Current Algebra beyond the Tree Approximation

HOWARD J. SCHNITZER*
The Rockefeller University, New York, New York 10021†
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A method is presented for constructing current-algebra amplitudes which satisfy (1) threshold theorems, (2) crossing symmetry, (3) approximate unitarity, and (4) cut-plane analyticity; and reduce to the usual tree approximation in the narrow-resonance limit. Pion-pion scattering is considered in detail to illustrate the method. A derivation of the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relation is provided which makes no reference to vector-meson dominance.

I. INTRODUCTION

THE current-algebra approach to hadron physics has been extremely fruitful as a unified phenomenology of weak, electromagnetic, and strong processes. The underlying assumptions, once accepted, lead to a number of sum rules and soft-pion threshold theorems.[1] However, these predictions do not exhaust the content of the theory, since the current algebra still constrains amplitudes away from threshold, much as does ordinary gauge invariance in hadron electrodynamics. An important question is how best to utilize the additional information contained in the current-commutation relations.

An ingenious way to approach this problem is to construct realizations of the current algebra by means of a phenomenological Lagrangian.[2] Such a Lagrangian, when evaluated in tree-graph approximation (no closed-loop graphs), satisfies all the current-algebra constraints and threshold theorems. This approximation is then the first term in a systematic expansion of the S matrix, with the number of closed loops characterizing the order of the perturbation, and the sum of all graphs of a fixed order separately satisfying the constraints.[3] In spite of the beauty of this approach, it is beset with a number of difficulties. Among these are the following:

(1) The tree approximation to the S matrix violates unitarity badly, since all amplitudes are real.
(2) It is very difficult (or impossible) to compute higher-order corrections, since the Lagrangians are nonlinear and nonrenormalizable.
(3) Even if such corrections could be computed, the resultant renormalized perturbation series would probably diverge, since the perturbation parameter has the strength characteristic of strong interactions.
(4) There is no reason why the tree approximation should be a good approximation to the complete theory even if it could be calculated by these techniques.

An alternative method for constructing realizations follows from the Ward-Takahashi identities of current algebra.[4–6] *Any* (coupled) set of amplitudes which satisfies the Ward identities automatically satisfies all the current-algebra constraints. It is obvious that the solution to the Ward identities is not unique, so that one must add additional dynamical information if there is to be any hope of completely specifying the amplitudes. The advantage of the Ward-identity method for such a program is that one may explore the consequences of various dynamical requirements without fear of violating current algebra. For example, if one postulates certain smoothness conditions for a set of amputated one-particle irreducible amplitudes which satisfy the Ward identities, one obtains the same S matrix as that of the tree approximation to phenomenological Lagrangians.[4–6] Both versions of the tree approximation have the same shortcomings.

In this paper we show that it is possible to avoid this approximation by making direct use of unitarity as a dynamical constraint. As a result we are able to formulate a method for constructing amplitudes which satisfy the following: (1) the Ward identities, (2) threshold theorems, (3) crossing symmetry, (4) approximate two-particle unitarity, and (5) cut-plane analyticity, and which reduce to the usual tree approximation in the narrow-resonance limit. The steps of the method for on-shell scattering processes are[7]:

(1) Solve the Ward identities connected to the amplitude of interest by the techniques of Ref. 5,

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[1] For a review of the subject, see S. L. Adler and R. F. Dashen, *Current Algebra* (Benjamin, New York, 1968); S. Weinberg, Rapporteur's talk, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 253.
[2] S. Weinberg, Phys. Rev. Letters **18**, 188 (1967); J. Schwinger, Phys. Letters **24B**, 473 (1967); J. Wess and B. Zumino, Phys. Rev. **163**, 1722 (1967). See S. Gasiorowicz and D. Geffin, Rev. Mod. Phys. **41**, 531 (1969), for a review of the subject.
[3] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, S. Coleman, J. Wess, and B. Zumino, *ibid.* **177**, 2247 (1969).
[4] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).
[5] I. S. Gerstein and H. J. Schnitzer, Phys. Rev. **170**, 1638 (1968). We use the normalizations and conventions of this paper.
[6] The Ward-identity method is summarized in H. J. Schnitzer, Proceedings of 1969 Erice Summer School (unpublished).
[7] For an abbreviated report, see H. J. Schnitzer, Phys. Rev. Letters **24**, 1384 (1970).

which explicitly displays all one-particle reducible amplitudes implied by the identities.

(2) Express the amplitude, evaluated on the mass shell of the external particles, in terms of propagators, form factors, and seagulls.

(3) Formulate the direct-channel two-particle partial-wave unitarity relations which couple the two-, three-, and four-point functions.

(4) Assume that the imaginary part of the "full seagull" is negligible in the elastic region, and construct the modified partial-wave amplitudes which result from this assumption and the unitarity equations.

(5) Solve for the form factors and two-point functions by means of the Omnès integral equation or an effective-range expansion, so as to satisfy the unitarity equations and guarantee correct analyticity properties.

(6) Assume that two-point functions and form factors so calculated may be analytically continued to the spacelike region, and insert them into the amplitude constructed in step (2).

(7) Estimate the seagull in terms of a simple model satisfying the hypothesis of step (4).

(8) Determine the free parameters of the model from unitarity and the consistency conditions of step (4).

The resulting amplitude has all the properties advertised above. One of the advantages of this technique is that there is no committment to the existence of resonances. Hopefully, they can be derived from the general structure of the theory, unitarity, and analyticity.[8,9]

Pion-pion scattering is a particularly interesting problem of current algebra.[10] We use it to illustrate the details of our point of view, and to discuss the justification of the various steps of the proposed method. Unfortunately, we are not able to comment as to the uniqueness of our amplitude.

II. STRUCTURE OF AMPLITUDE

The general structure of an amplitude satisfying current algebra can be expressed as the solution to a coupled set of Ward identities. For the particular case of $\pi\pi$ scattering, the techniques of Refs. 4 and 5, and Appendix A, lead to a simple and explicit structure for the amplitude describing $\Pi_a(q_1)+\Pi_b(q_2)\rightarrow\Pi_c(q_3)+\Pi_d(q_4)$, which on the mass shell is[11]

$$\begin{aligned}
T(s,t,u)^{abcd}&\\
&=q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}T_c(q_1,q_2;\,-q_3)^{\mu\nu\lambda\sigma}\\
&\quad-\tfrac{1}{3}F_\pi^{-2}[3N(N+2)-4]\\
&\quad\times[P_0+\tfrac{2}{5}P_2]^{ab,cd}+F_\pi^{-2}(-\tfrac{1}{3}+K_1)\\
&\quad\times\left\{[P_1]^{ab,cd}(u-t)+\begin{pmatrix}s\\b\end{pmatrix}\leftrightarrow\begin{pmatrix}t\\c\end{pmatrix}+\begin{pmatrix}s\\a\end{pmatrix}\leftrightarrow\begin{pmatrix}u\\c\end{pmatrix}\right\}\\
&\quad+\{[P_0]^{ab,cd}[F_0(s)-1]^2\Delta_0(s)^{-1}+\text{crossed terms}\}\\
&\quad+\{[P_2]^{ab,cd}[F_2(s)-1]^2\Delta_2(s)^{-1}+\text{crossed terms}\}\\
&\quad-2\{[P_1]^{ab,cd}(u-t)[F_1(s)-K_1]^2\Delta_V(s)^{-1}\\
&\qquad\qquad+\text{crossed terms}\}\\
&=(\text{full seagull})+(S+P\text{-wave polynomials})\\
&\qquad+(\text{full } S+P\text{-wave tree structures})\,. \qquad (2.1)
\end{aligned}$$

