When we write out the explicit form for f from Eq. (1), we see that

$$\cos\omega_i = \frac{(m^2-u_i-u_{i+1})\cos(\phi_i\pm\phi_{i+1})-2(u_iu_{i+1})^{1/2}}{(m^2-u_i-u_{i+1})-2(u_iu_{i+1})^{1/2}\cos(\phi_i\pm\phi_{i+1})}. \quad \text{(A3)}$$

From this one can see that as $\cos(\phi_{i-1}\pm\phi_{i+1})$ varies between $+1$ and -1, so does $\cos\omega_i$ and vice versa.

The two solutions arise because the ϕ_i angles are defined only in the range $0<\phi_i<\pi$. If we note that $\cos[\phi_i-(2\pi-\phi_{i+1})]=\cos(\phi_i+\phi_{i+1})$, then we can extend the range of the ϕ's to $0<\phi_i<2\pi$ and use only one of the solutions for $\cos\omega$ in (35). We choose the one with the difference $\phi_i-\phi_{i+1}$.

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Physical-Region Constraints on Low-Energy Partial-Wave Amplitudes. II. $\pi\pi$ Scattering*

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Physical-region constraints on the low partial waves of $\pi\pi$ scattering are derived on the basis of analyticity, crossing, and positivity of the absorptive parts. These constraints are sensitive only to the low-energy behavior of the partial-wave amplitudes. The necessity for subtractions in the dispersion relations implies that no results can be obtained for s and p waves.

I. INTRODUCTION

In a recent paper,[1] hereafter referred to as I, we developed a technique for obtaining rigorous constraints on the partial-wave amplitudes of $\pi^0\pi^0$ elastic scattering in the physical region. These were derived on the basis of analyticity, crossing, and positivity of the imaginary parts of the partial-wave amplitudes. They took the form of inequalities on integrals involving the imaginary part of the low partial waves and were sensitive only to the low-energy region. They could therefore be tested once a phase-shift analysis became available, or they could serve to discriminate between various proposed phase shifts.

In this paper, we shall extend the techniques to $\pi\pi$ scattering with isospin. In Sec. II, we obtain sum rules involving the absorptive parts of all partial waves for each isospin. Using the positivity of these absorptive parts, in Sec. III we rewrite the sum rules as inequalities involving integrals of the low partial-wave amplitudes which are sensitive only to the low-energy region.

One might have hoped that these inequalities would serve to discriminate between some of the low-energy $I=0$ s-wave phase shifts which have been proposed for $\pi\pi$ scattering. However, if one admits subtractions in the fixed-t dispersion relations for each isospin in the t channel, no information can be obtained on s and p waves. But the absence of exotic mesons has led many authors to suppose that the fixed-t dispersion relation corresponding to $I=2$ in the t channel, is unsubtracted. In our formalism this does lead to a family of sum rules involving s and p waves. However, we show in Sec. IV that these sum rules cannot be used to derive any useful information on the s and p waves.

Because no information is obtained on s and p waves, the simplest constraint involves $l=2$ and $l=3$, for which no phase shifts have yet been proposed. In a narrow-resonance scheme, we obtain

$$\frac{\Gamma_g}{\Gamma_{f^0}} < \frac{4}{21}\left(\frac{m_g}{m_{f^0}}\right)^9, \quad (1.1)$$

where m and Γ denote the mass and width of a resonance and g is the 3^- $\pi\pi$ resonance. Substituting the observed value $m_g=1663$ MeV, we obtain

$$\Gamma_g < 340\text{ MeV}, \quad (1.2)$$

whereas experimentally we have $\Gamma_g=111$ MeV. This numerical example suggests that the inequalities involving a small number of partial waves may not be very stringent. But by including more partial waves, the inequalities tend to become equalities, and therefore represent severe restrictions on the higher partial waves if the lower ones are given.

II. DERIVATION OF SUM RULES

On general axiomatic grounds,[2] one can write a Froissart-Gribov formula for the partial-wave amplitudes $f_l^{(i)}(t)$ for $l\geq 2$ in the region $0<t<4m_\pi^2$, where i

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[1] R. Roskies, Phys. Rev. D 2, 247 (1970).

[2] Y. S. Jin and A. Martin, Phys. Rev. 135, B1375 (1964).

