Lorentz-Invariant Algebraization of Pion-Hadron Vertex Functions*

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Making use of rather general dynamical assumptions, it is proven that the problem of determining relativistic pion transition amplitudes may be completely reduced to the study of unitary representations of the noncompact dynamical group $SO(3,1)\otimes SO(4,3)$, since matrix elements of physical observables are shown to form a closed algebra which is identical with the Lie algebra of this group. These assumptions are Lorentz and isospin invariance of strong interactions, the Lehmann-Symanzik-Zimmermann reduction technique, an effective-interaction Lagrangian or partial conservation of axial-vector current, the usual equal-time commutator algebra between axial charges, and the absence of exotic states. The connection with the dynamical group approach previously proposed by Barut is discussed.

I. INTRODUCTION

OVER the last few years the relativistic framework of dynamical groups proposed by Barut[1] has been successfully applied to strong decays of meson[2] and baryon[3] resonances as well as to the study of mass spectra and form factors of hadrons.[4] The essential assumptions made in such studies may be summarized as follows.

(a) Hadron states are assigned to unitary irreducible representations of some noncompact group G which contains the Lorentz group as a subgroup (in order to guarantee relativistic invariance of the theory). *A priori*, suitable candidates for G are, for example, the groups $SO(3,1)$, $SO(3,2)$, $SO(4,2)$, $SL(2,C)$, $SL(6,C)$, etc. The ultimate selection, however, is only to be dictated by results which agree with experiment.

(b) Once the group G has been selected then its generators (which are self-adjoint operators in the Hilbert space of physical states) are phenomenologically identified with physical observables such as, for example, momentum, angular momentum, electromagnetic current, etc. In this way matrix elements representing measurable quantities may then be easily calculated by group-theoretical considerations.

It is amazing that physical consequences following from the above set of assumptions agree quite well with experiments for suitable choices of the group G.

In fact, even the simplest possible dynamical group, $SO(3,1)$, has been able to describe very well the pion-baryon decay rates of many resonances[3] by making use of only two free parameters which are an effective coupling constant g and eigenvalue ν of one of the Casimir operators of the $SO(3,1)$ group. Of course, the parameter ν is adjusted phenomenologically by requiring a best fit to the experimental data. However, if one wishes to avoid the freedom in the choice of ν, one is then naturally led to the study of larger dynamical groups such as, for example, $SO(4,2)$, which was proposed by Barut and Tripathy.[5] This group has received a great deal of attention in the series of excellent papers by Barut *et al.*,[6] Nambu,[7] and Yao.[8]

Although all the calculations mentioned above show good agreement with experiment, it is of course not at all clear whether other choices for G might be more suitable. Furthermore, if one really expects the study of a given dynamical group to be physically meaningful one should be able to derive its Lie algebra from general physical assumptions.

It is the purpose of this paper to show that the dynamical group which describes the hadronic world may be rigorously derived starting from the following usually accepted physical hypotheses.

(a) Relativistic and isotopic invariance of the theory.

(b) Validity of the Lehmann-Symanzik-Zimmermann (LSZ) reduction technique.[9]

(c) Either an effective interaction Lagrangian $\mathcal{L}_I$ of the type[10] $\mathcal{L}_I=(F_\pi)^{-1}A_\mu{}^\alpha(x)\partial^\mu\Phi^\alpha(x)$ (where $F_\pi\simeq 190$ MeV is the pion decay amplitude, $A_\mu{}^\alpha(x)$ is the axial-vector current, $\Phi^\alpha(x)$ is the pion field, and $\alpha=1, 2, 3$ and $\mu=0, 1, 2, 3$ are isovector and space-time indices, respectively) or the validity of PCAC [i.e., the assumption that the soft-pion technique may be employed whenever $(F_\pi m_\pi{}^2)^{-1}\partial^\mu A_\mu{}^\alpha(x)$ is chosen as an interpolating pion field].[11]

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[10] S. Weinberg, Phys. Rev. **177**, 2604 (1969).

[11] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

(d) Validity of the usual equal-time commutators between axial charges.[12]

(e) Absence of exotic states having isospin $I=2$.

From the above assumptions it is then easy to reduce the dynamical problem of calculating decay amplitudes involving pions to the study of representations of a certain noncompact group. In fact, we show in Secs. II and III that the above assumptions lead us to conclude that hadron states form unitary representations of the noncompact group $SO(3,1)\otimes SO(4,3)$. In addition we show that the physical interpretation of the generators of this group is unique and unambiguous and that the relativistic transition amplitude is written as a sum of matrix elements of a certain class of generators of the group in question.