In this equation, $T_c(q_1,\,q_2;\,-q_3)^{\mu\nu\lambda\sigma}$ is the "seagull"[12] for axial-axial scattering evaluated on the pion mass shell; the $[P_I]^{ab,cd}$ are s-channel projection operators for isospin I; $F_1(s)$ is the electromagnetic form factor of the pion; $F_0(s)$ ($F_2(s)$) is the $I=0$ (2) form factor of the $\pi\pi\sigma$ vertex; $\Delta_I(s)$ is the two-point function for the scalar field σ^{ab} projected to isospin I;

$$\Delta_V(s)=\int\frac{dm^2\rho_V(m^2)}{m^2-s}$$

is the coefficient of $g_{\mu\nu}$ in the two-point function of isovector currents, $C_V=\Delta_V(0)$; $K_1=1+\frac{1}{2}C_VF_\pi^{-2}$; N denotes the representation $(\frac{1}{2}N,\frac{1}{2}N)$ of chiral $SU(2)\times SU(2)$ to which ∂A and the σ field belong, and $m_\pi=1$ throughout.

We emphasize that Eq. (2.1) is *exact* given the following hypotheses: (1) local, chiral $SU(2)\times SU(2)$ current-commutation relations, (2) the equal-time commutator $[A_0{}^a(x),\partial A^b(0)]\delta(x^0)=\sigma^{ab}(x)\delta^4(x)$ defines a local, scalar field, (3) $\partial A(x)$ and $\sigma^{ab}(x)$ transform according to the $(\frac{1}{2}N,\frac{1}{2}N)$ representation of chiral $SU(2)\times SU(2)$, (4) the Weinberg first sum rule is valid,[13] and (5) the two-point function $\int d^4x\, e^{iqx}\times\langle T\{\partial A(x)\partial A(0)\}\rangle_0$ is dominated by the pion pole. Assumptions (4) and (5) are made for technical convenience, and may be relaxed if need be.

[8] Other attempts to combine current algebra and unitarity are J. Iliopoulos, Nuovo Cimento **53A**, 552 (1968); L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 346 (1968); A. Amatya, A. Pagnamenta, and B. Renner, Phys. Rev. **172**, 1755 (1968); R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, *ibid.* **175**, 1820 (1968); J. J. Brehm, E. Golowich, and S. C. Prasad, Phys. Rev. Letters **23**, 666 (1969); R. Rockmore, *ibid.* **24**, 541 (1970); P. V. Collins, Nucl. Phys. **B21**, 577 (1970). These papers do *not* present a program to compute the complete $\pi\pi$ amplitude $T(s,t)$, valid for a range of energies, but concentrate on special features of low-energy $\pi\pi$ scattering.

[9] An alternative program to implement unitarity in current-algebra amplitudes has been developed by J. L. Basdevant and B. W. Lee, Phys. Letters **29B**, 437 (1969), who apply the Padé approximation to the renormalized σ model. However, this method seems to be limited to renormalizable field theories. Our techniques are closer to the spirit of S-matrix theory.

[10] S. Weinberg, Phys. Rev. Letters **17**, 616 (1967); see also N. N. Khuri, Phys. Rev. **153**, 1477 (1967).

[11] See Eq. (43) of Ref. 5.

[12] In perturbation theory it is conventional to call our "full trees" the amputated, one-particle *reducible* parts. Similarly our "full seagull" is the amputated, one-particle *irreducible* part.

[13] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

By construction, (2.1) satisfies the Ward identities, crossing symmetry, and the appropriate threshold theorems. This last property can easily be verified as a consequence of the normalization of the form factors, and the suppression of the seagull at threshold. That is, (2.1) predicts the Weinberg scattering lengths[10] to $O(m_\pi^2/M^2)$ since (a) $F_1(0)=1$ and $F_{0,2}(0)=1+O(m_\pi^2/M^2)$, which follows from the Ward identities for the three-point functions, (b) the seagull is $O(m_\pi^4/M^2F_\pi^2)$, and (c) $F(4m_\pi^2)=F(0)+O(m_\pi^2/M^2)$.

We characterize the corrections to the various threshold theorems by a typical mass M (not necessarily the same for each), which represents the energy at which important resonant effects appear in the channel in question.

One can get further insight into the structure of the $\pi\pi$ amplitude by considering the s-channel partial-wave projection of (2.1). The partial-wave amplitudes for S and P waves have the general form

$$T(s)=h(s)+F(s)^2\Delta(s)^{-1}, \tag{2.2}$$

where

$$h(s)=R(s)-2K\Gamma(s)+K^2\Delta(s)^{-1}, \tag{2.3}$$

with $\Gamma(s)=F(s)\Delta(s)^{-1}$, and

$$\begin{aligned} K&=1 \quad \text{for } S \text{ waves} \\ &=K_1 \quad \text{for } P \text{ waves.} \end{aligned}$$

We have defined $R(s)=$ the partial-wave projection of $\{t_c(s,t)+\text{polynomials}+(t)+(u) \text{ channel trees}\}$, and $t_c(s,t)=q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}T_c(q_1,q_2;-q_3)^{\mu\nu\lambda\sigma}$ for notational convenience. Stated in other words, $R(s)$ is the partial-wave projection of

$$\begin{aligned} R(s,t)^{ab,cd}=&\,t_c(s,t)^{ab,cd}-\tfrac{1}{3}F_\pi^{-2}[3N(N+2)-4] \\ &\times[P_0+\tfrac{2}{5}P_2]^{ab,cd}+F_\pi^{-2}(-\tfrac{1}{3}+K_1) \\ &\times\left\{[P_1]^{ab,cd}(u-t)+\binom{s}{b}\leftrightarrow\binom{t}{c}+\binom{s}{a}\leftrightarrow\binom{u}{c}\right\} \\ &+\left\{[P_0]^{ab,cd}[F_0(t)-1]^2\Delta_0(t)^{-1}+\binom{t}{c}\leftrightarrow\binom{u}{b}\right\} \\ &+\left\{[P_2]^{ac,bd}[F_2(t)-1]^2\Delta_2(t)^{-1}+\binom{t}{c}\leftrightarrow\binom{u}{b}\right\} \\ &-2\left\{[P_1]^{ac,bd}(u-s)[F_1(t)-K_1]^2\Delta_V(t)^{-1}\right. \\ &\left.\qquad+\binom{t}{c}\leftrightarrow\binom{u}{b}\right\}. \end{aligned} \tag{2.4}$$

From (2.4) it is clear that $R(s)$ has both right- and left-hand cuts, although the right-hand cuts are entirely due to $t_c(s,t)$. The self-consistent calculation of $R(s)$ plays a central role in our theory.