denotes the isospin. Denoting the full amplitudes by $T^{(i)}(t,s)$ (the first variable indicates the channel in which the isospin is measured), we can write[3] (in units such that $4m_\pi^2=1$)

$$T^{(i)}(t,s)=f_0^{(i)}(t)+\frac{1}{\pi}\int_1^\infty ds'\,A^{[i]}(s',t)\times\left(\frac{1}{s'-s}+\frac{1}{s'+s+t-1}-\frac{2}{1-t}\ln\frac{s'}{s'+t-1}\right),\quad i=0,2 \tag{2.1}$$

$$T^{(1)}(t,s)=3f_1^{(1)}(t)\left(1+\frac{2s}{t-1}\right)+\frac{1}{\pi}\int_1^\infty ds'\,A^{[1]}(s',t)\times\left[\frac{1}{s'-s}-\frac{1}{s'+s+t-1}+\frac{6(1-t-2s)}{(1-t)^2}\times\left(\frac{2s'+t-1}{1-t}\ln\frac{s'}{s'+t-1}-2\right)\right], \tag{2.2}$$

where

$$A^{[i]}(s',t)=\sum_l\sum_j C_{ij}(2l+1)\times\operatorname{Im}f_l^{(j)}(s')P_l\left(1+\frac{2t}{s'-1}\right) \tag{2.3}$$

and C_{ij} is the crossing matrix

$$C_{ij}=\begin{bmatrix}1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6\end{bmatrix}. \tag{2.4}$$

It is clear from (2.1) and (2.2) that we have enforced the proper s,u symmetry at fixed t. However, crossing symmetry implies that $T^{(0)}(s,t)$ and $T^{(2)}(s,t)$ are symmetric under t,u interchange, while $T^{(1)}(s,t)$ is antisymmetric. Recalling that

$$T^{(i)}(s,t)=\sum_j C_{ij}T^{(j)}(t,s), \tag{2.5}$$

these t,u symmetry properties first allow us to determine $f_0^{(0)}(t)$, $f_0^{(2)}(t)$, and $f_1^{(1)}(t)$ and then imply restrictions on $A^{[i]}(s',t)$. These give rise to the sum rules.

We can express the symmetry property of $T^{(0)}(s,t)$ in a different form. Let us consider $T^{(0)}(s,t)$ as a function of t and u, with s determined from

$$s=1-t-u. \tag{2.6}$$

[3] See, e.g., R. Roskies, Nuovo Cimento 65A, 467 (1970), Appendix II.

Defining

$$\tilde{T}^{(0)}(t,u)=T^{(0)}(s,t), \tag{2.7}$$

the symmetry of $\tilde{T}^{(0)}(t,u)$ can be expressed as follows:

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)\tilde{T}^{(0)}(t,u)\bigg|_{t=u}=0, \tag{2.8}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)^3\tilde{T}^{(0)}(t,u)\bigg|_{t=u}=0, \tag{2.9}$$

$$\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial u^2}\tilde{T}^{(0)}(t,u)\quad\text{symmetric in } t,u. \tag{2.10}$$

Returning to the variables s and t, these equations become

$$\frac{\partial}{\partial t}T^{(0)}(s,t)\bigg|_{s=1-2t}=0, \tag{2.11}$$

$$\frac{\partial^3}{\partial t^3}T^{(0)}(s,t)\bigg|_{s=1-2t}=0, \tag{2.12}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}\right)^2\frac{\partial^2}{\partial s^2}T^{(0)}(s,t)\quad\text{symmetric in } t,u. \tag{2.13}$$

The advantage of writing the symmetry conditions in this way is that the unknowns $f_0^{(0)}$, $f_0^{(2)}$, and $f_1^{(1)}$ do not enter into (2.13). Similarly, $T^{(2)}(s,t)$ satisfies

$$\frac{\partial}{\partial t}T^{(2)}(s,t)\bigg|_{s=1-2t}=0, \tag{2.14}$$

$$\frac{\partial^3}{\partial t^3}T^{(2)}(s,t)\bigg|_{s=1-2t}=0, \tag{2.15}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}\right)^2\frac{\partial^2}{\partial s^2}T^{(2)}(s,t)\quad\text{symmetric in } t,u, \tag{2.16}$$

while $T^{(1)}(s,t)$ satisfies

$$T^{(1)}(s,t)\big|_{s=1-2t}=0, \tag{2.17}$$

$$\frac{\partial^2}{\partial t^2}T^{(1)}(s,t)\bigg|_{s=1-2t}=0, \tag{2.18}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}\right)^2\frac{\partial^2}{\partial s^2}T^{(1)}(s,t)\quad\text{antisymmetric in } t,u. \tag{2.19}$$

Equations (2.11), (2.14), and (2.17) can be solved for $f_0^{(0)}$, $f_0^{(2)}$, and $f_1^{(1)}$. This solution is given in the Appendix. These solutions can then be inserted into (2.12), (2.15), and (2.18). A tedious calculation shows that

