PHYSICAL REVIEW D VOLUME 2, NUMBER 8 15 OCTOBER 1970

Feynman-Diagram Models of Fermion Daughter Regge Trajectories*†

DAVID STEELE‡
Department of Physics, University of Illinois, Urbana, Illinois 61801
(Received 22 April 1970)

Some Feynman-diagram models of Regge poles are used to study daughter trajectories of fermions in backward pion-nucleon scattering. Both the simple model with bare propagators and a generalized model that includes propagator self-energy insertions are considered. It is found that, in the simple model, the first daughter is a fixed (as a function of the squared momentum transfer u) double pole in the complex J plane. When self-energy corrections to the propagator are included, the fixed double pole breaks up into a moving daughter trajectory and a moving companion trajectory. It is found that away from $u=0$, the daughter and companion trajectories get mixed up in a complicated and model-dependent manner.

I. INTRODUCTION

THE daughter Regge trajectories found by Freedman and Wang[1] in unequal-mass scattering have been studied in a variety of models. Recently a Feynman-diagram model, originally suggested by Van Hove,[2] has been used to study this and other aspects of Regge-pole theory. This type of Feynman-diagram model has been used to study the first daughter of a boson trajectory,[3] the second daughter of a boson trajectory that closely parallels this calculation,[4] conspiracies in π-ρ scattering,[5] and multi-Regge couplings.[6,7] The work reported here applies the model to fermion Regge trajectories; in particular, we study the first daughter of a fermion trajectory.

The simplest case in which a fermion Regge pole occurs is in backward pion-nucleon scattering. Since the masses of the pion and nucleon are different, this case also gives rise to daughter trajectories. The amplitudes in pion-nucleon scattering have an important symmetry relation, first noted by MacDowell.[8] Since MacDowell symmetry follows from very general properties of field theory, any dynamical model of fermion Regge poles must be MacDowell symmetric. We have preserved MacDowell symmetry explicitly throughout this calculation. In this paper, we consistently ignore the complications of isospin and signature.

The infinite set of Feynman diagrams considered in the model consists of those in which a particle of spin $J=l+\frac{1}{2}$ is exchanged in a crossed channel (see Fig. 1). The matrix elements $\mathfrak{M}(J)$ are summed over J, and the sum converted to an integral in the complex J plane, using the Sommerfeld-Watson transform. Then, if certain very plausible assumptions are made about the coupling constant and mass as a function of J, one can distort the contour of integration and pick up Regge poles from the denominator of the propagator. The Feynman propagator leads to fixed daughter-trajectory poles (as a function of squared momentum transfer u) in the complex J plane which violate unitarity. To overcome these troubles, a more sophisticated model that includes self-energy corrections to the propagator is also considered; this changes all the fixed poles in the complex J plane into moving ones.

An off-mass-shell propagator of spin J carries lower spin components $J-1, J-2, \ldots$. These lower spin components are related to daughter trajectories, as emphasized by Durand.[9] In the fermion case, MacDowell symmetry is related to the presence of both parities in the off-mass-shell propagator. The off-mass-shell propagator can be expanded in a series of projection operators of definite spin and parity which are singular at $u=0$. Reggeizing this expansion leads to a parent trajectory

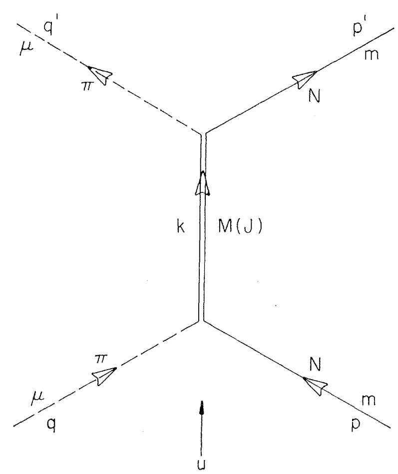

FIG. 1. Spin $J=l+\frac{1}{2}$ exchange contribution to π-N scattering, where $k=p+q$.

* This work is part of a thesis submitted to the University of Illinois in partial fulfillment of the requirements for the Ph.D. degree.

† Work supported in part by a National Science Foundation contract and in part by a NSF Graduate Fellowship.

‡ Present address: Department of Physics, University of Washington, Seattle, Wash. 98105.