III. UNITARITY

Let us consider the requirements of elastic unitarity. The two-particle partial-wave unitarity equations are

$$\operatorname{Im}T(s)=\rho(s)|T(s)|^2, \tag{3.1a}$$

$$\operatorname{Im}F(s)=\rho(s)T^*(s)F(s), \tag{3.1b}$$

and

$$\operatorname{Im}\Delta(s)=\rho(s)|F(s)|^2, \tag{3.1c}$$

where $\rho(s)$ is the appropriate phase-space factor. We define a partial-wave decomposition

$$T(s,t)^{ab,cd}=16\pi\sum_{l,T}(2l+1)P_l(\cos\theta)T_{l,T}(s)\times[P_T]^{ab,cd} \tag{3.2}$$

so that, with the normalization chosen in Appendix A and Ref. 5,

$$T_{l,T}(s)=-2\left(\frac{s}{s-4}\right)^{1/2}e^{i\delta_{lT}}\sin\delta_{lT},$$

$$\begin{aligned} \rho(s)&=-\frac{1}{32\pi}\left(\frac{s-4}{s}\right)^{1/2}\theta(s-4) \quad \text{for } S \text{ waves} \\ &=-\frac{1}{48\pi}\frac{(s-4)^{3/2}}{\sqrt{s}}\theta(s-4) \quad \text{for } P \text{ waves} \end{aligned} \tag{3.3}$$

for $4\leq s\leq 16$. From Eqs. (2.2), (2.3), and (3.1) it is straightforward to show that

$$\operatorname{Im}h(s)=\rho(s)|h(s)|^2 \tag{3.4a}$$

and

$$\operatorname{Im}\Gamma(s)=\rho(s)h^*(s)\Gamma(s). \tag{3.4b}$$

Recall that the seagull $t_c(s,t)$ is single-particle irreducible with respect to the σ field and the isovector current, and is $O(m_\pi^4)$ at threshold, which means that it should be negligible at sufficiently low energies. Hence, it is plausible that all important low-energy effects are well represented by the tree structures and polynomials, and that the characteristic mass associated with the seagull is large. These assumptions lead to the hypothesis that

$$\operatorname{Im}t_c(s,t)\simeq 0 \quad \text{for } 4\leq s\leq s_0,$$

where hopefully s_0 is at least as large as the energy of the first $\pi\pi$ resonance. If one wishes, one might understand this assumption to mean that the important contributions to the seagull come from resonances of large mass, which is neither unreasonable nor inconsistent with other phenomenological approaches to $\pi\pi$ scattering. This hypothesis can be translated into the following result:

Lemma 1. If $\mathrm{Im}t_c(s)=0$ to $O(\epsilon^2)$ for $4\leq s\leq s_0$, then[14]

$$h(s)=-K\Gamma(s)+O(\epsilon)\,, \quad (3.5a)$$

$$T(s)=[F(s)-K]\Gamma(s)+O(\epsilon)\,, \quad (3.5b)$$

$$R(s)=KT(s)F(s)^{-1}+O(\epsilon)\,, \quad (3.5c)$$

and

$$R(s) \text{ is real to } O(\epsilon^2) \quad (3.5d)$$

for this energy region.[15]

Proof. The conclusions follow from the identity

$$\mathrm{Im}t_c(s)=|h(s)+K\Gamma(s)|^2 \quad \text{for} \quad 4\leq s\leq 16.$$

From (3.5b) and (3.5c), we have

$$R(s)=K[F(s)-K]\Delta(s)^{-1} \quad \text{for} \quad 4\leq s\leq s_0, \quad (3.6)$$

which is an important dynamical constraint on the theory. Equation (3.6) is a self-consistency condition which requires $R(s)$, as computed from $F(s)$ and $\Delta(s)$, to be identical to the partial-wave projection of (2.4). Equation (3.6) rewritten is[16]

$$F(s)=K+K^{-1}R(s)\Delta(s)\,, \quad (3.7)$$

from which we obtain

$$\mathrm{Im}F(s)=K^{-1}R(s)\rho(s)|F(s)|^2 \quad (3.8)$$

by virtue of (3.1c) and (3.5d). Equation (3.8) enables us to construct an effective-range approximation for $F(s)$ in the usual way, valid for $4\leq s\leq s_0$. However, to make proper use of (3.6) as a self-consistency condition, we must know $F(s)$ and $\Delta(s)$ for $s<0$. It is not obvious that (3.7) can be continued to $s<0$ because of the contradictory analytic properties attributed to $R(s)$ in (2.4) and (3.6). Although (3.6) continued implies that $R(s)$ remains real for $s<0$, (2.4) clearly implies that $R(s)$ has a left-hand cut. In spite of this difficulty, we are anxious to continue (3.7), since we have no other way of determining $F(s)$ and $\Delta(s)$ for $s<0$. A compromise solution is to continue (3.7) in such a way as to minimize the inconsistency, by requiring the discontinuity of $R(s)$ across the nearby portion of the left-hand cut to be negligible for S and P waves. A sufficient condition for this is the requirement that the characteristic masses associated with the (t) and (u) channel trees be large compared to m_π. This then also guarantees that the correction to the PCAC (partial conservation of axial-vector current) prediction, $F_{0,2}(0)=1+O(m_\pi^2/M^2)$, is small. We adopt these conditions in all that follows.

[14] An estimate gives $\mathrm{Re}t_c(s)\sim(s/M^2)T(s)$, and $\mathrm{Im}t_c(s)\sim[\gamma(s)/M^2]\,\mathrm{Re}t_c(s)$, where M ($\gamma(s)$) is a characteristic mass (width) associated with the partial-wave projection of the seagull. This implies that $\epsilon(s)^2\sim s\gamma(s)/M^4$.

[15] The detailed error estimates will be suppressed in what follows. *A postiori* it turns out that they are negligible in the elastic region.

[16] $F_1(0)=1$ implies $R_1(0)=-K_1/2F_\pi^2$, which gives Weinberg's prediction for the P-wave scattering lengths, with a correction of order $m_\pi^2 dF_1(0)/ds$.