(2.12) becomes

$$\int_1^\infty ds'\Bigg\{\left(\frac{d^2}{dt^2}[A^{[0]}(s',t)+5A^{[2]}(s',t)]\right)\frac{(1-3t)(2s'+t-1)}{(s'-1+2t)^2(s'-t)^2}+\left(\frac{d^2}{dt^2}A^{[1]}(s',t)\right)\frac{3(1-3t)^2}{(s'-t)^2(s'-1+2t)^2}$$
$$+\left(\frac{d}{dt}A^{[0]}(s',t)\right)\times\frac{2}{3}\left(\frac{1}{(s'-1+2t)^3}+\frac{5}{(s'-t)^3}+(1-3t)\frac{6t^2-9t+2+15s't-3s'^2-3s'}{(s'-t)^3(s'-1+2t)^3}\right)$$
$$+\left(\frac{d}{dt}A^{[2]}(s',t)\right)\times\frac{5}{3}\left(\frac{4}{(s'-t)^3}-\frac{10}{(s'-1+2t)^3}-(1-3t)\frac{6t^2-9t+2+15s't-3s'^2-3s'}{(s'-t)^3(s'-1+2t)^3}\right)$$
$$+\left(\frac{d}{dt}A^{[1]}(s',t)\right)\frac{3(1-3t)(3s'^2+15s't-7s'-5t+2)}{(s'-t)^3(s'-1+2t)^3}+[\tfrac{2}{3}A^{[0]}(s',t)-(5/3)A^{[2]}(s',t)+A^{[1]}(s',t)]$$
$$\times\Bigg[\frac{3}{(s'-t)^4}-\frac{12}{(s'-1+2t)^4}+(1-3t)\Bigg(\frac{-4}{(s'-1+2t)^3(s'-t)^2}+\frac{2}{(s'-1+2t)^2(s'-t)^3}$$
$$+\frac{12(2s'+t-1)}{(s'-1+2t)^4(s'-t)^2}-\frac{8(2s'+t-1)}{(s'-1+2t)^3(s'-t)^3}+\frac{3(2s'+t-1)}{(s'-1+2t)^2(s'-t)^4}\Bigg)\Bigg]\Bigg\}=0,\quad (2.20)$$

while (2.18) becomes

$$\int_1^\infty ds'\Bigg[\left(2\frac{dA^{[0]}(s',t)}{dt}-5\frac{dA^{[2]}(s',t)}{dt}\right)\frac{2s'+t-1}{(s'-1+2t)^2(s'-t)^2}-3\frac{dA^{[1]}(s',t)}{dt}\frac{(3t-1)}{(s'-1+2t)^2(s'-t)^2}$$
$$-[2A^{[0]}(s',t)-5A^{[2]}(s',t)+3A^{[1]}(s',t)]\frac{3s'^2-s'(1+3t)+t}{(s'-1+2t)^3(s'-t)^3}\Bigg]=0.\quad (2.21)$$

From (2.13) we find

$$S(s,t,u)=S(s,u,t)\,,\quad (2.22a)$$

where

$$S(s,t,u)=\int_1^\infty ds'\Bigg\{\left(\frac{d^2}{dt^2}[A^{[0]}(s',t)+5A^{[2]}(s',t)]\right)\left(\frac{1}{(s'-s)^3}+\frac{1}{(s'+s+t-1)^3}\right)$$
$$+3\left(\frac{d^2}{dt^2}A^{[1]}(s',t)\right)\left(\frac{1}{(s'-s)^3}-\frac{1}{(s'+s+t-1)^3}\right)-6\left(\frac{d}{dt}[A^{[0]}(s',t)+5A^{[2]}(s',t)+3A^{[1]}(s',t)]\right)\frac{1}{(s'-s)^4}$$
$$+12[A^{[0]}(s',t)+5A^{[2]}(s',t)+3A^{[1]}(s',t)]\frac{1}{(s'-s)^5}\Bigg\}\,,\quad (2.22b)$$

while

$$A(s,t,u)=-A(s,u,t)\,,\quad (2.23a)$$

where

$$A(s,t,u)=\int_1^\infty ds'\Bigg\{\left(\frac{d^2}{dt^2}[2A^{[0]}(s',t)-5A^{[2]}(s',t)]\right)\left(\frac{1}{(s'-s)^3}+\frac{1}{(s'+s+t-1)^3}\right)$$
$$+3\left(\frac{d^2}{dt^2}A^{[1]}(s',t)\right)\left(\frac{1}{(s'-s)^3}-\frac{1}{(s'+s+t-1)^3}\right)-6\left(\frac{d}{dt}[2A^{[0]}(s',t)-5A^{[2]}(s',t)+3A^{[1]}(s',t)]\right)\frac{1}{(s'-s)^4}$$
$$+12[2A^{[0]}(s',t)-5A^{[2]}(s',t)+3A^{[1]}(s',t)]\frac{1}{(s'-s)^5}\Bigg\}.\quad (2.23b)$$

TABLE I. Expressions for $\gamma_q^{(i)}(m)$.