II. REDUCTION OF DYNAMICAL PROBLEM TO ALGEBRA OF MATRIX ELEMENTS

We start by considering a general pion transition process

$$a(p)\to b(p')+\pi(q,\alpha)\,, \tag{2.1}$$

where $a(p)$ and $b(p')$ denote arbitrary hadron states with momenta p and p', respectively, while $\pi(q,\alpha)$ denotes a pion with momentum q and isospin index α. The S matrix for this process is defined by

$$^{\rm in}\langle b\mathbf{p}';\mathbf{q}\alpha|S|a\mathbf{p}\rangle^{\rm in}={}^{\rm out}\langle b\mathbf{p}';\mathbf{q}\alpha|a\mathbf{p}\rangle^{\rm in}\,, \tag{2.2}$$

(where a and b denote all other quantum numbers) and is related to the invariant Feynman amplitude $M_\alpha(p',q;p)$ by

$$^{\rm in}\langle b\mathbf{p}',\mathbf{q}\alpha|S|a\mathbf{p}\rangle^{\rm in}=-(2\pi)^4\delta^4(p'+q-p)(2\pi)^{-9/2}\times(8q^0p^0p'^0)^{-1/2}M_\alpha(p'q;p)\,. \tag{2.3}$$

By use of LSZ reduction technique,[9] one may then write

$$^{\rm in}\langle b\mathbf{p}';\mathbf{q}\alpha|S|a\mathbf{p}\rangle^{\rm in}=-\frac{i}{(2\pi)^{3/2}}\frac{1}{(2q^0)^{1/2}}\int d^4x\times e^{-iqx}(\Box-m_\pi^2)\langle b\mathbf{p}'|\Phi^\alpha(x)|a\mathbf{p}\rangle\,, \tag{2.4}$$

where the states $|a\mathbf{p}\rangle^{\rm in}$ and $|b\mathbf{p}'\rangle^{\rm in}$ have been normalized to

$$^{\rm in}\langle b\mathbf{p}'|a\mathbf{p}\rangle^{\rm in}=\delta_{ab}\delta^3(\mathbf{p}-\mathbf{p}')\,. \tag{2.5}$$

We can then proceed further either by making use of an effective interaction Lagrangian[10] of the type

$$\mathfrak{L}_I=(F_\pi)^{-1}A_\mu{}^\alpha(x)\partial^\mu\Phi^\alpha(x) \tag{2.6}$$

and its corresponding equations of motion

$$(\Box-m_\pi^2)\Phi^\alpha(x)=-(F_\pi)^{-1}\partial^\mu A_\mu{}^\alpha(x) \tag{2.7}$$

or PCAC[11] [i.e., $(F_\pi m_\pi^2)^{-1}\partial^\mu A_\mu{}^\alpha(x)$ is chosen as an interpolating pion field and the soft-pion technique is then employed] and rewrite Eq. (2.4) in the following form:

$$^{\rm in}\langle b\mathbf{p}',\mathbf{q}\alpha|S|a\mathbf{p}\rangle^{\rm in}=F_\pi{}^{-1}\frac{(2\pi)^{5/2}}{(2q^0)^{1/2}}\delta^4(p'+q-p)\times(p-p')^\mu\langle b\mathbf{p}'|A_\mu{}^\alpha(0)|a\mathbf{p}\rangle\,. \tag{2.8}$$

In the derivation of the last equation, we have also used translation invariance, i.e.,

$$\langle b\mathbf{p}'|A_\mu{}^\alpha(x)|a\mathbf{p}\rangle=\exp[-i(p'-p)\cdot x]\langle b\mathbf{p}'|A_\mu{}^\alpha(0)|a\mathbf{p}\rangle\,.$$

Comparing Eq. (2.8) with (2.3), we note that the invariant Feynman amplitude $M_\alpha(p',q;p)$ may be written as

$$M_\alpha(p'q;p)=F_\pi{}^{-1}(2\pi)^3(4p^0p'^0)^{1/2}(p-p')^\mu\times\langle b\mathbf{p}'|A_\mu{}^\alpha(0)|a\mathbf{p}\rangle\,. \tag{2.9}$$

From the above equation it is now evident that $M_\alpha(p',q;p)$ may be obtained by calculating the matrix elements of $A_\mu{}^\alpha(0)$ between two hadron states $|b\mathbf{p}'\rangle$ and $|a\mathbf{p}\rangle$. It should also be mentioned at this point that the state $|a\mathbf{p}\rangle$ representing a hadron of momentum $\mathbf{p}$ may be obtained from the state $|a\rangle$ at rest by means of a homogenous Lorentz transformation, i.e.,

$$|a\mathbf{p}\rangle=e^{i\boldsymbol{\xi}\cdot\mathbf{M}}|a\rangle\,, \tag{2.10}$$

where $\mathbf{M}$ denotes the boost operator and $\boldsymbol{\xi}$ is a vector in the direction of $\mathbf{p}$ with magnitude given by

$$\tanh|\boldsymbol{\xi}|=|\mathbf{p}|/p^0\,. \tag{2.11}$$

Use of Eq. (2.10) simplifies extremely the calculations of the matrix element $\langle b\mathbf{p}'|A_\mu{}^\alpha(0)|a\mathbf{p}\rangle$. In fact, since the invariant amplitude (2.9) is Lorentz invariant, we may assume without loss of generality that the initial state $|a\mathbf{p}\rangle$ is at rest. The final state $|b\mathbf{p}'\rangle$ is then obtained by boosting the state $|b\rangle$ at rest to momentum $\mathbf{p}'$. The calculation of the invariant amplitude is then reduced to the determination of matrix elements of the type

$$\langle b|e^{-i\boldsymbol{\xi}\cdot\mathbf{M}}A_\mu{}^\alpha(0)|a\rangle\,. \tag{2.12}$$