To be specific we assume that $F(s)$ and $\Delta(s)$ are determined for $s<0$ by analytic continuation of (3.7). Since $R(s)$ must then have the characteristics described in the previous paragraph, we also assume that

$$R(s)=-K(d+fs) \quad (3.9)$$

is a reasonable approximation for all values of s of interest, both positive and negative. These additional assumptions enable us to compute $R(s)$ from (2.4), which now must equal $R(s)$ as computed from (3.9). If this self-consistency condition is satisfied, we can determine all elements of (2.1). The amplitude so constructed will have the properties described in the Introduction. It is convenient to divide the remaining discussion according to energy region: threshold, elastic, and resonance regions.

IV. LOW-ENERGY AMPLITUDE

A. Threshold

Equations (3.5c) and (3.9) imply that the S- and P-wave amplitudes are of the form

$$T(s)=-(d+fs)F(s) \quad \text{for} \quad 4\leq s\leq s_0. \quad (4.1)$$

However, there is no guarantee that the partial-wave projection of (2.1) will agree with (4.1). They will be identical for $s\geq 4$, only if (a) $\mathrm{Im}t_c(s)=0$, and (b) $R(s)$ satisfies the self-consistency conditions. Since we can always arbitrarily choose a parametrization of $t_c(s)$ in which the first is true for sufficiently low energies, we shift the entire weight of the discussion to $R(s)$. If these two conditions are satisfied we will have a crossing-symmetric, approximately unitary model. Therefore

$$\{\text{partial-wave projection of (2.4)}\}=-K(d+fs) \quad (4.2)$$

is the *self-consistency condition* which will determine the free parameters of the theory.

For s sufficiently close to threshold, one can expand the left-hand side of (4.2) in powers of $(s-4)$, retaining only the constant and linear terms. The results are[17]:

constant terms

I=1:

$$F_1(0)=1 \text{ implies } d_1=\tfrac{1}{2}F_\pi^{-2}, \quad (4.3)$$

I=0:

$$[d_0+4f_0]=F_\pi^{-2}[N(N+2)+4]+\tfrac{2}{3}d_0F_0(0)[F_0(0)-1]+(10/3)d_2F_2(0)\times[F_2(0)-1]-t_0(4)\,, \quad (4.4)$$

I=2:

$$[d_2+4f_2]=\tfrac{2}{5}F_\pi^{-2}[N(N+2)-8]+\tfrac{2}{3}d_0F_0(0)[F_0(0)-1]+\tfrac{1}{3}d_2F_2(0)\times[F_2(0)-1]-t_2(4)\,; \quad (4.5)$$

[17] The subscripts denote the isospin associated with the various parameters.

terms linear in $(s-4)$

$I=0$:

$$-f_0=-2F_\pi^{-2}+4[f_1(1-K_1)+\tfrac{1}{2}F_\pi^{-2}\times dF_1(0)/dt]+\tfrac{1}{3}F_0(0)[f_0(F_0(0)-1)+d_0dF_0(0)/dt]+(5/3)F_2(0)[f_2(F_2(0)-1)+d_2dF_2(0)/dt]+dt_0(4)/ds\,, \quad (4.6)$$

$I=2$:

$$-f_2=F_\pi^{-2}-4[f_1(1-K)+\tfrac{1}{2}F_\pi^{2}dF_1(0)/dt]+\tfrac{1}{3}F_0(0)[f_0(F_0(0)-1)+d_0dF_0(0)/dt]+\tfrac{1}{6}F_2(0)[f_2(F_2(0)-1)+d_2dF_2(0)/dt]+dt_2(4)/ds\,, \quad (4.7)$$

and

$I=1$:

$$-[\tfrac{1}{2}F_\pi^{-2}+4f_1]K_1=-\tfrac{1}{2}K_1F_\pi^{-2}-4[f_1(1-K_1)+\tfrac{1}{2}F_\pi^{-2}dF_1(0)/dt]-\tfrac{1}{3}F_0(0)[f_0(F_0(0)-1)+d_0dF_0(0)/dt]+\tfrac{5}{6}F_2(0)[f_2(F_2(0)-1)+d_2dF_2(0)/dt]+3dt_1(4)/ds\,. \quad (4.8)$$

As a result we find

$$d_0=F_\pi^{-2}[N(N+2)-4]+O(m_\pi^2/M^2)\,, \quad (4.9a)$$

$$f_0=2F_\pi^{-2}+O(m_\pi^2/M^2)\,, \quad (4.9b)$$

$$d_2=\tfrac{2}{5}F_\pi^{-2}[N(N+2)+2]+O(m_\pi^2/M^2)\,, \quad (4.9c)$$

$$f_2=-F_\pi^{-2}+O(m_\pi^2/M^2)\,, \quad (4.9d)$$

and

$$F_\pi^2f_1=O(m_\pi^2/M^2)f_0\,, \quad (4.9e)$$

where the $O(m_\pi^2/M^2)$ are corrections which can be characterized in terms of typical masses for the form factors and seagull.

There is an additional question that must be considered before we are assured of the consistency of the theory. From (3.7) and (3.9),

$$F(s)=K-(d+fs)\Delta(s)\,, \quad (4.10)$$

which means $\Delta(s)$ will have a spurious pole at $s=-d/f$ unless $F(-d/f)=K$, or $f=0$. To avoid the spurious pole in the S-wave propagators, we must set

$$F_0(-d_0/f_0)=1 \quad\text{and}\quad F_2(-d_2/f_2)=1\,. \quad (4.11)$$

This is perfectly feasible, since $m_\pi^2f_0/d_0$ and $m_\pi^2f_2/d_2$ are both $O(1)$. Then

$$F_0(0)-1\simeq\frac{d_0}{f_0}\frac{dF_0(0)}{dt}+\cdots=O\left(\frac{m_\pi^2}{M^2}\right)$$

and (4.12)

$$F_2(0)-1\simeq\frac{d_2}{f_0}\frac{dF_2(0)}{dt}+\cdots=O\left(\frac{m_\pi^2}{M^2}\right).$$

Since $F_\pi^2f_1$ is already of $O(m_\pi^2/M^2)f_0$, we remove the spurious singularity from $\Delta_V(s)$ by choosing

$$f_1=0\,. \quad (4.13)$$

Thus the additional conditions provided by (4.11) and (4.13) remove the spurious poles from $\Delta(s)$ in a way which is completely compatible with assumptions underlying the results of (4.9). The model now satisfies approximate unitarity and crossing in the neighborhood of threshold.