	$\gamma_q^{(0)}(m)$	$\gamma_q^{(1)}(m)$	$\gamma_q^{(2)}(m)$
From (2.20)	$\frac{1}{3}m\{(m-1)[(-2)^{q-m+2}(9q-9m+23) -(3q-3m+11)] +(q-m+2)[3q-3m+7 -(-2)^{q-m+1}(9q-9m+19)]\}$	$m(m-1)[3q-3m+8+(-2)^{q-m+3}] -m[3(q-m+2)^2+2(q-m)+2 -(-2)^{q-m+3}(2q-2m+5)] +2(q-m+3)(q-m+1)[1-(-2)^{q-m+2}]$	$(5/3)m\{(m-1)[3m-3q-8 -(-2)^{q-m+3}]+(q-m+2) \times[3q-3m+10-(-2)^{q-m+3}]\}$
From (2.21)	$\frac{1}{9}m[(-2)^{q-m+4}-6q+6m-16]$	$\frac{1}{3}m[3q-3m+10+(-2)^{q-m+3}(3q-3m+8)] -\frac{2}{3}\{q-m+3-(-2)^{q-m+1} \times[3(q-m+2)^2+q-m]\}$	$(5/9)m[3q-3m+8+(-2)^{q-m+3}]$

We can omit the implications of (2.15) and (2.16) because they can be recovered by combining the results of I on the $\pi^0\pi^0$ amplitude with those involving only $T^{(0)}(s,t)$.

Equations (2.20)–(2.23) are one form of the sum rules on the absorptive parts. We can cast them in a form analogous to those in Sec. IV of I by expanding (2.20) and (2.21) in a power series around $t=\frac{1}{3}$. As in I, we can expand $S(s,t,u)$ and $A(s,t,u)$ as double power series around $t=\frac{1}{3}$ and $u=\frac{1}{3}$ and retain those terms with the proper symmetry. The results are as follows: From (2.20) and (2.21),

$$\sum_l \int_1^\infty ds' \frac{2l+1}{(s'-\frac{1}{3})^{q+4}} \sum_{m=0}^{q+1} \frac{(2z)^m P_l^{(m)}(z)}{m!} \times \sum_{i=0}^{2} \gamma_q^{(i)}(m)\, \mathrm{Im} f_l^{(i)}(s')=0, \quad (2.24)$$

where

$$z=(s'-\tfrac{1}{3})/(s'-1), \quad q=0,\,1,\,2,\,\ldots, \quad (2.25)$$

and $\gamma_q^{(i)}(m)$ is given in Table I. $P_l^{(m)}$ denotes the mth

TABLE II. Expressions for $\epsilon_{qr}^{(0)}(m)$.

	$\epsilon_{qr}^{(0)}(m)$	$\epsilon_{qr}^{(1)}(m)$	$\epsilon_{qr}^{(2)}(m)$
From (2.22); $q=2,3,\ldots$; $r=0,1,\ldots,[q/2]-1$	$\frac{3m(m-1)(-1)^{q-m-1}}{2} \times\left[\binom{q-m+1}{r}-\binom{q-m+1}{q-r-1}\right] +3m(q-m+1)(-1)^{q-m-1} \times\left[\binom{q-m}{r}-\binom{q-m}{q-r-1}\right] +\frac{3}{2}(q-m+1)(q-m)(-1)^{q-m-1} \times\left[\binom{q-m-1}{r}-\binom{q-m-1}{q-r-1}\right] +\frac{1}{2}m(m-1)(\delta_{m,q-r+1}-\delta_{m,r+2})$	$-\frac{3}{2}m(m-1)(\delta_{m,q-r+1}-\delta_{m,r+2})$	$\frac{5}{2}m(m-1)(\delta_{m,q-r+1}-\delta_{m,r+2})$
From (2.23); $q=1,2,\ldots$; $r=0,1,\ldots,[(q-1)/2]$	$-m(m-1)(\delta_{m,q-r+1}+\delta_{m,r+2})$	$3m(m-1)(-1)^{q-m-1} \times\left[\binom{q-m+1}{r}+\binom{q-m+1}{q-r-1}\right] +6m(q-m+1)(-1)^{q-m-1} \times\left[\binom{q-m}{r}+\binom{q-m}{q-r-1}\right] +3(q-m+1)(q-m)(-1)^{q-m-1} \times\left[\binom{q-m-1}{r}+\binom{q-m-1}{q-r-1}\right] +\frac{3}{2}m(m-1)(\delta_{m,q-r+1}+\delta_{m,r+2})$	$\frac{5}{2}m(m-1)(\delta_{m,q-r+1}+\delta_{m,r+2})$

derivative of the Legendre polynomial.

$$\sum_l \int_1^\infty ds' \frac{2l+1}{(s'-\frac{1}{3})^{4+q}} \sum_{m=0}^{q+1} \frac{(2z)^m P_l^{(m)}(z)}{m!} \times \sum_{i=0}^{2} \delta_{qr}^{(i)}(m) \operatorname{Im} f_l^{(i)}(s') = 0, \quad (2.26)$$

$$q=1, 2, \ldots; \quad r=0, 1 \ldots [q/2],$$
$$\delta_{qr}^{(i)}(m) = (q-m+3)(q-m+2)\epsilon_{qr}^{(i)}(m), \quad (2.27)$$

and $\epsilon_{qr}^{(i)}(m)$ is given in Table II. Here $[q/2]$ means the largest integer $\leqslant q/2$.