Since the matrices of finite Lorentz transformations

$$\langle b|e^{-i\boldsymbol{\xi}\cdot\mathbf{M}}|a\rangle \tag{2.13}$$

can be found in the literature,[13] the preceding problem is then reduced to the determination of the matrix elements of $A_\mu{}^\alpha(0)$ between two states at rest, i.e., to the calculation of

$$\langle b|A_\mu{}^\alpha(0)|a\rangle\,. \tag{2.14}$$

In order to determine these matrix elements, we start by assuming the validity of the usual equal-time commutation relations between axial charges,[12] i.e.,

$$\left[\int d^3x\,A_0{}^\alpha(\mathbf{x},t),\ \int d^3y\,A_0{}^\beta(\mathbf{y},t)\right]=i\epsilon^{\alpha\beta\gamma}I^\gamma\,, \tag{2.15}$$

where I^γ denotes the generator of isospin transformations.

[12] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[13] S. Ström, Arkiv Fysik **29**, 467 (1965); **33**, 465 (1966).

Let us next consider the matrix element of the commutator (2.15) between hadron states $|b\mathbf{p}'\rangle$ and $|a\mathbf{p}\rangle$. We obtain

$$\langle b\mathbf{p}'|\left[\int d^3x\, A_0(\mathbf{x},0),\ \int d^3y\, A_0{}^{\beta}(\mathbf{y},0)\right]|a\mathbf{p}\rangle = i\epsilon^{\alpha\beta\gamma}(I^{\gamma})_{ba}\delta^3(\mathbf{p}-\mathbf{p}')\,. \quad (2.16)$$

To evaluate the left-hand side of Eq. (2.16), one inserts a complete set of intermediate states $|n\mathbf{p}_n\rangle$ and uses translational invariance to carry out the spatial and momentum $\mathbf{p}_n$ integrations. This yields

$$(2\pi)^6\sum_n \left[\langle b\mathbf{p}|A_0{}^{\alpha}(0)|n\mathbf{p}\rangle\langle n\mathbf{p}|A_0{}^{\beta}(0)|a\mathbf{p}\rangle - \langle b\mathbf{p}|A_0{}^{\beta}(0)|n\mathbf{p}\rangle\langle n\mathbf{p}|A_0{}^{\alpha}(0)|a\mathbf{p}\rangle\right] = i\epsilon^{\alpha\beta\gamma}(I^{\gamma})_{ba}\,, \quad (2.17)$$

where we have canceled a common factor of $\delta^3(\mathbf{p}-\mathbf{p}')$ on both sides. It should be also stressed that the above relation can only be derived for states $|b\mathbf{p}\rangle$ and $|a\mathbf{p}\rangle$ with the same momentum $\mathbf{p}$. Therefore, we shall restrict ourselves to hadron states at rest without loss of generality.

To proceed further, let us define three matrices $x_0{}^{\alpha}$ ($\alpha=1,\,2,\,3$) by

$$(x_0{}^{\alpha})_{bn}=\langle b|A_0{}^{\alpha}(0)|n\rangle(2\pi)^3\,, \quad (2.18)$$

where b and n denote the bth row and nth column of the matrix $x_0{}^{\alpha}$. With this notation, Eq. (2.17) may then be written in matrix form as

$$x_0{}^{\alpha}x_0{}^{\beta}-x_0{}^{\beta}x_0{}^{\alpha}=i\epsilon^{\alpha\beta\gamma}I^{\gamma}\,, \quad (2.19)$$

where also I^{γ} is a matrix. Introducing the usual abbreviation

$$[x_0{}^{\alpha},x_0{}^{\beta}]\equiv x_0{}^{\alpha}x_0{}^{\beta}-x_0{}^{\beta}x_0{}^{\alpha}\,, \quad (2.20)$$

Eqs. (2.17) and (2.19) may then be written as

$$[x_0{}^{\alpha},x_0{}^{\beta}]=i\epsilon^{\alpha\beta\gamma}I^{\gamma}\,. \quad (2.21)$$

The above equation will play an important role in the rest of our discussion. In fact, we proceed Sec. III to make use of this equation along with Lorentz invariance and absence of exotic states to study relations among the matrix elements $\langle b|A_{\mu}{}^{\alpha}(0)|a\rangle$ and show that they form a closed algebra which is isomorphic to the Lie algebra of the group $SO(3,1)\otimes SO(4,3)$.

