*, Phys. Rev. Letters **24**, 781 (1970); their result is $|g_1/g_0|=0.89_{-0.06}^{+0.07}$.

[24] M. Roos and J. Pisut, Nucl. Phys. **B10**, 563 (1969).

ratio, entry (f), is also sensitive to δ and presently demands $\delta \gtrsim -\frac{1}{2}$. The peak value of $|F|^2$ (or, equivalently, the branching ratio) is directly obtainable from the colliding-beam cross section, and leads to the cleanest determination of δ. On the other hand, $|g_1/g_0|$ is extracted from A_1 production data via a more difficult analysis entailing an A_1 production-plus-background hypothesis. If this quantity could be obtained from some other A_1 production experiment, such as $\bar{p}p$ annihilation into πA_1, and were to confirm the result of Ballam *et al.*,[23] then it would constitute strong evidence that the SW parametrization is inadequate and in need of serious modification. Along these lines, Brown and West[25] have used the Bjorken limit[26] and the algebra of fields[27] in a modified pole-dominated hard-pion analysis of the $A_1\rho\pi$ system. This analysis contains two parameters and allows greater freedom in fitting the data. However, if it is the result of Crennell *et al.*[23] that survives future experimental tests, then the SW parametrization would be entirely adequate. The calculations presented in this paper would then stand, without exception, in good agreement with a wide range of experimental results.

Analytic methods have been applied previously to problems in current algebra. For example, there is the work of Amatya, Pagnamenta, and Renner[28] in which σ-pole dominance of the two-point function $\langle 0|T\sigma(x)\sigma(0)|0\rangle$ is replaced by the continuum contribution from $\pi\pi$ states in order to learn something about $T=J=0$ $\pi\pi$ scattering. Also, there exists a large body of literature[29] on the unitarization of soft-pion calculations, the spirit of which differs considerably from that of the work described here. In this paper, special emphasis has been placed on the *consistent* treatment of vector current matrix elements with analytic methods. We believe that the calculation of the p-wave $\pi\pi$ phase shift from the effective-range formula for the pion form factor (see Fig. 4) is of particular interest, although we hasten to emphasize that this result is meaningful only within the effective-range approximation. Clearly the p-wave amplitude cannot be strictly proportional to $F(t)$ because the former has left-hand cuts whereas the latter does not. However, the existence of such an approximate relation between $\cot\delta_{11}$ and $F(t)$ suggests that the unitarity constraint can be employed in conjunction with the Ward identities of current algebra to generate relations of a more general nature. We shall now show that this is the case.

We begin by returning to Eq. (41), rewritten for $p^2=q^2=-m_\pi^2$ as follows:

$$F_\pi^2[F_\lambda(q,p)-Q_\lambda]-q_\mu p_\nu F_{\mu\nu\lambda}(q,p)=\tfrac{1}{2}tQ_\lambda\int\frac{dx}{x}\frac{\rho_V(x)}{x-t}. \quad (41')$$

As already noted, F_λ and $F_{\mu\nu\lambda}$ may be expressed in terms of form factors which are analytic in the cut t plane. Thus we may view (41′) as a relation among analytic functions. Considering the discontinuity across the cut for $(2m_\pi)^2<t<(4m_\pi)^2$ where only the two-pion intermediate state contributes, we have from Eqs. (1), (16), (18), and (41′)

$$\begin{aligned}\mathrm{disc}_t[i\epsilon_{abc}F_\pi^2F_\lambda]_{p^2=q^2=-m_\pi^2}&=\sum_n (2\pi)^4\delta(k-P_n)\langle 0|V_\lambda^c(0)|n\rangle\\&\times\left[\frac{(m_\pi^2+p^2)(m_\pi^2+q^2)}{m_\pi^4}\int dz\, e^{-iqz}\right.\\&\left.\times\langle n|T\partial_\mu A_\mu^a(z)\partial_\nu A_\nu^b(0)|0\rangle\right]_{p^2=q^2=-m_\pi^2}, \quad (89)\end{aligned}$$

$$\begin{aligned}\mathrm{disc}_t[i\epsilon_{abc}q_\mu p_\nu F_{\mu\nu\lambda}(q,p)]_{p^2=q^2=-m_\pi^2}&=\mathrm{disc}_t[q_\mu p_\nu W_{\mu\nu\lambda}{}^{abc}(q,p)]'_{p^2=q^2=-m_\pi^2}\\&=\sum_n (2\pi)^4\delta(k-P_n)\langle 0|V_\lambda^c(0)|n\rangle\\&\times\left[\int dz\, e^{-iqz}q_\mu p_\nu\langle n|TA_\mu^a(z)A_\nu^b(0)|0\rangle\right]'_{p^2=q^2=-m_\pi^2}, \quad (90)\end{aligned}$$

in which

$$|n\rangle=|\pi(p_1d_1)\pi(p_2d_2)\rangle,$$
$$P_n=p_1+p_2,$$
$$\sum_n=\frac{1}{(2\pi)^6}\int d^3p_1d^3p_2\sum_{d_1d_2}.$$