B. Elastic Region

Let us consider our assumptions to be valid through the entire elastic region. Again (4.2) is the key equation to be satisfied, although we now require more detailed expressions for $F(s)$, $\Delta(s)$, and $t_c(s)$. Equations (3.8) and (3.9), (4.9), and (4.13) enable us to construct an effective-range expansion for $F(s)$ with the correct analyticity properties. In the absence of CDD (Castillejo-Dalitz-Dyson) singularities,[18]

$$F_0(s)=F_0(0)\{1+B_0s+F_\pi^{-2}[N(N+2)-4]\times[g_0(s)-g_0(0)-sg_0'(0)]+2sF_\pi^{-2}[g_0(s)-g_0(0)]\}^{-1}, \quad (4.14a)$$

$$F_2(s)=F_2(0)\{1+B_2s+\tfrac{2}{5}F_\pi^{-2}[N(N+2)+2]\times[g_0(s)-g_0(0)-sg_0'(0)]-sF_\pi^{-2}[g_0(s)-g_0(0)]\}^{-1}, \quad (4.14b)$$

and

$$F_1(s)=\{1+B_1s+\tfrac{1}{2}F_\pi^{-2}[g_1(s)-g_1(0)-sg_1'(0)]\}^{-1}, \quad (4.14c)$$

with

$$32\pi^2g_0(s)=2\left(\frac{s-4}{s}\right)^{1/2}\ln\left[\frac{s^{1/2}+(s-4)^{1/2}}{2}\right]-i\pi\left(\frac{s-4}{s}\right)^{1/2},$$

$$g_1(s)=\tfrac{2}{3}(s-4)g_0(s)\,, \quad (4.15)$$

$$B=-F(0)dF(0)/ds\,,$$

and with $\Delta(s)$ determined from Eq. (4.10).

We can simplify the discussion somewhat by means of a simple hypothesis for the P-wave vertex, which has no analog for S waves. Motivated by the hard-pion philosophy, we assume $\mathrm{Im}\Gamma_1(s)=0$ for $4\leq s\leq s_1\leq s_0$. The consequences of this additional assumption are:

Lemma 2 (KSFR relation).[19] If

$$\mathrm{Im}t_1(s)=0 \quad\text{for}\quad 4\leq s\leq s_0$$

---

[18] It is illusory that B_0 can be fixed by the requirement that $F_0(-d_0/f_0)=1$, since $F_0(-d_0/f_0)=F_0(0)[1-(d_0/f_0)B_0]^{-1}$ to a high degree of accuracy, as can be verified by a numerical evaluation of the omitted terms. Then $F_0(0)=1-(d_0/f_0)B_0$ is consistent with the definition $B_0=-F_0(0)dF_0(0)/ds\simeq-dF_0(0)/ds$. There is no *new* information. Similarly for $I=2$.

[19] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 225 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966); F. Gilman and H. J. Schnitzer, *ibid.* **150**, 1362 (1966); J. J. Sakurai, Phys. Rev. Letters **17**, 552 (1966); M. Ademollo, Nuovo Cimento **46**, 156 (1966).

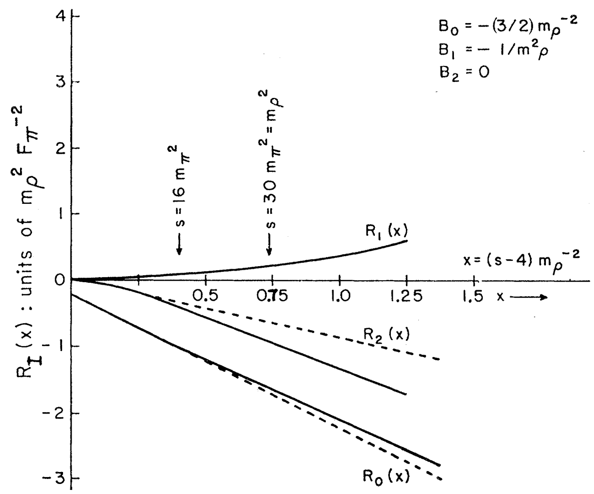

FIG. 1. Comparison of the two sides of Eq. (4.2) for S and P waves. $R_I(s)$ denotes the function defined in (2.3). One requires $R_0(s)$ and $R_2(s)$, as computed from the left-hand side of (4.2), to coincide with the straight lines, and $R_1(s)=0$ for self-consistency.

and

$$\mathrm{Im}\Gamma_1(s)=0 \quad \text{for} \quad 4\leq s\leq s_1<s_0,$$

then

$$K_1=0 \quad \text{and} \quad s_1=s_0. \tag{4.16}$$

That is,

$$-2F_\pi{}^2=\int\frac{dm^2\rho_V(m^2)}{m^2}$$

and

$$\mathrm{Im}\Gamma_1(s)=0 \quad \text{for} \quad 4\leq s\leq s_0.$$

Proof. From Lemma 1 (suppressing the error estimates), we have

$$h_1(s)=-K_1\Gamma_1(s) \quad \text{for } 4\leq s\leq s_0.$$

Then $\mathrm{Im}\Gamma_1(s)=0$ for $4\leq s\leq s_1$, which implies $\rho_1(s)h_1(s)=0$ in the same (finite) interval. Since $\rho_1(s)\neq 0$, one must have $h_1(s)=0$. Then either $\Gamma_1(s)=0$ or $K_1=0$. But if both $h_1(s)=0$ and $\Gamma_1(s)=0$, there is no P-wave scattering, which is unacceptable. Hence

$$K_1=0,$$

which implies

$$h_1(s)=0 \quad \text{for} \quad 4\leq s\leq s_0.$$

Remarks. It is known that at least two additional assumptions must be added to current algebra to derive the KSFR relation. Our two hypotheses are considerably weaker than usual, as our derivation makes no reference to vector-meson dominance. All that is required is the hypotheses be true in a finite interval of the elastic region. Our treatment of the KSFR relation is compatible with vector-meson dominance if our underlying equations can be extended to the resonance region. Since $K=1$ for S waves, there is no analogous discussion for the $\pi\pi\sigma$ vertex.

We now assume $K_1=0$ in all that follows. The remaining undetermined quantities are B_0, B_1, B_2, N, and the seagull $T_c(s,t)^{ab,cd}$. The detailed consideration of the self-consistency condition (4.2) at finite energies depends on an estimate of the seagull. For this we take over the model described in Ref. 5, which expresses the seagull in terms of two real parameters, ξ_1 and ξ_2, which are expected to be of order $F_\pi{}^{-2}O(m_\pi{}^2/M^2)$, i.e.,

$$\begin{aligned}4T_c(s,t)^{ab,cd}=&[P_0]^{ab,cd}\{2\xi_1[3(s-2)^2+(t-2)^2\\&+(u-2)^2]-\xi_2[2(s-2)^2+4(t-2)^2+4(u-2)^2]\}\\&+[P_1]^{ab,cd}(2\xi_1+\xi_2)[(t-2)^2-(u-2)^2]\\&+[P_2]^{ab,cd}\{2\xi_1[(t-2)^2+(u-2)^2]\\&-\xi_2[2(s-2)^2+(t-2)^2+(u-2)^2]\}.\end{aligned} \tag{4.17}$$

As a further simplification, we fix $N=1$ and $B_2=0$, consistent with the absence of any significant structure in the $I=2$ channel. (Some discussion of this choice is to be found in Appendix B.)