[1] D. Z. Freedman and J.-M. Wang, Phys. Rev. Letters **17**, 569 (1966); Phys. Rev. **153**, 1596 (1967).
[2] L. Van Hove, Phys. Letters **24B**, 183 (1967).
[3] R. L. Sugar and J. D. Sullivan, Phys. Rev. **166**, 1515 (1968).
[4] D. Steele, following paper, Phys. Rev. D **2**, 1610 (1970).
[5] R. L. Sugar and R. Blankenbecler, Phys. Rev. Letters **20**, 1014 (1968); R. Blankenbecler, R. L. Sugar, and J. D. Sullivan, Phys. Rev. **172**, 1451 (1968).
[6] R. Blankenbecler and R. L. Sugar, Phys. Rev. **168**, 1597 (1968); J. M. Kosterlitz, Nucl. Phys. **B9**, 273 (1969).
[7] I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Nucl. Phys. **B11**, 384 (1969); Phys. Letters **28B**, 676 (1969).
[8] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).
[9] L. Durand III, Phys. Rev. **154**, 1537 (1967).

and a series of daughters such that, while each term is singular at $u=0$, the sum is finite. One of the virtues of this type of model is that it provides a natural explanation of how daughter trajectories arise and how they cancel the singularities of the leading pole trajectory.

The structure of the first daughter of a fermion trajectory is more complicated than that of a boson trajectory. The simple model with Feynman propagators, considered in Sec. II, leads to fixed single and double poles for the fermion first daughter. When self-energy corrections to the propagator are included, in Sec. III, the fixed single pole becomes a moving one, and the fixed double pole becomes two moving ones, one of which is the daughter and the other of which has a nonsingular residue at $u=0$. Away from $u=0$, the two trajectories get mixed up in a complicated manner reminiscent of the trajectories calculated by Cutkosky and Deo[10] using the Bethe-Salpeter equation with a potential. These calculations suggest that it is unlikely that the daughter trajectories are parallel to the parent trajectory.

II. FEYNMAN-DIAGRAM MODEL OF FERMION REGGE POLES WITHOUT SELF-ENERGY BUBBLES

The class of Feynman diagrams that we consider in this section consists of those in which a particle of spin $J=l+\frac{1}{2}=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$ is exchanged in the u channel. We use the model to study the first daughter trajectory, which comes from the $l-\frac{1}{2}$ spin component carried by the off-mass-shell propagator of spin $J=l+\frac{1}{2}$. In this model, we find that the first daughter consists of a fixed single pole and a coincident fixed double pole in the complex J plane. Since fixed poles in the complex J plane violate unitarity and hence are unphysical, we will consider a more sophisticated model in Sec. III in which the fixed poles become poles that move as a function of $W=u^{1/2}$.

We begin by defining the notation to be used (see Fig. 1) and giving a few kinematical formulas for reference. The nucleon center-of-mass energy in the u channel is given by

$$E=(W^2+m^2-\mu^2)/2W=(p\cdot k)/W=(p'\cdot k)/W, \quad (1)$$

where $W^2=k^2=u$. The momentum and scattering angle in the center-of-mass system are given by

$$p_u^2=-m^2+(u+m^2-\mu^2)^2/4u, \quad (2)$$

$$p_u^2\cos\theta_u=\mu^2-\tfrac{1}{2}(s+u)+(u+m^2-\mu^2)^2/4u. \quad (3)$$

It is convenient to work with a set of amplitudes suggested by Gribov, Okun, and Pomeranchuk.[11] They write the scattering amplitude as

$$\mathfrak{M}=\bar{u}(p')[A_1(s,u)(\boldsymbol{k}+W)+A_2(s,u)(\boldsymbol{k}-W)]u(p), \quad (4)$$

where $k=q+p$. The A_1 and A_2 amplitudes are related to the usual invariant amplitudes $A(s,u)$ and $B(s,u)$, defined by

$$\mathfrak{M}=\bar{u}(p')[A(s,u)+\tfrac{1}{2}B(s,u)\gamma\cdot(q+q')]u(p),$$

by the relations

$$B=A_1+A_2, \quad (5a)$$

$$A=(A_1+A_2)m+(A_1-A_2)W, \quad (5b)$$

and to the conventional amplitudes f_1 and f_2 by

$$f_1=[(E+m)/4\pi]A_1, \quad (6a)$$

$$f_2=[(E-m)/4\pi]A_2. \quad (6b)$$

The statement of MacDowell symmetry in terms of A_1 and A_2 is

$$A_1(s,W)=A_2(s,-W). \quad (7)$$

To investigate daughter trajectories, we need to know the analytic properties of A_1 and A_2.[11] Since $A(s,u)$ and $B(s,u)$ are analytic functions of s and u, we see from (5) that $A_1(W)+A_2(W)$ and $(A_1-A_2)W$ must be analytic functions of u. Since $A_1(W)=A_2(-W)$, either $A_1(W)\sim 1/W$ or $A_1(W)\sim W$ at $W=0$. Because it is unphysical to have $\mathfrak{M}$ vanish at $W=0$, we assume that