III. DERIVATION OF DYNAMICAL ALGEBRA

Any reasonable theory describing strong interactions of pions with hadrons must be Lorentz and isotopic-spin invariant. This then implies that the invariance symmetry group K is the direct product of the isospin group $SU(2)_I$ and the Lorentz group $SO(3,1)$ i.e., $K=SU(2)_I\otimes SO(3,1)$. Clearly, the group K is generated by the following Lie algebra:

$$[I^{\alpha},I^{\beta}]=i\epsilon^{\alpha\beta\gamma}I^{\gamma}\,, \quad (3.1)$$

$$[J_{\mu\nu},J_{\rho\sigma}]=i(g_{\nu\rho}J_{\mu\sigma}-g_{\nu\sigma}J_{\mu\rho}-g_{\mu\rho}J_{\nu\sigma}+g_{\mu\sigma}J_{\nu\rho})\,, \quad (3.2)$$

and

$$[I^{\alpha},J_{\mu\nu}]=0\,, \quad (3.3)$$

where $\alpha,\ \beta,\ \gamma=1,\,2,\,3$ are isospin indices, $\mu,\ \nu,\ \rho,\ \sigma=1,\,2,\,3,\,0$ are space-time indices, and I^{α} and $J_{\mu\nu}$ are the generators of the groups $SU(2)_I$ and $SO(3,1)$, respectively. The metric tensor $g_{\mu\nu}$ is defined by

$$g_{00}=1\,,\quad g_{11}=g_{22}=g_{33}=-1\,,\quad g_{\mu\nu}=0\,,\quad \text{if}\quad \mu\neq\nu\,.$$

The most important operator in our theory is the axial-vector current $A_{\mu}{}^{\alpha}(0)$ taken at the origin of a reference frame. This operator is, of course, an isovector and a Lorentz four-vector and therefore obeys the following set of commutation relations:

$$[I^{\alpha},A_{\mu}{}^{\beta}(0)]=i\epsilon^{\alpha\beta\gamma}A_{\mu}{}^{\gamma}(0) \quad (3.4a)$$

and

$$[J_{\mu\nu},A_{\rho}{}^{\alpha}(0)]=i[g_{\nu\rho}A_{\mu}{}^{\alpha}(0)-g_{\mu\rho}A_{\nu}{}^{\alpha}(0)]\,. \quad (3.4b)$$

We next consider the sum rules obtained by taking matrix elements of the commutators (3.1)–(3.4) between two hadron states $|f\rangle$ and $|i\rangle$ at rest. Thus we define twelve matrices $x_{\mu}{}^{\alpha}$ by

$$(x_{\mu}{}^{\alpha})_{fi}=\langle f|A_{\mu}{}^{\alpha}(0)|i\rangle(2\pi)^3\,, \quad (3.5)$$

where the subscripts f and i denote the fth row and ith column of the matrix $x_{\mu}{}^{\alpha}$. The matrix relations following from the commutators (3.4) then take the form

$$[I^{\alpha},x_{\mu}{}^{\beta}]=i\epsilon^{\alpha\beta\gamma}x_{\mu}{}^{\gamma} \quad (3.6)$$

and

$$[J_{\mu\nu},x_{\rho}{}^{\alpha}]=i(g_{\nu\rho}x_{\mu}{}^{\alpha}-g_{\mu\rho}x_{\nu}{}^{\alpha})\,, \quad (3.7)$$

where we have used the abbreviation defined in Eq. (2.20). It should also be mentioned that the symbols I^{α} and $J_{\mu\nu}$ occurring in (3.6) and (3.7) are not operators but matrices. However, we have used the same symbols for the linear operators I^{α} and $J_{\mu\nu}$ as for their algebraical realizations since these matrices will, of course, satisfy the same algebraic relations as those given by Eqs. (3.1)–(3.3). To avoid any confusion, we stress that from now on, any commutator which will be derived must be understood as a matrix relation.

If we were now able to construct uniquely the set of 12 matrices $x_{\mu}{}^{\alpha}$, then we would be finished and the invariant Feynman amplitude (2.9) would then be uniquely determined. To proceed further in this construction, we next write the most general form for the commutators $[x_{\mu}{}^{\alpha},x_{\nu}{}^{\beta}]$. In order to do this, we note that this quantity is antisymmetric with respect to the interchange of pairs of indices $(\alpha\mu)$ and $(\beta\nu)$. Therefore, the most general decomposition of this commutator is

$$[x_{\mu}{}^{\alpha},x_{\nu}{}^{\beta}]=i\epsilon^{\alpha\beta\gamma}Y_{\{\mu\nu\}}{}^{\gamma}+iZ_{[\mu\nu]}{}^{\{\alpha\beta\}}\,, \quad (3.8)$$

where $Y_{\{\mu\nu\}}{}^{\gamma}$ and $Z_{[\mu\nu]}{}^{\{\alpha\beta\}}$ are matrices, and the symbols

[··] and {··} are abbreviations for antisymmetricity and symmetricity in the corresponding pairs of indices. It is now a simple matter to prove that $Y_{\{\mu\nu\}}{}^{\gamma}$ is an isovector and a symmetric Lorentz tensor while $Z_{[\mu\nu]}{}^{\{\alpha\beta\}}$ is a reducible symmetric isotensor and an antisymmetric Lorentz tensor. This proof can be found in Appendix A.