We now have all the ingredients for the calculation of Eq. (4.2). It turns out that one can satisfy (4.2) reasonably well for a broad range of values for B_0, ξ_1, and ξ_2, which can be characterized by $B_0\lesssim B_1<0$, $m_\pi{}^2|B_1|\ll 1$, and $F_\pi{}^2\xi_1\simeq F_\pi{}^2\xi_2=O(B_1)$. This kind of parametrization ensures that the S waves satisfy unitarity throughout the elastic region. For the P waves, the self-consistency condition is not as well satisfied, while the D-wave and higher partial-wave amplitudes are purely real and not unitary, although they are the projections of a crossing-symmetric amplitude. The violation of unitarity in the elastic region for higher waves is not serious if the amplitudes are sufficiently small, which is ensured by our earlier requirement that the characteristic masses M for the (t) and (u) channel trees are sufficiently large.

To be more specific, consider a definite choice of parameters,

$$B_0=\tfrac{3}{2}B_1=-(1/20)m_\pi{}^{-2} \tag{4.18a}$$

and

$$\xi_1=\xi_2=-(1/60)m_\pi{}^{-2}F_\pi{}^{-2}. \tag{4.18b}$$

(The motivation for this choice will become apparent in the next section.) We have computed both sides of (4.2) with these parameters, with the results for the S and P waves shown in Fig. 1. It is clear that the two ways of computing $R(s)$ are in excellent agreement throughout the elastic region, verifying the self-consistency of the model. This also means that Eq. (4.1) is a valid representation of the S- and P-wave amplitudes in the interval $4\leq s\leq 16$. Since (4.1) is a good approximation to the partial-wave projection of our complete amplitude (in the elastic region), the phase shifts can be conveniently computed from

$$\cot\delta=\mathrm{Re}F(s)/\mathrm{Im}F(s),\quad 4\leq s\leq 16, \tag{4.19}$$

with the numerical results displayed in Fig. 2. Similarly, this allows us to obtain the scattering lengths from

(4.14), namely,[10]

$$a_0 \simeq \frac{m_\pi F_\pi^{-2}}{32\pi}[N(N+2)+4](1+4m_\pi^2 B_0)^{-1}, \quad (4.20a)$$

$$a_2 \simeq \frac{2}{5}\frac{m_\pi F_\pi^{-2}}{32\pi}[N(N+2)-8](1+4m_\pi^2 B_2)^{-1}, \quad (4.20b)$$

and

$$a_1 \simeq (1/24)m_\pi^{-1}F_\pi^{-2}(1+4m_\pi^2 B_1)^{-1}. \quad (4.20c)$$

The parameters of (4.18) imply a 15% correction to the Weinberg scattering lengths in the $I=0$ and 1 states.

V. UNITARIZED TREE APPROXIMATION

We have succeeded in constructing a representation of $\pi\pi$ scattering which satisfies current algebra, crossing, and unitarity throughout the elastic region. Let us now be more speculative and extend the model to higher energies by assuming that (1) elastic unitarity is a good approximation for $16 \leq s \leq 32$ (for which we have no *a priori* explanation), and (2) the model constructed in the previous sections is still valid in the inelastic region. One criterion for the validity of these assumptions is the self-consistency condition for $R(s)$. Figure 1 shows the comparison of the two sides of (4.2), including the extension to the inelastic region. Since the self-consistency condition is qualitatively satisfied for $s \geq 16$, our model can serve as a reasonable guide to $\pi\pi$ scattering above the inelastic threshold, although the quantitative details may not be precisely correct.

The lack of complete self-consistency implies an ambiguity in the computation of phase shifts for $s \geq 16$. We have two options: (1) to represent the partial-wave amplitude by (4.1), in which case elastic unitarity is satisfied, but crossing symmetry is violated for $s \geq 16$, or (2) to project Eq. (2.1) into partial waves, which preserves crossing but violates unitarity for $s \geq 16$. The two procedures agree qualitatively in their prediction of phase shifts for $s < 32$. Therefore, we extend the S- and P-wave phase shifts to $16 \leq s \leq 32$, with the results plotted in Fig. 2. Our choice of parameters in (4.18) produces two resonances at $s=30$, in $I=0$ and $I=1$, with the $I=0$ resonance being very broad, while the ρ meson is quite narrow.[20] In fact the ρ-meson width predicted by (4.14c) is identical to that of the hard-pion analysis (with $\delta = -1$).[4,21]

We now define what we call the *unitarized tree approximation*, which we assume up to energies of $s \lesssim 32$. The components of this approximation are:

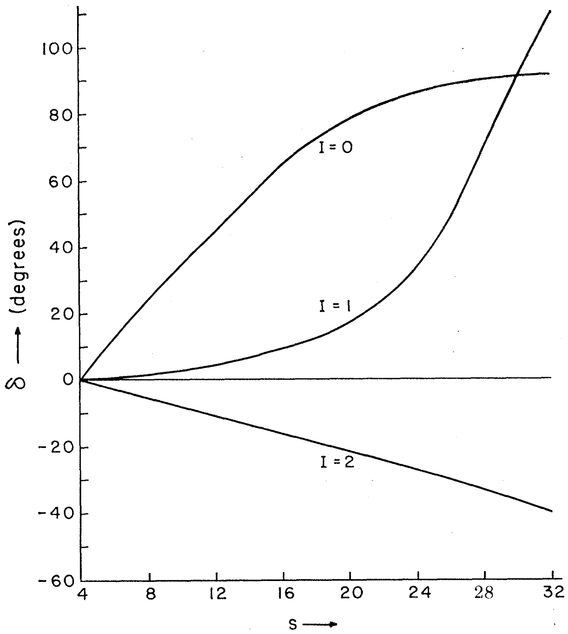

FIG. 2. S and P-wave phase-shifts as computed from projection of (2.1).

(1) the representation (2.1), (2) the form factors, as given by (4.14), analytically continued, as required by (2.1), (3) $\Delta(s)$ as determined by (4.10), (4.11), (4.13), and (4.16), (4) $K_1=0$, (5) $B_1=\frac{2}{3}B_0=-m_\rho^{-2}$; $B_2=0$, and (6) $T_c(s,t)$ as in (4.17), with $\xi_1=\xi_2=-\frac{1}{2}m_\rho^{-2}F_\pi^{-2}$. We have already given a discussion of the properties of this model in various energy regions. Now further note that the unitarized tree approximation reduces to the usual tree approximation in the narrow-resonance limit. Our theory thus gives some sense of the accuracy of the tree approximation of the phenomenological Lagrangian technique.

VI. CONCLUSIONS

We have described a method to construct current-algebra amplitudes which satisfy unitarity, crossing symmetry, and threshold theorems. As a specific result, we found a parametrization of Eq. (2.1) which we call the *unitarized tree approximation*. It defines a scattering amplitude $T(s,t)$, whose partial-wave projection coincides with (4.1) for $4 \leq s \leq 16$,[22] but *improves upon* (4.1) in $0 \leq s \leq 4$ and $s \geq 16$. To take a broader view, the power of the method can be traced directly to the exact representation given by (2.1), which gives a very simple structure to that which can be calculated, and suppresses that which is more difficult to calculate. Although we have worked exclusively with $\pi\pi$ scattering, these ideas should be applicable to other problems for which the Ward identities of chiral current algebra

[20] We do not predict these resonances, but merely find their existence is compatible with our hypotheses.