The primes in Eq. (90) denote contributions containing no pion poles in p^2 and q^2; i.e., $F_{\mu\nu\lambda}$ is defined in terms of that part of $W_{\mu\nu\lambda}{}^{abc}$ which remains when all pion poles in p^2 and q^2 are removed [see Eq. (18)], and the term in (90) which contains the matrix element of two axial-vector currents is analogously defined. Note that the factor $q_\mu p_\nu$ in (90) prevents the appearance of A_1 poles in p^2 and q^2 if we were to A_1-dominate the matrix elements. The vector current matrix element contains $F^*(t)$:

$$\langle 0|V_\lambda^c(0)|\pi(p_1d_1)\pi(p_2d_2)\rangle=(p_1-p_2)_\lambda(4\omega_1\omega_2)^{-1/2}F^*i\epsilon_{cd_1d_2}.$$

The integration over the two-pion phase space projects the $T=J=1$ partial waves of the four-point functions in square brackets in (89) and (90); let these be denoted by $M_{\pi\pi}{}^{11}(t)$ and $M_{\pi A}{}^{11}(t)$. The discontinuity of

[25] S. G. Brown and G. B. West, Ref. 21; see also P. Horwitz and P. Roy, Phys. Rev. 180, 1430 (1969).
[26] J. D. Bjorken, Phys. Rev. 148, 1467 (1966).
[27] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).
[28] A. Amatya, A. Pagnamenta, and B. Renner, Phys. Rev. 172, 1755 (1968).
[29] See Ref. 3 of Ref. 28.

the right-hand side of (41′) is given by $\rho_V^{\pi\pi}$, Eq. (55). Since F^* appears in the discontinuity of all three terms of (41′), we have the result that $M_{\pi\pi}^{11}(t)$, $M_{\pi A}^{11}(t)$, and $F(t)$ are *linearly related* in the region $(2m_\pi)^2<t<(4m_\pi)^2$. Since this is a relation among analytic functions of t, it holds throughout their common domain of analyticity. This result cannot be obtained in any way from conventional dispersion theory alone. Given the usual hypotheses of current algebra, it is an *exact* result, obtained from three-point function Ward identities and unitarity. A similar relation can also be deduced in a current algebraic analysis of four-point functions.[30] In this case the relation is among $M_{\pi\pi}^{T=1}(s,t)$, $M_{\pi A}^{T=1}(s,t)$, where $s=-(q+p_2)^2$, and $F(t)$, and the soft-pion limit, p or $q\to 0$, is taken to eliminate $M_{\pi A}$. The Veneziano representation,[31] used off the pion mass shell, for $M_{\pi\pi}$ then in principle provides a determination of $F(t)$. Our relation gives $F(t)$ in terms of partial-wave amplitudes. If we wished to invoke A_1 dominance in p^2 and q^2 for $M_{\pi A}^{11}$, we would have a prescription for obtaining $F(t)$ from on-shell $\pi\pi$ p-wave elastic scattering and from the $T=J=1$ amplitude for $\pi\pi$ on shell $\to A_1A_1$ off shell. We regard this result as an interesting example of what can be learned from the simultaneous application of the constraints of current algebra and of unitarity. Further consequences that it may have towards improving our knowledge of the t-dependent form factors and phase shifts for the $A_1\rho\pi$ system are actively being examined.

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ACKNOWLEDGMENT

Part of the work of one of us (E.G.) was carried out at the Aspen Center for Physics. Their hospitality is gratefully acknowledged.

PHYSICAL REVIEW D VOLUME 2, NUMBER 8 15 OCTOBER 1970

Pion-Pion Dynamics in the σ Model

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We propose an unambiguous way of constructing amplitudes which satisfy both unitarity and the current-algebra constraints. This consists in working out higher-order corrections on a Lagrangian which produces the correct soft-pion limit in the tree approximation. We consider $\pi\pi$ scattering in the σ model, and we compute the perturbation series up to second order. The renormalization procedure preserves the partially conserved axial-vector current condition and the current-algebra constraints at each order. In order to sum the strong-coupling perturbation series, we use the Padé-approximation technique. Thereby, our partial-wave amplitudes satisfy unitarity. The ρ and f_0 resonances are generated, although they were not present in the Lagrangian. Our unitary amplitudes satisfy crossing symmetry to a very good accuracy, showing the consistency of the results. Our results are in agreement with the "up-down" solution of the $I=0$, s-wave $\pi\pi$ phase shift, with a very broad σ resonance; the $I=2$ s-wave phase shift is repulsive, and agrees very well with experiment.

I. INTRODUCTION

ALTHOUGH current algebra has been successful in in describing low-energy pion processes, the predictive power of the theory in the form used so far becomes weakened as soon as the energy increases beyond the threshold, since the unitarity is not taken into account in the usual treatments. With the help of chiral Lagrangians, one can realize the results of current algebra within the framework of Lagrangian field theory; based on this observation, we have proposed[1] an unambiguous way of unitarizing the current-algebra amplitude. This consists in taking a Lagrangian which is renormalizable and which produces the correct soft-pion limit, and in computing higher-order corrections and summing the presumably divergent perturbation series by the Padé algorithm.

The σ model of Gell-Mann and Lévy[2] is ideally suited for implementing this program. The Lagrangian

* Laboratoire associé au CNRS.
† Supported in part by U. S. Atomic Energy Commission under Contract No. AT(30-1)3668B.

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