For completeness, we add the results of I:

$$\sum_l \int_1^\infty ds' \frac{2l+1}{(s'-\frac{1}{3})^{4+9}} \sum_{m=1}^{q+1} \frac{(2z)^m P_l^{(m)}(z)}{m!} \times \sum_{i=0}^{2} \mu_{qr}^{(i)}(m) \operatorname{Im} f_l^{(i)}(s') = 0, \quad (2.28)$$

with

$$\mu_{qr}^{(1)}(m) = 0, \quad (2.29)$$

$$\mu_{qr}^{(2)}(m) = 2\mu_{qr}^{(0)}(m) = (q-m+3)\Bigg\{ m\delta_{m,r+1} - m\delta_{m,q-r+2} + (-1)^{q-m-1}\left[(r+1)\binom{q-m+2}{q+1-r} - (q+2-r)\binom{q-m+2}{r} \right]\Bigg\}, \quad (2.30)$$

$$q=1, 2, \ldots; \quad r=0, 1, \ldots, [q/2].$$

Equations (2.24), (2.26), and (2.28) are not all independent. In fact, one can show that for a given q there are altogether $q+1$ independent equations.

It is easily verified that $\operatorname{Im} f_0^{(0)}$, $\operatorname{Im} f_0^{(2)}$, and $\operatorname{Im} f_1^{(1)}$ always enter into the sum rules with coefficient zero, so that the s and p waves are never involved in the sum rules.

III. DERIVATION OF INEQUALITIES

We shall now combine the results of Sec. II with the positivity of $\operatorname{Im} f_l^{(i)}(s)$ in order to derive inequalities involving only the low partial waves. If we have a relation

$$\sum_l (2l+1) \sum_{i=0}^{2} \int_1^\infty ds' \operatorname{Im} f_l^{(i)}(s') F_l^{(i)}(s') = 0, \quad (3.1)$$

where $F_l^{(i)}(s')$ are known, and if $F_l^{(i)}(s')$ satisfy

$$F_l^{(i)}(s') \geqslant 0, \quad l>l_0, \quad i=0, 1, 2, \quad (3.2)$$

then we have

$$\sum_{l=0}^{l_0} (2l+1) \sum_{i=0}^{2} \int_1^\infty ds' \operatorname{Im} f_l^{(i)}(s') F_l^{(i)}(s') \leqslant 0. \quad (3.3)$$

This relation involves only $l<l_0$, and is the desired inequality.

For example, consider the terms (2.24), (2.26), and (2.28) with $q=1$. There are two linearly independent relations which give

$$\sum_l (2l+1) \int_1^\infty \frac{ds'}{(s'-\frac{1}{3})^5} \{ z^2 P_l''(z)[-\operatorname{Im} f_l^{(0)}(s') + \tfrac{9}{2} \operatorname{Im} f_l^{(1)}(s') + \tfrac{5}{2} \operatorname{Im} f_l^{(2)}(s')] - 9zP_l'(z) \operatorname{Im} f_l^{(1)}(s') + 9P_l(z) \operatorname{Im} f_l^{(1)}(s') \} = 0 \quad (3.4)$$

and

$$\sum_l (2l+1) \int_1^\infty \frac{ds'}{(s'-\frac{1}{3})^5} \times \{ z^2 P_l''(z)[\operatorname{Im} f_l^{(0)}(s') + 2 \operatorname{Im} f_l^{(2)}(s')] - \tfrac{3}{2} zP_l'(z)[\operatorname{Im} f_l^{(0)}(s') + 2 \operatorname{Im} f_l^{(2)}(s')] \} = 0. \quad (3.5)$$

Multiplying (3.5) by α and adding to (3.4) gives

$$\sum_l (2l+1) \int_1^\infty \frac{ds'}{(s'-\frac{1}{3})^5} \times \{ \operatorname{Im} f_l^{(0)}(s')[(\alpha-1)z^2 P_l''(z) - \tfrac{3}{2}\alpha z P_l'(z)] + \operatorname{Im} f_l^{(1)}(s')[\tfrac{9}{2} z^2 P_l''(z) - 9zP_l'(z) + 9P_l(z)] + \operatorname{Im} f_l^{(2)}(s')[(\tfrac{5}{2}+2\alpha) z^2 P_l''(z) - 3\alpha z P_l'(z)] \} = 0. \quad (3.6)$$

This is of the form (3.1) with

$$F_l^{(0)}(s') = (\alpha-1)z^2 P_l''(z) - \tfrac{3}{2}\alpha z P_l'(z),$$
$$F_l^{(1)}(s') = \tfrac{9}{2}[z^2 P_l''(z) - 2zP_l'(z) + 2P_l(z)], \quad (3.7)$$
$$F_l^{(2)}(s') = (\tfrac{5}{2}+2\alpha) z^2 P_l''(z) - 3\alpha z P_l'(z).$$