[21] Therefore, ρ dominance is a good approximation to (4.14c) and similar equations. However, the σ-dominance or narrow-width approximation to (4.14a), (4.14b), etc. is highly suspect. We also doubt whether the hard-pion approximation is very accurate when applied to σ decays since $\Gamma_0(s)=F_0(s)\Delta_0(s)^{-1}$ is not well represented by a polynomial.

[22] The various calculations of Ref. 8 have not gone beyond this first approximation.

play a role, such πN and $\pi\pi$ scattering. Other problems such as $\pi N \to 2\pi N$ and $3\pi \to 3\pi$ would provide interesting exercises in the combination of current-algebra and three-particle unitarity. Further questions are raised by our intuitive treatment of the seagull, which we would hope could be put on a firmer footing.

Finally, we feel our method should be regarded as an alternative to the program of Lee.[9] Our results suggest that his findings are probably a consequence of current algebra, crossing symmetry, unitarity, and analyticity, rather than the specific field-theoretic model. Further, since his method appears to be applicable only for renormalized field theories, it is not clear what one is to do if the theory underlying current algebra is nonrenormalizable. Here such difficulties are avoided, but the price we pay is the appearance of free parameters such as B_0, B_1, B_2, ξ_1, and ξ_1. Hopefully, these parameters can be determined by a sharper application of S-matrix methodology.

ACKNOWLEDGMENT

I wish to thank the Physics Department of Rockefeller University for the warm hospitality extended during the academic year 1969–70.

APPENDIX A

We sketch the derivation of Eq. (2.1) of the text. One begins with Eq. (43) of Ref. 5, and then displays the structure for the S waves, due to the σ field, according to the method of Ref. 5. On the pion mass shell, the result is

$$m_\pi{}^8F_\pi{}^8M(q_1,q_2,q_3)^{abcd}=C_A{}^2q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}M_c{}^{(0)}(q_1,q_2,q_3)_{\mu\nu\lambda\sigma}{}^{abcd}+\Big\{m_\pi{}^4F_\pi{}^4[m_\pi{}^2F_\pi{}^2\Gamma_\Sigma{}^{(0)}(q_3+q_4,\,q_1)^{ef,ab}+L^{ef,ba}]$$
$$\times\Sigma^{-1}(q_1+q_2)^{ef,gh}[m_\pi{}^2F_\pi{}^2\Gamma_\Sigma{}^{(0)}(q_1+q_2,\,q_3)^{gh,cd}+L^{gh,dc}]+\binom{q_2}{b}\leftrightarrow\binom{q_3}{c}+\binom{q_1}{a}\leftrightarrow\binom{q_3}{c}\Big\}-i\epsilon^{abe}\epsilon^{cde}\Big\{[-C_A{}^2q_{3\lambda}$$
$$\times q_{4\sigma}\Gamma^{(1)}(q_3,q_4)_{\lambda\sigma\alpha}+\tfrac{1}{2}(q_3-q_4)_\alpha]\Delta_V(q_1+q_2)_{\alpha\alpha'}[-C_A{}^2q_{1\mu}q_{2\nu}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\alpha'}+\tfrac{1}{2}(q_1-q_2)_{\alpha'}]+[-\tfrac{1}{6}F_\pi{}^2+\tfrac{1}{2}(C_A-\tfrac{1}{2}C_V)]$$
$$\times(q_3-q_4)\cdot(q_2-q_1)+\binom{q_2}{b}\leftrightarrow\binom{q_2}{c}+\binom{q_1}{a}\leftrightarrow\binom{q_3}{c}\Big\}\,. \quad \text{(A1)}$$

The next step is to relate these proper vertex parts and two-point functions to conventional form factors and propagators. The Ward identities for the $\pi\pi V_\mu$ vertex imply

$$\begin{aligned}&-C_A{}^2q_{1\mu}q_{2\nu}\Gamma^{(1)}(q_1,q_2)_{\mu\nu\lambda}+\tfrac{1}{2}(q_1-q_2)_\lambda\\&\qquad+[C_A-\tfrac{1}{2}C_V](q_2-q_1)_\alpha\Delta_V(q_1+q_2)^{-1}{}_{\alpha\lambda}\\&=m_\pi{}^4F_\pi{}^4\Gamma^{(1)}(q_1,q_2)_\lambda\,,\end{aligned} \quad \text{(A2)}$$

which is related to the pion electromagnetic form factor by

$$\begin{aligned}&-(q_1+q_2)_\lambda F_1((q_1+q_2)^2)\\&\qquad=m_\pi{}^4F_\pi{}^2\Delta_V((q_1+q_2)^2)\Gamma^{(1)}(q_1,q_2)_\lambda\end{aligned} \quad \text{(A3)}$$

on the pion mass shell, where $F_1(q^2)$ and $\Delta_V(q^2)$ are as in Eq. (2.1).

We must also consider the $\pi\pi\sigma$ vertices and σ propagators, with the relevant Ward identity being

$$\begin{aligned}&C_A{}^2q_{2\lambda}(q_1+q_2)_\mu\Gamma_\Sigma{}^{(0)}(q_1,q_2)_{\lambda\mu}{}^{abcd}\\&=m_\pi{}^4F_\pi{}^4\Gamma_\Sigma{}^{(0)}(q_1,q_2)^{abcd}+m_\pi{}^2F_\pi{}^2L^{abdc}\\&\quad+\Sigma^{abcd}(q_1)m_\pi{}^2F_\pi{}^2[\Delta_\pi(q_2)^{-1}+\Delta_\pi(q_1+q_2)^{-1}\\&\qquad\qquad-\Delta_\pi(0)^{-1}].\end{aligned} \quad \text{(A4)}$$

The limit $q_1,q_2\to 0$ implies that

$$\begin{aligned}&m_\pi{}^4F_\pi{}^4\Gamma_\Sigma{}^{(0)}(0,0)^{abcd}+m_\pi{}^2F_\pi{}^2L^{abdc}\\&\qquad+m_\pi{}^2F_\pi{}^2\Sigma^{abcd}(0)\Delta_\pi(0)^{-1}=0\,.\end{aligned} \quad \text{(A5)}$$

The $\sigma\pi\pi$ form factor is defined by

$$\begin{aligned}&[(2\pi)^64q_2{}^0q_3{}^0]^{1/2}\langle\Pi_d(q_1+q_2)\,|\,\sigma^{ab}(0)\,|\,\Pi_c(q_2)\rangle\\&\quad=m_\pi{}^2F_\pi{}^2\Gamma_\Sigma{}^{(0)}(q_1,q_2)^{ab,cd}\big|_{q_2{}^2=(q_2+q_3)^2=m_\pi{}^2}\\&\quad=F_\Sigma(q_1{}^2)^{ab,cd}.\end{aligned} \quad \text{(A6)}$$

Therefore, PCAC implies

$$m_\pi{}^{-2}F_\Sigma(0)^{ab,cd}=-L^{abdc}+O(m_\pi{}^2/M^2)\,, \quad \text{(A7)}$$

where the correction is characterized by the extrapolation to the pion mass shell, and the last term in Eq. (A5).