The decomposition of the commutator $[x_\mu{}^\alpha, x_\nu{}^\beta]$, given by Eq. (3.8), reminds us very strongly of the algebraic structure of Weinberg's superconvergence conditions[10] analyzed in a series of papers.[14–16] In these works, the left-hand side of Eq. (3.8) is interpreted as the s- and u-channel contributions to a superconvergence sum rule while the right-hand side correspond to t-channel meson-exchange contributions. In accordance with this approach the matrices $Z_{[\mu\nu]}{}^{\{\alpha\beta\}}$ are only related to the exchange of mesons with isospin $I=0$, 2 since $Z_{[\mu\nu]}{}^{\{\alpha\beta\}}$ is a symmetric isotensor. It is usually assumed that isospin-two states (which belong to the class of so-called exotic states) do not exist. Therefore, we require that the part of $Z_{[\mu\nu]}{}^{\{\alpha\beta\}}$ which transforms under $SU(2)_I$ as an irreducible symmetric tensor with $I=2$ must vanish. This implies immediately that $Z_{[\mu\nu]}{}^{\{\alpha\beta\}}$ is only an isospin scalar, namely,

$$Z_{[\mu\nu]}{}^{\{\alpha\beta\}} = -\delta^{\alpha\beta} T_{\mu\nu}, \tag{3.9}$$

where the minus sign is only a convention and $T_{\mu\nu}$ is a matrix which transforms as an antisymmetric Lorentz tensor and obeys the following set of matrix relations:

$$[J_{\mu\nu}, T_{\rho\sigma}] = i(g_{\nu\rho} T_{\mu\sigma} - g_{\nu\sigma} T_{\mu\rho} - g_{\mu\rho} T_{\nu\sigma} + g_{\mu\sigma} T_{\nu\rho}) \tag{3.10}$$

and

$$[I^\alpha, T_{\mu\nu}] = 0. \tag{3.11}$$

We stress that the application of Eq. (3.8) to the time components

$$[x_0{}^\alpha, x_0{}^\beta] = i\epsilon^{\alpha\beta\gamma} Y_{\{00\}}{}^{\gamma} \tag{3.12a}$$

must give the same result as Eq. (2.21), i.e.,

$$[x_0{}^\alpha, x_0{}^\beta] = i\epsilon^{\alpha\beta\gamma} I^\gamma, \tag{3.12b}$$

and which follow from the equal-time commutator algebra which the axial-vector charges satisfy.[12]

Comparing Eqs. (3.12), we obtain

$$Y_{\{00\}}{}^{\gamma} = I^\gamma. \tag{3.12c}$$

Since I^γ is a Lorentz scalar it follows immediately (by making use of the commutators $[J_{\mu\nu}, Y_{\{00\}}{}^\gamma]=0$; see Appendix A) that $A_{\{\mu\nu\}}{}^\gamma$ is also a Lorentz scalar and therefore we conclude

$$Y_{\{\mu\nu\}}{}^\gamma = g_{\mu\nu} I^\gamma. \tag{3.13}$$

Using the results given by Eqs. (3.9) and (3.13), we rewrite the important relation (3.8) as follows:

$$[x_\mu{}^\alpha, x_\nu{}^\beta] = i g_{\mu\nu} \epsilon^{\alpha\beta\gamma} I^\gamma - i\delta^{\alpha\beta} T_{\mu\nu}. \tag{3.14}$$

From (3.14) it is now simple to express $T_{\mu\nu}$ in terms of the $x_\mu{}^\alpha$'s and make use of the Jacobi identity to determine the commutators $[T_{\mu\nu}, x_\rho{}^\alpha]$ and $[T_{\mu\nu}, T_{\rho\sigma}]$. These calculations can be found in Appendix B, where the following results are derived:

$$[T_{\mu\nu}, x_\rho{}^\alpha] = i(g_{\nu\rho} x_\mu{}^\alpha - g_{\mu\rho} x_\nu{}^\alpha) \tag{3.15}$$

and

$$[T_{\mu\nu}, T_{\rho\sigma}] = i(g_{\nu\rho} T_{\mu\sigma} - g_{\nu\sigma} T_{\mu\rho} - g_{\mu\rho} T_{\nu\sigma} + g_{\mu\sigma} T_{\nu\rho}). \tag{3.16}$$

The commutators (3.1)–(3.3), (3.6), (3.7), (3.10), (3.11), and (3.14)–(3.16) show that the 27 matrices I^α, $J_{\mu\nu}$, $T_{\mu\nu}$, and $x_\mu{}^\alpha$ form a closed algebra which may be identical to the Lie algebra of some dynamical group G. If we are able to find the structure of this group G, then our dynamical problem will be completely reduced to the study of unitary representations of this group.