To convert this to an inequality we must satisfy (3.2). But for large l, $F_l^{(1)}(s')>0$, independent of α. Consequently, α must be chosen so that $F_l^{(0)}(s')$, $F_l^{(2)}(s')>0$ for large l, i.e.,

$$\alpha>1. \quad (3.8)$$

With this choice,

$$F_l^{(2)}(s') > 2F_l^{(0)}(s'), \quad (3.9)$$

so it suffices to choose α such that

$$F_l^{(0)}(s') \geqslant 0, \quad l>l_0 \quad (3.10)$$

i.e.,

$$\frac{zP_l''(z)}{P_l'(z)} \geqslant \frac{3\alpha}{2(\alpha-1)}, \quad l>l_0, \quad \text{all } z\geqslant 1. \quad (3.11)$$

The left-hand side of (3.11) is a minimum at $z=\infty$, and there the equation becomes

$$l-1 \geqslant \frac{3\alpha}{2(\alpha-1)}. \tag{3.12}$$

One cannot satisfy this equation for $l=2$, since $\alpha>1$, but for

$$\alpha \geqslant 2 \tag{3.13}$$

we have

$$\frac{zP_l''(z)}{P_l'(z)} \geqslant \frac{3\alpha}{2(\alpha-1)}, \quad l \geqslant 4. \tag{3.14}$$

Choosing $\alpha=2$, we derive the inequality (reinserting the dimension of mass)

$$\begin{aligned} 5\int_{4\mu^2}^{\infty} \frac{ds'}{(s'-4\mu^2/3)^5} 6\,\mathrm{Im} f_2^{(0)}(s')\left(\frac{s'-4\mu^2/3}{s'-4\mu^2}\right)^2 \\ > 5\int_{4\mu^2}^{\infty} \frac{ds'}{(s'-4\mu^2/3)^5} \frac{3}{2}\,\mathrm{Im} f_2^{(2)}(s')\left(\frac{s'-4\mu^2/3}{s'-4\mu^2}\right)^2 \\ +7\int_{4\mu^2}^{\infty} \frac{ds'}{(s'-4\mu^2/3)^5} \frac{45}{2}\,\mathrm{Im} f_3^{(1)}(s')\left(\frac{s'-4\mu^2/3}{s'-4\mu^2}\right)^3. \end{aligned} \tag{3.15}$$

Notice that because of the weight functions $1/(s'-4\mu^2/3)^5$, the integral cuts off rapidly for large s' and is therefore sensitive only to the low-energy region. In the narrow-resonance approximation, saturating the left-hand side with the f^0 resonance, and the $I=1$ part of the right-hand side with the g meson, one obtains the result

$$\Gamma_g < \frac{4}{21}\left(\frac{m_{f^0}}{m_g}\right)^9 \Gamma_{f^0}. \tag{3.16}$$

Experimentally, this equation reads

$$111 \text{ MeV} < 340 \text{ MeV}.$$

Of course, by including more terms in the inequality it becomes stronger and stronger. However, the inequality then involves higher partial waves, and so is harder to verify experimentally. For example, including an $I=0$, $l=4$ resonance, the relation becomes

$$\frac{\Gamma_{f^0}}{(m_{f^0})^9} > \frac{21}{4}\frac{\Gamma_g}{(m_g)^9} + \frac{9}{2}\frac{\Gamma_{I=0,l=4}}{(m_{I=0,l=4})^9}. \tag{3.17}$$

One can apply these techniques to the family of relations (2.24), (2.26), and (2.28). As α increases, the weight functions cut off more and more rapidly, and therefore the results will be sensitive to the partial-wave amplitudes at lower and lower energies. It also appears that as q increases, so does the value of l_0 beyond which (3.2) holds, so that more partial waves will enter into the left-handside of (3.3). For example, if $q=2$, one cannot satisfy $F_l^{(i)}(s')>0$ for $l \geqslant 4$, so that no relation involving only $l=2$ and $l=3$ can be obtained.

IV. NO SUBTRACTIONS FOR ISOSPIN 2

We have seen in Sec. II that none of the sum rules involves s or p waves. This can be traced to the assumption of subtractions for the fixed t dispersion relations $T^{(i)}(t,s)$. If we interpret the absence of $I=2$ mesons to mean that $T^{(2)}(t,s)$ is unsubtracted [in Regge language, we assume that $\alpha_{I=2}(0)<0$], we should be able to recover some information on s and p waves. These waves are currently of great interest since they dominate the low-energy region of $\pi\pi$ scattering. There are various proposed s-wave phase shifts[4] up to about 1 BeV, and it would be useful to be able to discriminate between them on the basis of our sum rules.

In this section, we shall show that the no subtraction ansatz does lead to sum rules involving s and p waves, but that these cannot be reformulated as inequalities involving only low partial waves. This ansatz is therefore of very little use for our purposes.