$$A_1(W)\sim 1/W \quad \text{at} \quad W=0. \quad (8)$$

We follow, for the most part, the conventions and Feynman rules given in the work of Scadron.[12] The exchanged particle of spin $J=l+\frac{1}{2}$ can have either parity $(-)^l$ or parity $-(-)^l$, so we have to consider both kinds of vertices. Following Scadron, we call the former vertex "abnormal" and the latter vertex "normal"; recall that the pion is a pseudoscalar particle. Note that the diagram with "normal" vertices (no γ_5), denoted by $+$, corresponds to the exchange of $\frac{1}{2}^-$, $\frac{3}{2}^+$, $\frac{5}{2}^-$, ... particles, while the diagram with "abnormal" vertices, denoted by $-$, corresponds to the exchange of $\frac{1}{2}^+$, $\frac{3}{2}^-$, $\frac{5}{2}^+$, ... particles. As we shall show, the two diagrams are related by a simple transformation, so that we only need to compute one of them explicitly.

Our Feynman rules come from the effective interaction Hamiltonian

$$\mathcal{H}^{(+)}=g_+(J)\left(\frac{2l+1}{c_l}\right)^{1/2}\bar{\psi}_{(J)}{}^{\alpha_1\cdots\alpha_l}(x)\times\partial_{\alpha_1}\cdots\partial_{\alpha_l}\psi_N(x)\varphi_\pi(x)+\text{H.c.} \quad (9a)$$

for the vertex with $\frac{1}{2}^-$, $\frac{3}{2}^+$, ... particles, and

$$\mathcal{H}^{(-)}=g_-(J)\left(\frac{2l+1}{c_l}\right)^{1/2}\bar{\psi}_{(J)}{}^{\alpha_1\cdots\alpha_l}(x)\times(i\gamma_5)\partial_{\alpha_1}\cdots\partial_{\alpha_l}\psi_N(x)\varphi_\pi(x)+\text{H.c.} \quad (9b)$$

for the vertex with $\frac{1}{2}^+$, $\frac{3}{2}^-$, ... particles, where

$$c_l=2^l(l!)^2/(2l)!, \quad (10)$$

and where φ_π is the pion field, ψ_N is the nucleon field,

[10] R. E. Cutkosky and B. B. Deo, Phys. Rev. Letters 19, 1256 (1967).

[11] V. Gribov, L. Okun, and I. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. 45, 1114 (1963) [Soviet Phys. JETP 18, 769 (1964)].

[12] M. Scadron, Phys. Rev. 165, 1640 (1968).

$\psi_J{}^{\alpha_1\cdots\alpha_l}$ is the four-component spinor field of the spin $J=l+\frac{1}{2}$ particle, and $\partial_{\alpha_1}=\partial/\partial x^{\alpha_1}$.

The diagram we wish to compute is shown in Fig. 1. The numerator for a high-spin propagator is given in the Appendix in both contracted and uncontracted forms. The matrix element for the normal vertex is

$$\mathfrak{M}_J{}^{(+)}=g_+{}^2(J)\left(\frac{2l+1}{c_l}\right)\bar{u}(p')\left(\frac{p'^{\mu_1}\cdots p'^{\mu_l}(-)^l T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)p^{\nu_1}\cdots p^{\nu_l}}{u-M^2(J)}\right)u(p)\,,\tag{11a}$$

where $J=l+\frac{1}{2}$, and for the abnormal vertex is

$$\mathfrak{M}_J{}^{(-)}=g_-{}^2(J)\left(\frac{2l+1}{c_l}\right)\bar{u}(p')\left(\frac{p'^{\mu_1}\cdots p'^{\mu_l}(-)^l(i\gamma_5)T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)(i\gamma_5)p^{\nu_1}\cdots p^{\nu_l}}{u-M^2(J)}\right)u(p)\,,\tag{11b}$$

where $i(-1)^l T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)$ is the numerator of the Feynman propagator and μ stands for $\mu_1\cdots\mu_l$. The argument M of T means that all momentum factors enter as $k^\mu k^\nu/M^2$ instead of $k^\mu k^\nu/W^2$.