In order to find the structure of G, we find it convenient to introduce six matrices $F_{\mu\nu}$ defined as follows:

$$F_{\mu\nu} = J_{\mu\nu} - T_{\mu\nu}. \tag{3.17}$$

It is then a simple matter to verify that the matrices $F_{\mu\nu}$ commute with all the 21 matrices I^α, $T_{\mu\nu}$, and $x_\mu{}^\alpha$, and that they satisfy the Lie algebra of the $SO(3,1)$ group, namely,

$$[F_{\mu\nu}, F_{\rho\sigma}] = i(g_{\nu\rho} F_{\mu\sigma} - g_{\nu\sigma} F_{\mu\rho} - g_{\mu\rho} F_{\nu\sigma} + g_{\mu\sigma} F_{\nu\rho}). \tag{3.18}$$

This implies that the group G is the direct product of $SO(3,1)$ with a group G_0 which is generated by the Lie algebra given by commutators (3.1), (3.6), (3.11), and (3.14)–(3.16). Thus the problem is now reduced to finding the group structure of G_0. This can be done quite easily if we define a metric tensor g_{ab} for

$$a, b = \mu, \nu, \rho, \sigma, \ldots = 1, 2, 3, 0$$

and

$$a, b = \alpha, \beta, \gamma, \ldots = 5, 6, 7$$

by

$$\begin{aligned} &g_{11} = g_{22} = g_{33} = -1, \\ &g_{00} = g_{55} = g_{66} = g_{77} = +1, \\ \text{and} \quad &g_{ab} = 0 \quad \text{if} \quad a \neq b, \end{aligned} \tag{3.19}$$

and introduce, in addition, matrices

$$L_{ab} = -L_{ba}$$

defined by

$$\begin{aligned} &L_{\alpha\beta} \equiv -\epsilon^{\alpha\beta\gamma} I^\gamma, \\ &L_{\alpha\mu} \equiv x_\mu{}^\alpha, \\ \text{and} \quad &L_{\mu\nu} \equiv T_{\mu\nu}. \end{aligned} \tag{3.20}$$

With the above definitions the commutators (3.1), (3.6), (3.11), and (3.14)–(3.16) may then be compactly rewritten in the form

$$[L_{ab}, L_{cd}] = i(g_{bc} L_{ad} - g_{bd} L_{ac} - g_{ac} L_{bd} + g_{ad} L_{bc}). \tag{3.21}$$

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[14] M. Noga and C. Cronström, Nucl. Phys. B15, 61 (1970); Phys. Rev. D 1, 2414 (1970).

[15] A. McDonald, Phys. Rev. D 1, 721 (1970).

[16] L. R. Ram Mohan, Phys. Rev. D 2, 299 (1970).

The above commutation relations define the well-known Lie algebra of the noncompact rotational group $SO(4,3)$.

To conclude this section, we would like to stress once again that the dynamical problem of determining the Feynman invariant amplitude for processes involving pions has been completely reduced to the study of the algebra of the noncompact group $SO(3,1)\otimes SO(4,3)$. Since operators representing physical observables operate on the Hilbert space of physical states this then implies that hadron states must form a representation space of the dynamical algebra of observables, i.e., of the Lie algebra of the group $SO(3,1)\otimes SO(4,3)$. From this it then follows that any unitary (reducible or irreducible) representation of this group may correspond to possible physical states. Of course, there is no reason at all to demand that physical states transform according to unitary irreducible representations of this group, since the required Lie algebra relations are also fulfilled if one considers unitary reducible representations.

IV. CONNECTION WITH DYNAMICAL GROUPS PROPOSED BY BARUT

We have proved that matrix elements of physical observables form the closed algebra of a dynamical group which combines in a nontrivial way internal (isospin) symmetry with space-time. Originally the dynamical groups proposed by Barut *et al.*[1,6] were only restricted to the external (space-time) properties of hadrons, while later these groups were combined with internal symmetries by taking their direct products.[2,3]

We would next like to discuss what happens if we restrict ourselves to matrix elements of physical observables describing external properties of hadrons, i.e., to sets of hadrons with the same internal quantum numbers. This is equivalent to considering hadron families with the same third component of isospin and thus implies that we rule out all matrices I^α connected with internal symmetries as well as the matrices $x_\mu{}^1$ and $x_\mu{}^2$ which change the charges of the hadrons under consideration. Thus we shall only deal now with the 16 matrices $J_{\mu\nu}$, $T_{\mu\nu}$, and $X_\mu{}^3\equiv\Gamma_\mu$. It is then simple to verify that they form a closed algebra which is identical with the Lie algebra of the group $SO(3,1)\otimes SO(3,2)$. This result tells us that hadron states with the same third component of isospin must transform according to unitary (reducible or irreducible) representations of this group.

The dynamical group $SO(3,2)$ was proposed by Barut, Corrigan, and Kleinert[4] in order to calculate mass spectra and electromagnetic form factors of hadrons. In their framework, hadron states are assumed to transform according to unitary irreducible representations of this group and the matrix Γ_μ introduced above plays the role of their so-called algebraic current. They then consider one class of representations of the group $SO(3,2)$, which, of course, are also representations of the group $SO(3,1)\otimes SO(3,2)$ which we have derived here by identifying the matrices $T_{\mu\nu}$ with the matrices $J_{\mu\nu}$. Thus we have shown that the assumptions made by the preceding authors on the basis of an excellent physical intuition can, in fact, be uniquely derived making use of usually accepted dynamical assumptions.