If $T^{(2)}(t,s)$ is unsubtracted, we can write

$$f_0^{(2)}(t) = \frac{2}{\pi(1-t)}\int ds'\, A^{[2]}(s',t) \ln\frac{s'}{s'+t-1}. \tag{4.1}$$

Combining this relation with our solution for $f_0^{(2)}(t)$ in the Appendix, and differentiating twice with respect to t to eliminate the arbitrary constants appearing there, we find the following sum rule:

$$\begin{aligned} \int_1^{\infty} ds' \Bigg\{ & \frac{d^2A^{[2]}(s',t)}{dt^2}\left(\frac{1}{s'-1+2t}+\frac{1}{s'-t}\right)+\frac{1}{6}\frac{dA^{[2]}(s',t)}{dt}\left[\frac{5}{(s'-t)^2}-\frac{17}{(s'-1+2t)^2}\right] \\ & +\frac{1}{2}\frac{dA^{[1]}(s',t)}{dt}\left(\frac{1}{s'-t}-\frac{1}{s'-1+2t}\right)^2+\frac{1}{3}\frac{dA^{[0]}(s',t)}{dt}\left[\frac{1}{(s'-1+2t)^2}-\frac{1}{(s'-t)^2}\right] \\ & +\frac{A^{[0]}(s',t)}{3}\left[\frac{2s'+t-1}{(s'-t)^2(s'-1+2t)^2}-\frac{2}{(s'-t)^3}-\frac{4}{(s'-1+2t)^3}\right] \\ & +\frac{A^{[1]}(s',t)}{2}\left[\frac{2s'+t-1}{(s'-1+2t)^2(s'-t)^2}+\frac{2}{(s'-t)^3}-\frac{4}{(s'-1+2t)^3}\right] \\ & +\frac{A^{[2]}(s',t)}{6}\left[\frac{20}{(s'-1+2t)^3}-\frac{2}{(s'-t)^3}-\frac{5(2s'+t-1)}{(s'-1+2t)^2(s'-t)^2}\right]\Bigg\}=0. \end{aligned} \tag{4.2}$$

[4] See, e.g., Proceedings of the Argonne Conference on $\pi\pi$ and $K\pi$ Interactions, 1969 (unpublished).

Substituting (2.3) for $A^{[i]}(s,t)$, one easily sees that this relation involves s and p waves. As it stands, however, for large l, it involves $I=1$ waves with an opposite sign from $I=0$ and $I=2$ waves, because for large l and fixed s' and t, the term with

$$\frac{d^2A^{[2]}(s',t)}{dt^2}=\frac{4}{(s'-1)^2}\sum_l (2l+1)\times[\tfrac{1}{3}\operatorname{Im}f_l^{(0)}(s')-\tfrac{1}{2}\operatorname{Im}f_l^{(1)}(s')+\tfrac{1}{6}\operatorname{Im}f_l^{(2)}(s')]P_l''\left(1+\frac{2t}{s'-1}\right) \quad (4.3)$$

dominates.

The question then arises as to whether this realtion can be combined with those of Sec. II [i.e., (2.24), (2.26), or (2.28)] to produce an expression in which all partial waves enter with the same sign for large l. We now show that this is impossible.

It is easily seen that for large s', the coefficient of $\operatorname{Im}f_l^{(i)}(s')$ in (4.2) is $O(1/s'^3)$. In the relations of Sec. II, the coefficient is at least $O(1/s'^4)$. Consequently, in any linear combination of (4.2) with any relation of Sec. II, the term involving (4.2) will dominate for large s', and this gives opposite signs to $I=1$ and $I=0$ partial waves for large l. However, writing (4.2) as

$$\int_1^\infty ds'\, a(s',t)=0, \quad (4.4)$$

we see that this relation holds for t in the interval [0,1]. Could we then choose a weight function $w(t)$ so that

$$\int_0^1 dt\, w(t)\int_1^\infty ds'\, a(s',t)=0 \quad (4.5)$$

could be combined with equations of Sec. II to yield partial-wave amplitudes of a given sign for large l?

For large l, the term involving (4.2) would be approximately

$$\int_0^1 dt\, w(t)\int_1^\infty ds'\frac{4}{(s'-1)^2}\sum_l (2l+1)\times[\tfrac{1}{3}\operatorname{Im}f_l^{0}(s')-\tfrac{1}{2}\operatorname{Im}f_l^{(1)}(s')+\tfrac{1}{6}\operatorname{Im}f_l^{(2)}(s')]\times P_l''\left(1+\frac{2t}{s'-1}\right)\left(\frac{1}{s'-1+2t}+\frac{1}{s'-t}\right), \quad (4.6)$$

whereas those from Sec. II would involve

$$\sum_l\int_1^\infty (2l+1)\frac{1}{(s'-1)^{q+1}}\frac{1}{(s'-\frac{1}{3})^3}P_l^{(q+1)}\left(\frac{s'-\frac{1}{3}}{s'-1}\right)\times\sum_i \operatorname{Im}f_l^{(i)}(s')\beta_q^{(i)},\quad q=0,1,\ldots, \quad (4.7)$$

where $\beta_q^{(i)}$ are some constants, independent of l and s'. In order that (4.6) be comparable to (4.5) for large l, we need

$$\left|\int_0^1 dt\, w(t)P_l''\left(1+\frac{2t}{s'-1}\right)\left(\frac{1}{s'-1+2t}+\frac{1}{s'-t}\right)\right| < C\frac{1}{(s'-\frac{1}{3})^3}\frac{1}{(s'-1)^{q-1}}P_l^{(q+1)}\left(\frac{s'-\frac{1}{3}}{s'-1}\right) \quad (4.8)$$

for some constant C, for all large s' and large l, and some q.