The contracted form of the off-mass-shell propagator is evaluated in the Appendix. Since $(i\gamma_5)(\boldsymbol{k}\pm M)(i\gamma_5)=\boldsymbol{k}\mp M$ and $(i\gamma_5)\gamma_\mu(i\gamma_5)=\gamma_\mu$, we can get the matrix element with abnormal-parity vertices from the normal one by changing $M\to -M$. The matrix elements then become

$$\mathfrak{M}_J{}^{(+)}=\frac{g_+{}^2(J)}{W^2-M^2(J)}\bar{u}(p')\left[\bar{p}^{2l}P_{l+1}{}'(\bar{z})(\boldsymbol{k}+M)-\left(m+\frac{p\cdot k}{M}\right)^2(\boldsymbol{k}-M)\bar{p}^{2l-2}P_l{}'(\bar{z})\right]u(p)\,,\tag{12}$$

where

$$\bar{p}^2=-m^2+(W^2+m^2-\mu^2)^2/4M^2\,,\tag{13}$$

$$\bar{p}^2\bar{z}=\mu^2-\tfrac{1}{2}(s+u)+(W^2+m^2-\mu^2)^2/4M^2\,.\tag{14}$$

The off-mass-shell propagator of spin $J=l+\frac{1}{2}$ carries lower-spin components $l-\frac{1}{2}$, $l-\frac{3}{2}$, $\ldots$, $\frac{1}{2}$. To study the first daughter trajectory, we expand Eq. (12) in first derivatives of Legendre polynomials with argument $\cos\theta_u$, the center-of-mass scattering angle in the u channel. We get the contribution of the leading trajectory by replacing $\bar{p}^2$ by $p_u{}^2$ and $\bar{z}$ by z_u. When Reggeized, this reproduces the usual result. To get the first daughter contributions, we keep all terms containing $P_{l+1}{}'(z_u)$ and $P_l{}'(z_u)$ in the expansion of (12).

From Eqs. (3) and (14) we get

$$\bar{p}^2\bar{z}=p_u{}^2\cos\theta_u+\frac{W^2-M^2}{M^2}\frac{(W^2+m^2-\mu^2)^2}{4W^2}\,.\tag{15}$$

This is just the decomposition of the off-mass-shell spin-1 propagator into spin-1 and spin-0 parts. Let

$$\Delta=\frac{W^2-M^2}{M^2}\frac{(W^2+m^2-\mu^2)^2}{4W^2}\,.\tag{16}$$

Using the expansion of $P_{l+1}{}'(z)$ in powers of z, we can write $\bar{p}^{2l}P_{l+1}{}'(z)$ as

$$\begin{aligned}\bar{p}^{2l}P_{l+1}{}'(\bar{z})=&\,p_u{}^{2l}P_{l+1}{}'(z_u)+(2l+1)\Delta p_u{}^{2l-2}P_l{}'(z_u)\\&+l(2l-1)\Delta^2p_u{}^{2l-4}P_{l-1}{}'(z_u)\\&-(2l-1)(p_u{}^2\Delta)p_u{}^{2l-4}P_{l-1}{}'(z_u)\\&+(\text{lower-spin terms})\,.\end{aligned}\tag{17}$$

Substituting (17) into (12) using (16), one has

$$\begin{aligned}\mathfrak{M}_J{}^{(\pm)}=&\,g_\pm{}^2\frac{p_u{}^{2l}P_{l+1}{}'(z_u)(\boldsymbol{k}\pm M)}{W^2-M^2}\\&+(2l+1)g_\pm{}^2\frac{(W^2+m^2-\mu^2)^2}{4W^2}(p_u{}^2)^{l-1}P_l{}'(z_u)\left(\frac{\boldsymbol{k}}{M^2}\pm\frac{1}{M}\right)\\&-g_\pm{}^2\frac{(p_u{}^2)^{l-1}P_l{}'(z_u)}{W^2-M^2}\left(m\pm\frac{p\cdot k}{M}\right)^2(\boldsymbol{k}\mp M)+\cdots\,.\end{aligned}\tag{18}$$

With the help of the identity

$$\boldsymbol{k}\pm M=\frac{W\pm M}{2W}(\boldsymbol{k}+W)+\frac{W\mp M}{2W}(\boldsymbol{k}-W)\,,$$

we see that (18) is MacDowell symmetric. The operator $(\boldsymbol{k}\pm W)$, acting on a state of the π-N system in the center-of-mass system, picks out states of definite parity.[11] Summing (18) over J and converting the sum to an integral in the complex J plane, using the Sommerfeld-Watson transformation, preserves MacDowell symmetry, since each $\mathfrak{M}_J$ is MacDowell symmetric. The complete Regge representation of this integral is therefore also MacDowell symmetric.