One next decomposes $F_\Sigma(q^2)^{abcd}$ and $\Sigma(q^2)^{abcd}$ into isospin components, for which Weinberg's analysis[23] becomes extremely useful. We find

$$\begin{aligned}iL^{abcd}=&\tfrac{1}{5}(N^2+2N+2)\delta^{ab}\delta^{cd}\\&+\tfrac{1}{5}(N^2+2N-3)(\delta^{ac}\delta^{bd}+\delta^{bc}\delta^{ad})\,,\end{aligned} \quad \text{(A8)}$$

and similarly decompose σ^{ab}. We define the propagators and form factors for the σ field, normalized so as to satisfy Eqs. (3.1c) and (3.3), i.e.,

$$\begin{aligned}i\Sigma^{ab,cd}(s)=&N^2(N+2)^2\Delta_0(s)[P_0]^{ab,cd}\\&+[\tfrac{5}{2}N^2(N-1)/(N+3)]^2\Delta_2(s)[P_2]^{ab,cd}\end{aligned} \quad \text{(A9)}$$

and

$$\begin{aligned}-im_\pi{}^{-2}F_\Sigma(s)^{ef,ab}=&N(N+2)F_0(s)[P_0]^{ab,ef}\\&+\tfrac{2}{5}(N+3)(N-1)F_2(s)[P_2]^{ab,ef}.\end{aligned} \quad \text{(A10)}$$

[23] S. Weinberg, Phys. Rev. 166, 1568 (1968).

By virtue of (A8)–(A10),

$$F_{0,2}(0)=1+O(m_\pi^2/M^2). \qquad \text{(A11)}$$

When the results of this section are combined with (A1), one arrives at Eq. (2.1), where

$$T(s,t,u)^{abcd}=iF_\pi^8 m_\pi^8 M(q_1, q_2; -q_3, -q_4)^{abcd}. \qquad \text{(A12)}$$

APPENDIX B: EXACT CONSTRAINTS

Martin has found exact constraints for the partial-wave amplitudes in the region $0\leqslant s\leqslant 4$. For the S-wave, $\pi^0\pi^0$ amplitude they are[24,25]

$$T_0^{00}(s)>T_0^{00}(4), \quad 0\leqslant s\leqslant 4, \qquad \text{(B1)}$$

$$T_0^{00}(0)\leqslant T_0^{00}(2(1+1/\sqrt{3}))\simeq T_0^{00}(3.155), \qquad \text{(B2)}$$

$$dT_0^{00}(s)/ds<0, \quad 1.7\leqslant s\leqslant 4, \qquad \text{(B3)}$$

$$dT_0^{00}(s)/ds>0, \quad 0\leqslant s\leqslant 1.29, \qquad \text{(B4)}$$

and

$$T_0^{00}(3.205)<T_0^{00}(0.2134)<T_0^{00}(2.9863). \qquad \text{(B5)}$$

In the text we verified that

$$T(s)=-(d+fs)F(s) \qquad \text{(4.1)}$$

adequately represents the S-wave amplitude for $4\leqslant s\leqslant 16$. One can ask further whether (4.1) can be continued to $0\leqslant s\leqslant 4$; i.e., does (4.1) satisfy the Martin constraints? As an example, consider (4.1), together with (4.9), (4.14), and $B_2=0$, with undetermined parameters B_0 and N. A summary of the results follows:

(B3) requires $dT_0^{00}(4)/ds<0$, which implies

$$B_0\lesssim 0.05/[N(N+2)+4]. \qquad \text{(B6)}$$

(B1) requires $T_0^{00}(0)>T_0^{00}(4)$, whence

$$\frac{1.4+0.1[N(N+2)-4]}{32+8[N(N+2)-4]}\gtrsim B_0, \qquad \text{(B7)}$$

which implies $B_0\lesssim 0.05$ for $N=1$.

[24] Note our sign convention.

[25] A. Martin, Nuovo Cimento **47A**, 265 (1967); **58A**, 303 (1968); see also G. Auberson, G. Mahoux, O. Brander, and A. Martin, CERN Report No. TH-1032 (unpublished); and R. Roskies, Nuovo Cimento **65A**, 467 (1970); **66A**, 494(E) (1970).

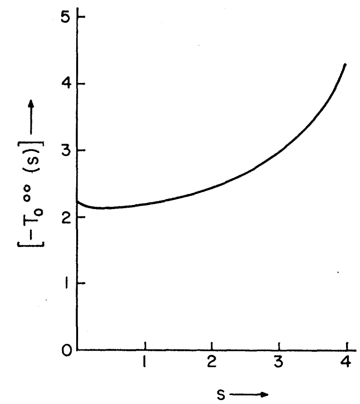

FIG. 3. Graph of $[-T_0^{00}(s)]$ versus s for the interval $0<s<4$, where $T_0^{00}(s)$ is the S-wave $\pi^0-\pi^0$ scattering amplitude as calculated from Eq. (4.1).

(B4) requires $dT_0^{00}(0)/ds>0$, so that

$$B_0[N(N+2)-4]>\tfrac{1}{4}[N(N+2)-4]^2B_0^2, \qquad \text{(B8)}$$

which implies

$$-\tfrac{1}{4}<B_0<0 \quad \text{for} \quad N=1 \qquad \text{(B9)}$$

and

$$0<B_0\leq 1 \quad \text{for} \quad N\geq 2. \qquad \text{(B10)}$$

To visualize the meaning of the constraints, we have plotted $-T_0^{00}(s)$ [as computed from (4.1), $N=1$, $B_0=-(1/20)m_\pi^{-2}$, and $B_2=0$] in Fig. 3. Although the qualitative behavior is in agreement with (B1)–(B5), inequalities (B2) and $T_0^{00}(3.205)<T_0^{00}(0.2134)$ are violated, since the minimum shown in Fig. 3 is at $s\simeq 0.5$ instead of $1.29<s<1.7$ as required. Further the exact amplitude should have a larger rise near $s=0$. The quantitative failure of (4.1) near $s=0$ should not be entirely unexpected, since (4.1) does not have a left-hand cut.

We conjecture that the partial-wave projection of our *complete* amplitude, the unitarized tree approximation, will lead to better agreement with (B1)–(B5), since it coincides with (4.1) near $s=4$, but also has a left-hand cut, and satisfies crossing. Therefore, it should improve the agreement with the Martin inequalities, although we have not carried out the tedious numerical work to verify this.