V. SUMMARY AND CONCLUSIONS

Several dynamical models for the description of hadron states which lead naturally to relations identical to the algebra of certain Lie groups have been proposed over the last few years. Among them we start by mentioning the popular Chew static bootstrap model,[17] which was completely reworded in group-theoretic language by Cook, Goebel, and Sakita.[18] Next we mention the work of Capps,[19] who has shown under fairly general assumptions that if one saturates superconvergence sum rules with single-particle states, one is naturally led to models in which hadron states are associated with unitary representations of certain Lie groups. More recently, algebraic superconvergence conditions for the forward scattering of massless pions with hadrons have been derived by Weinberg,[10,20] making use of the effective chiral Lagrangian formalism.

All the preceding treatments led to the conclusion that hadron states form a basis for unitary representations of certain Lie groups. On the other hand, in the framework of dynamical groups one usually makes the *ad hoc* assumption that hadron states form unitary irreducible representations of some noncompact group. Since this approach has been rather successful, one is then led to conjecture that these dynamical groups might in fact be derived from generally accepted physical assumptions. We have shown that this is actually the case. In fact, we have derived relations identical to the Lie algebra of the group $SO(3,1)\otimes SO(4,3)$ merely by assuming isospin and Lorentz invariance, usual equal-time commutator algebra between axial charges, absence of exotic states, and either an effective interaction Lagrangian or PCAC. Since physical observables are self-adjoint operators in the Hilbert space $\mathcal{H}$ of hadronic physical states, it then follows that $\mathcal{H}$ is the representation space of the Lie group $SO(3,1)\otimes SO(4,3)$. Thus hadron states must form a basis for unitary (irreducible or reducible) representations of this group, which is a nontrivial combination of the isospin group $SU(2)$ with the Lorentz group $SO(3,1)$. The generalization to larger internal symmetry groups [for example, $SU(3)$] is straightforward and may be done along the lines discussed in this paper.

The dynamical groups proposed by Barut and his collaborators were $SO(3,1)$, $SO(3,2)$, and $SO(4,2)$, which

[17] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).

[18] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).

[19] R. H. Capps, Phys. Rev. **168**, 1731 (1968); **171**, 1591 (1968).

[20] S. Weinberg, Phys. Rev. Letters **22**, 1023 (1969); in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 253.

are all subgroups of $SO(3,1)\otimes SO(4,3)$, so that all representations of the latter are also reducible representations of the former groups. If we only restrict ourselves to the external properties of hadrons, we have found that hadron states with the same third components of isospin are classified according to unitary representations of the group $SO(3,1)\otimes SO(3,2)$. Note that the group $SO(3,2)$ is exactly the one proposed by Barut *et al.*[4] in their calculations of electromagnetic form factors and mass spectra of physical states.

To conclude this discussion, we stress that the dynamical calculation of the pion-hadron vertex function was reduced to a set of algebraic relations which turned out to be the same as the Lie algebra of the group $SO(3,1)\otimes SO(4,3)$. Finally, it should also be mentioned that an algebraic treatment to the dynamical problem of pion-hadron coupling constants has also been extensively developed in a series of papers by Sugawara,[21] who makes use of the LSZ reduction technique and the assumption that the dispersive part of the three-point function may be completely saturated by single-particle intermediate states.

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APPENDIX A

In this appendix we show that the matrices $Y_{\{\mu\nu\}}{}^{\gamma}$ and $Z_{[\mu\nu]}{}^{\{\alpha\beta\}}$ introduced in Eq. (3.8) transform as tensors under Lorentz and isospin transformations. We start with the matrix relation (3.8), which is of the form

$$[x_\mu{}^\alpha,x_\nu{}^\beta]=i\epsilon^{\alpha\beta\gamma}Y_{\{\mu\nu\}}{}^{\gamma}+iZ_{[\mu\nu]}{}^{\{\alpha\beta\}}. \tag{A1}$$

Our first step is to express $Y_{\{\mu\nu\}}{}^{\gamma}$ in terms of $x_\mu{}^\alpha$. This can simply be done and one obtains the following result:

$$Y_{\{\mu\nu\}}{}^{\gamma}=-\tfrac{1}{2}i\epsilon^{\alpha\beta\gamma}[x_\mu{}^\alpha,x_\nu{}^\beta]. \tag{A2}$$

The commutator

$$[J_{\rho\sigma},Y_{\{\mu\nu\}}{}^{\gamma}]=-\tfrac{1}{2}i\epsilon^{\alpha\beta\gamma}[J_{\rho\sigma},[x_\mu{}^\alpha,x_\nu{}^\beta]] \tag{A3}$$

can then be rewritten by making use of the Jacobi identity as

$$[J_{\rho\sigma},Y_{\{\mu\nu\}}{}^{\gamma}]=\tfrac{1}{2}i\epsilon^{\alpha\beta\gamma}\{[x_\nu{}^\beta,[J_{\rho\sigma},x_\mu{}^\alpha]]+[x_\mu{}^\alpha,[x_\nu{}^\beta,J_{\rho\sigma}]]\}. \tag{A4}$$