The amplitude for Regge exchange is

$$R^{(\pm)}=\sum_{l=0}\mathfrak{M}_{l+\frac{1}{2}}{}^{(\pm)}$$
$$=-\tfrac{1}{2}i\int\frac{dJ}{\cos\pi J}p_u{}^{2(J-\frac{1}{2})}P_{J+\frac{1}{2}}{}'(-z_u)\bigg\{g_\pm{}^2(J)\frac{\boldsymbol{k}\pm M(J)}{W^2-M^2(J)}$$
$$-2g_\pm{}^2(J+1)(J+1)\frac{(W^2+m^2-\mu^2)^2}{4W^2}\left(\frac{\boldsymbol{k}}{M^2(J+1)}\pm\frac{1}{M(J+1)}\right)$$
$$-g_\pm{}^2(J+1)\left[m^2\pm\frac{2mp\cdot k}{M(J+1)}+\frac{(p\cdot k)^2}{M^2(J+1)}\right]\frac{[\boldsymbol{k}\mp M(J+1)]}{W^2-M^2(J+1)}\bigg\}+[\text{spin-}(j-2),\ldots,\text{ terms}],\quad(19)$$

where we have changed variables from l to $J=l+\frac{1}{2}$.

We assume that the coupling constants $g_\pm{}^2(J)$ have no singularities which prevent us from opening up the original contour C to some vertical line in the left-hand J plane and picking up the Regge poles in the upper right-hand quadrant. This gives contour integrals of the form

$$\int_\gamma\frac{F(J)dJ}{M^2(J)-W^2},\quad\int_\gamma\frac{F(J)dJ}{M(J)},\quad\text{and}\quad\int_\gamma\frac{F(J)dJ}{M^2(J)},$$

where γ is a closed contour in the J plane that includes the zeros of the denominator. Since for fermions, the Regge trajectory $\alpha(W)$ is a function of $u^{1/2}$ instead of $u=W^2$, we factor $M^2(J)-W^2$ into $[M(J)-W][M(J)+W]$ and treat the propagator as the product of two poles—one at $W=M(J)$ with solution $J=\alpha(W)$ and the other at $W=-M(J)$ with solution $J=\alpha(-W)$. This method preserves MacDowell symmetry, whereas treating $[W^2-M^2(J)]^{-1}$ as a single pole at $u=M^2(J)$ does not. In the boson problem,[3] the propagator is interpreted as a single pole at $t=M^2(J)$. If we wish to do this in the fermion problem and still maintain MacDowell symmetry, we must add the normal and abnormal parity contributions $\mathfrak{M}_J{}^{(+)}$ and $\mathfrak{M}_J{}^{(-)}$ and assume that $g_+{}^2$ and $g_-{}^2$ are related. The interpretation used here seems the most natural and satisfactory one.

The first daughter trajectory consists of all the fixed-pole terms, $M(J)^{-1}$ and $M(J)^{-2}$, in Eq. (19). The second term in (19) is part of the first daughter since it contains first- and second-order poles at $M(J+1)=0$ which can be inverted to $J=\alpha(0)-1$. The third term in (19) has a moving pole that is part of the leading trajectory and fixed poles that are further parts of the first daughter.

If we now open up the contour C in Eq. (19) and evaluate the contour integrals, we get

$$R^{(\pm)}=Y_1{}^{(\pm)}(W)(\boldsymbol{k}+W)+Y_2{}^{(\pm)}(W)(\boldsymbol{k}-W)\mp\pi\frac{g_\pm{}^2(\alpha(0))\alpha'(0)}{\cos\pi\alpha(0)}(p_u{}^2)^{\alpha(0)-\frac{1}{2}}P_{\alpha(0)-\frac{1}{2}}{}'(-z_u)\left(E^2-\frac{2m\boldsymbol{k}E}{W}\right)$$
$$+\pi E^2\boldsymbol{k}\frac{d}{dW}\left[\frac{\alpha'(W)g_\pm{}^2(\alpha(W))}{\cos\pi\alpha(W)}[2\alpha(W)+1]p_u{}^{2[\alpha(W)-\frac{3}{2}]}P_{\alpha(W)-\frac{1}{2}}{}'(-z_u)\right]_{W=0}$$
$$\mp E^2 2\pi\frac{g_\pm{}^2(\alpha(0))\alpha'(0)\alpha(0)}{\cos\pi\alpha(0)}p_u{}^{2[\alpha(0)-\frac{3}{2}]}P_{\alpha(0)-\frac{1}{2}}{}'(-z_u)+(\text{lower-spin terms}),\quad(20\text{a})$$