Let $s'=l^2/\alpha^2$ in (4.8), and consider the limit $l\to\infty$, α fixed. Using the formula[5]

$$\lim_{n\to\infty} P_n\left(\cos\frac{z}{n}\right)=J_0(z), \quad (4.9)$$

we find that the left-hand side of (4.8) becomes for large l

$$\tfrac{1}{2}\alpha l^2\int_0^1 dt\frac{w(t)}{t}J_2(2i\sqrt{(\alpha t)}), \quad (4.10)$$

while the right-hand side is $O(1/l^2)$. For large l, (4.8) then implies that the integral in (4.10) vanishes for all α, and therefore that $w(t)$ vanishes. Consequently, (4.2) cannot be combined with any of the relations of Sec. II to give partial waves with the same sign for all isospins for large l.

APPENDIX

The equations

$$\left.\frac{d}{dt}T^{(0)}(s,t)\right|_{s=1-2t}=\left.\frac{d}{dt}T^{(2)}(s,t)\right|_{s=1-2t}=0, \quad (A1)$$

$$T^{(1)}(s,t)|_{s=1-2t}=0, \quad (A2)$$

$$T^{(i)}(s,t)=\sum C_{ij}T^{(j)}(t,s), \quad (A3)$$

[5] See, for example, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, p. 147.

where C_{ij} is given by (2.4) and $T^{(j)}(t,s)$ by (2.1)–(2.3), can be solved for $f_0^{(0)}(t)$, $f_0^{(2)}(t)$, and $f_1^{(1)}(t)$ to give

$$f_0^{(0)}(t)=(1-3t)\int^t X(t')dt'+5\int^t Y(t')dt'+4Z(t)+5a+2b(3t-1)\,, \tag{A4}$$

$$f_0^{(2)}(t)=-\tfrac{1}{2}(1-3t)\int^t X(t')dt'+2\int^t Y(t')dt'-2Z(t)+2a-b(3t-1)\,, \tag{A5}$$

$$f_1^{(1)}(t)=-\tfrac{1}{2}(t-1)\int^t X(t')dt'-b(1-t)\,, \tag{A6}$$

where a and b are arbitrary constants and

$$\begin{aligned}X(t)=\frac{1}{9\pi}\int_1^\infty ds'\bigg\{&\left(\frac{d}{dt}6A^{[1]}(s',t)\right)\left[\frac{1}{(s'-1+2t)(s'-t)}-\frac{6}{(1-t)^2}\left(\frac{2s'+t-1}{1-t}\ln\frac{s'}{s'+t-1}-2\right)\right]\\&+3A^{[1]}(s',t)\left[\frac{1-t-2s'}{(s'-1+2t)^2(s'-t)^2}-12\frac{d}{dt}\frac{1}{(1-t)^2}\left(\frac{2s'+t-1}{1-t}\ln\frac{s'}{s'+t-1}-2\right)\right]\\&-[2A^{[0]}(s',t)-5A^{[2]}(s',t)]\frac{2s'+t-1}{(s'-1+2t)^2(s'-t)^2}\bigg\}\,,\end{aligned} \tag{A7}$$

$$\begin{aligned}Y(t)=\frac{1}{9\pi}\int_1^\infty ds'\bigg\{&[A^{[0]}(s',t)+2A^{[2]}(s',t)]\left(\frac{1}{(s'-t)^2}+\frac{d}{dt}\frac{2}{1-t}\ln\frac{s'}{s'+t-1}\right)\\&-\left[\frac{d}{dt}A^{[0]}(s',t)+2\frac{d}{dt}A^{[2]}(s',t)\right]\left(\frac{1}{s'-1+2t}+\frac{1}{s'-t}-\frac{2}{1-t}\ln\frac{s'}{s'+t-1}\right)\bigg\}\,,\end{aligned} \tag{A8}$$

$$\begin{aligned}Z(t)=-\frac{1}{3\pi}\int_1^\infty ds'\bigg\{&\left[\frac{2A^{[0]}(s',t)-5A^{[2]}(s',t)}{6}\right]\left(\frac{1}{s'-1+2t}+\frac{1}{s'-t}-\frac{2}{1-t}\ln\frac{s'}{s'+t-1}\right)\\&+\frac{A^{[1]}(s',t)}{2}\left[\frac{1}{s'-1+2t}-\frac{1}{s'-t}+\frac{6(3t-1)}{(1-t)^2}\left(\frac{2s'+t-1}{1-t}\ln\frac{s'}{s'+t-1}-2\right)\right]\bigg\}\,.\end{aligned} \tag{A9}$$