Carrying out the algebraic reduction using the commutation relations (3.7) and (3.8), we finally obtain the result

$$[J_{\rho\sigma},Y_{\{\mu\nu\}}{}^{\gamma}]=i(g_{\sigma\mu}Y_{\{\rho\nu\}}{}^{\gamma}+g_{\sigma\nu}Y_{\{\rho\mu\}}{}^{\gamma}-g_{\rho\mu}Y_{\{\sigma\nu\}}{}^{\gamma}-g_{\rho\nu}Y_{\{\sigma\mu\}}{}^{\gamma}). \tag{A5}$$

From the above equation, we see that the matrices $Y_{\{\mu\nu\}}{}^{\gamma}$ transform as a symmetric Lorentz tensor. The same procedure can be used to prove that $A_{\{\mu\nu\}}{}^{\gamma}$ transforms as an isovector while $Z_{[\mu\nu]}{}^{\{\alpha\beta\}}$ transforms as a symmetric isotensor and an antisymmetric Lorentz tensor.

APPENDIX B

The purpose of this appendix is to derive the commutators $[T_{\mu\nu},x_\rho{}^\alpha]$ and $[T_{\mu\nu},T_{\rho\sigma}]$. We start from relation (3.14) and obtain

$$T_{\mu\nu}=\tfrac{1}{3}i[x_\mu{}^\alpha,x_\nu{}^\alpha]. \tag{B1}$$

By making use of the above equation we may then write

$$[T_{\mu\nu},x_\rho{}^\beta]=-\tfrac{1}{3}i[x_\rho{}^\beta,[x_\mu{}^\alpha,x_\nu{}^\alpha]]. \tag{B2}$$

We next apply the Jacobi identity to the double commutator given above and obtain

$$[T_{\mu\nu},x_\rho{}^\beta]=\tfrac{1}{3}i\{[x_\nu{}^\alpha,[x_\rho{}^\beta,x_\mu{}^\alpha]]+[x_\mu{}^\alpha,[x_\nu{}^\alpha,x_\rho{}^\beta]]\}. \tag{B3}$$

Making use of the Eq. (3.14), we then carry out the algebraic reduction of the double commutators on the right-hand side of Eq. (B3). This yields the result

$$[T_{\mu\nu},x_\rho{}^\beta]=\tfrac{2}{3}i\{g_{\nu\rho}x_\mu{}^\beta-g_{\mu\rho}x_\nu{}^\beta\}-\tfrac{1}{3}\{[T_{\rho\mu},x_\nu{}^\beta]+[T_{\nu\rho},x_\mu{}^\beta]\}. \tag{B4}$$

The preceding commutator is then used to calculate the sum $[T_{\rho\mu},x_\nu{}^\beta]+[T_{\nu\rho},x_\mu{}^\beta]$. After some simple algebra, we obtain

$$[T_{\rho\mu},x_\nu{}^\beta]+[T_{\nu\rho},x_\mu{}^\beta]=\tfrac{1}{2}i\{g_{\rho\mu}x_\nu{}^\beta-g_{\rho\nu}x_\mu{}^\beta\}-\tfrac{1}{2}[T_{\mu\nu},x_\rho{}^\beta]. \tag{B5}$$

Inserting Eq. (B5) into Eq. (B4), we then obtain the relation

$$[T_{\mu\nu},x_\rho{}^\beta]=i(g_{\nu\rho}x_\mu{}^\beta-g_{\mu\rho}x_\nu{}^\beta), \tag{B6}$$

which has been used in Sec. III.

We can now proceed further and calculate

$$[T_{\mu\nu},T_{\rho\sigma}]=\tfrac{1}{3}i[T_{\mu\nu},[x_\rho{}^\alpha,x_\sigma{}^\alpha]]. \tag{B7}$$

In order to do this, we make once again use of the Jacobi identity for the double commutator and obtain, upon using Eq. (B6), the result

$$[T_{\mu\nu},T_{\rho\sigma}]=\tfrac{1}{3}\{g_{\nu\rho}[x_\sigma{}^\alpha,x_\mu{}^\alpha]-g_{\mu\rho}[x_\sigma{}^\alpha,x_\nu{}^\alpha]+g_{\sigma\mu}[x_\rho{}^\alpha,x_\nu{}^\alpha]-g_{\nu\sigma}[x_\rho{}^\alpha,x_\mu{}^\alpha]\}. \tag{B8}$$

Combining the above relation with Eq. (B1), we then find

$$[T_{\mu\nu},T_{\rho\sigma}]=i(g_{\nu\rho}T_{\mu\sigma}-g_{\nu\sigma}T_{\mu\rho}-g_{\mu\rho}T_{\nu\sigma}+g_{\mu\sigma}T_{\nu\rho}). \tag{B9}$$

Thus the matrices $T_{\mu\nu}$ form a closed algebra identical to the Lie algebra of the group $SO(3,1)$. Relation (B6) then tells us that the matrices $x_\mu{}^\alpha$ transform as four-vectors with respect to the group in question.

[21] M. Sugawara, in *Particle Physics*, edited by Paul Urban (Springer-Verlag, Berlin, 1969), p. 310; Phys. Rev. Letters **24**, 691 (1970).