where

$$Y_1{}^{(+)}(W)=\frac{-\pi}{2W}\left[\frac{g_+{}^2(\alpha(W))\alpha'(W)}{\cos\pi\alpha(W)}p_u{}^{2[\alpha(W)-\frac{1}{2}]}P_{\alpha(W)+\frac{1}{2}}{}'(-z_u)\right.$$
$$\left.-\frac{g_+{}^2(\alpha(-W))\alpha'(-W)}{\cos\pi\alpha(-W)}p_u{}^{2[\alpha(-W)-\frac{3}{2}]}P_{\alpha(-W)-\frac{1}{2}}{}'(-z_u)(m-E)^2\right],\quad(20\text{b})$$

$$Y_1{}^{(-)}(W)=\frac{\pi}{2W}\left[\frac{g_-{}^2(\alpha(-W))\alpha'(-W)}{\cos\pi\alpha(-W)}P_u{}^{2[\alpha(-W)-\frac{1}{2}]}P_{\alpha(-W)+\frac{1}{2}}{}'(-z_u)\right.$$
$$\left.-\frac{g_-{}^2(\alpha(W))\alpha'(W)}{\cos\pi\alpha(W)}p_u{}^{2[\alpha(W)-\frac{3}{2}]}P_{\alpha(W)-\frac{1}{2}}{}'(-z_u)(m-E)^2\right],\quad(20\text{c})$$

$$Y_2{}^{(\pm)}(W)=Y_1{}^{(\pm)}(-W).\quad(20\text{d})$$

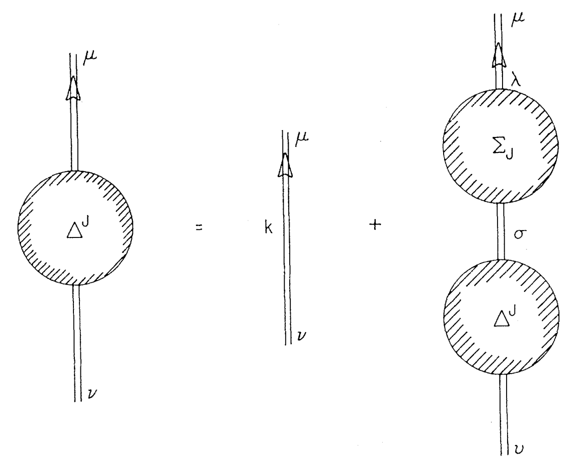

FIG. 2. Full propagator for a particle of spin $J=l+\frac{1}{2}$.

The first two terms, $Y_1(W)$ and $Y_2(W)$, in Eq. (20a) are the contributions of the leading trajectory; the remaining terms come from the first daughter trajectory.

In comparing Eqs. (20) with more conventional treatments, one should look at $R^{(-)}$, since it corresponds to the exchange of $\frac{1}{2}^+$, $\frac{3}{2}^-$, $\frac{5}{2}^+$, ... particles. The $1/2W$ and $(m-E)^2/2W$ factors in $Y_1(W)$, Eqs. (20b) and (20c), are kinematical singularities. The second term represents an additional complication that is not present in the boson problem. For large s and small u,

$$(m-E)^2(p_u{}^2)^{\alpha(W)-\frac{3}{2}}P_{\alpha(W)-\frac{1}{2}}{}'(-z_u)$$
$$\sim\left[\frac{(m^2-\mu^2)^2}{4W^2}-\frac{m(m^2-\mu^2)}{W}\right]s^{\alpha(W)-\frac{3}{2}},$$

giving additional, second-order, singularities at $W=0$ which the first daughter trajectory must cancel. Clearly the first daughter of a fermion trajectory is much more complicated than that of a boson trajectory.

Equation (20) can be shown to be correct by expanding $[Y_1(W)+Y_2(W)]$ and $W[Y_1(W)-Y_2(W)]$ in powers of s and showing that the $1/W$ and $1/W^2$ singularities are canceled by the first daughter terms in Eq. (20a).

III. FEYNMAN-DIAGRAM MODEL OF FERMION REGGE POLE WITH SELF-ENERGY BUBBLES

The fixed poles found in Sec. II violate u-channel unitarity. In this section we extend the model so that it satisfies two-particle unitarity in the u channel by including self-energy corrections to the bare propagator.[3,5] In particular, we can study what happens to the fixed double pole when self-energy corrections are included.

The full propagator for a fermion of spin $J=l+\frac{1}{2}$ satisfies (see Fig. 2)

$$\Delta_{\mu;\nu}{}^J(W)=\frac{T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)}{W^2-M^2}+\frac{T_{\mu;\lambda}{}^J(M)(\boldsymbol{k}+M)}{W^2-M^2}\Sigma_J{}^{\lambda;\sigma}(W)\Delta_{\sigma;\nu}{}^J(W),\quad(21)$$

where μ stands for the set of indices $\mu_1\cdots\mu_l$. The off-mass-shell propagator $T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)$ is discussed in the Appendix. In order to calculate $\Delta_{\mu;\nu}{}^J(W)$ from (21), we must adopt a model that enables us to determine the self-energy function $\Sigma_J{}^{\lambda;\sigma}(W)$. We assume that two-particle intermediate states dominate and we neglect multiparticle intermediate states. This means that we include only the bubble diagrams. In general, such diagrams are divergent. Since we are primarily interested in the qualitative features of the result, we can adopt a model to handle these divergences. We define various amplitudes, $A_J(W)$, $B_J(W)$, ..., derived from $\Sigma_J{}^{\lambda;\sigma}(W)$. We then calculate the imaginary part of $\Sigma_J{}^{\lambda;\sigma}(W)$ using Cutkosky's rules and write dispersion relations for the amplitudes $A_J(W)$, ... When J is sufficiently large, the dispersion integrals diverge; we handle this by using a cutoff on all divergent integrals. The amplitudes we define are related at $W=0$ because of the $O(4)$ invariance of the Feynman integral at $W=0$. We show this by introducing a cutoff in the Feynman integral. After Reggeizing, we can let the cutoff $\to\infty$ since $\alpha(0)<\frac{1}{2}$; in a sense, the Regge behavior provides its own cutoff.

We calculate $\Sigma_J{}^{\lambda;\sigma}(W)$ by calculating the diagram with one bubble (Fig. 3), using Feynman rules and comparing the result with (21). The result for normal vertices is

$$\Sigma_J{}^{\lambda;\sigma}(W)=i(-1)^lg^2\left(\frac{2l+1}{c_l}\right)\int\frac{d^4Q}{(2\pi)^4}\times\frac{Q^{\lambda_1}\cdots Q^{\lambda_l}(\boldsymbol{Q}+m)Q^{\sigma_1}\cdots Q^{\sigma_l}}{(Q^2-m^2+i\epsilon)[(Q-k)^2-\mu^2+i\epsilon]}.\quad(22)$$

Equation (22) also holds for abnormal vertices, since it is independent of M. It is shown in the Appendix that $(i\gamma_5)T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)(i\gamma_5)$ can be calculated by sub-

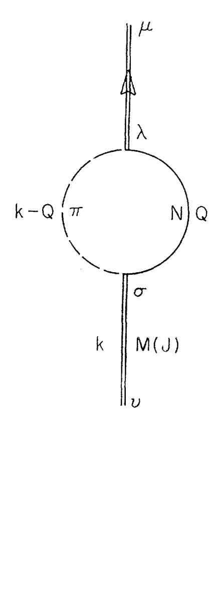

FIG. 3. Bubble diagram used to calculate $\Sigma_J{}^{\lambda;\sigma}(W)$.

stituting $M\to -M$. From Eq. (21), it is clear that $(i\gamma_5)\Delta_{\mu;\nu}{}^J(W)(i\gamma_5)$ can also be calculated by taking $M\to -M$.

In order to solve (21) for Δ^J, we expand all quantities in projection operators of definite spin and parity. In the Appendix the bare off-mass-shell propagator $T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)$ is expanded in a series of projection operators:

$$T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)=T_{\mu;\nu}{}^J(W)\left[\frac{W+M}{2W}(\boldsymbol{k}+W)+\frac{W-M}{2W}(\boldsymbol{k}-W)\right]-\frac{1}{2l+1}\frac{W^2-M^2}{2MW^2}$$
$$\times\Big[\sum_n k_{\mu_n}\gamma^{\mu_n'}\Gamma_{\mu_1\cdots\mu_n'\cdots\mu_l;\nu}{}^l(W^2)-\sum_n \Gamma_{\mu;\nu_1\cdots\nu_n'\cdots\nu_l}{}^l(W^2)\gamma^{\nu_n'}k_{\nu_n}\Big]\big[(\boldsymbol{k}+W)+(\boldsymbol{k}-W)\big]$$
$$-\frac{2l}{2l+1}\frac{W^2-M^2}{M^2}S_{\mu;\nu}{}^{J-1}(W)\left[\frac{W+M}{2W}(\boldsymbol{k}+W)+\frac{W-M}{2W}(\boldsymbol{k}-W)\right]+(\text{spin}\leq l-\tfrac{3}{2}),\quad(23)$$

where

$$S_{\mu;\nu}{}^{J-1}(W)=\frac{1}{l}\sum_{i,j}\frac{k_{\mu_i}k_{\nu_j}}{W^2}T_{\mu[\mu i];\nu[\nu j]}{}^{J-1}(W)$$

is a spin-$(J-1)$ projection operator defined in the Appendix that is orthogonal to $T_{\mu;\nu}{}^J(W)$ [see Eq. (A7)].

We can also expand the full propagator $\Delta_{\mu;\nu}{}^J(W)$ in a series of projection operators. In the Appendix it is shown that

$$\Theta_{\mu;\nu}{}^{J-1}(W)=\frac{1}{2l+1}\sum_n \gamma_{\mu_n}(W^2)\gamma^{\mu_n'}\Gamma_{\mu_1\cdots\mu_n'\cdots\mu_l;\nu}{}^l(W^2)\quad(24)$$

is a spin-$(J-1)$ projection operator and is orthogonal to $T_{\mu;\nu}{}^J(W)$ and $S_{\mu;\nu}{}^{J-1}(W)$. Although it is not present in (23), it must be included in the expansion of $\Delta_{\mu;\nu}{}^J(W)$. We write the expansion as

$$\Delta_{\mu;\nu}{}^J(W)=\frac{1}{2W}T_{\mu;\nu}{}^J(W)\big[D(W)(\boldsymbol{k}+W)-D(-W)(\boldsymbol{k}-W)\big]-\frac{1}{2W}S_{\mu;\nu}{}^{J-1}(W)\big[E(W)(\boldsymbol{k}+W)-E(-W)(\boldsymbol{k}-W)\big]$$
$$-\frac{1}{2l+1}\frac{1}{2W^2}\sum_n k_{\mu_n}\gamma^{\mu_n'}\Gamma_{\mu_1\cdots\mu_n'\cdots\mu_l;\nu}{}^l(W^2)\big[F(-W)(\boldsymbol{k}+W)+F(W)(\boldsymbol{k}-W)\big]$$
$$+\frac{1}{2l+1}\frac{1}{2W^2}\sum_n \Gamma_{\mu;\nu_1\cdots\nu_n'\cdots\nu_l}{}^l(W^2)\gamma^{\nu_n'}k_{\nu_n}\big[F(W)(\boldsymbol{k}+W)+F(-W)(\boldsymbol{k}-W)\big]$$
$$+\frac{1}{2W}\Theta_{\mu;\nu}{}^{J-1}(W)\big[G(-W)(\boldsymbol{k}+W)-G(W)(\boldsymbol{k}-W)\big]+(\text{spin}\leq l-\tfrac{3}{2}).\quad(25)$$

Equation (25) satisfies MacDowell symmetry,

$$\Delta_{\mu;\nu}{}^J(W)=\Delta_{\mu;\nu}{}^J(-W),\quad(26)$$

and is symmetric under the interchange of the μ and ν indices,

$$\Delta_{\mu_1\cdots\mu_l;\nu_1\cdots\nu_l}{}^J(W)=\Delta_{\nu_1\cdots\nu_l;\mu_1\cdots\mu_l}{}^J(W).\quad(27)$$

We have used (26) and (27) to reduce the number of independent self-energy functions in (25).

We solve for the self-energy function in (25) by substituting (22), (23), and (25) into (21) and reducing the resulting tensor equation to a series of linear algebraic equations. This is done by applying various projection operators, contracting the μ indices with the ν indices, and taking the trace in the spinor indices. The projection operators are chosen so that the resulting algebraic equation involves the minimum number of unknown self-energy functions.

Since $T_{\mu;\nu}{}^J(W)$ is orthogonal to all lower-spin projection operators in the expansion of $T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)$ and $\Delta_{\mu;\nu}{}^J(W)$, we get an equation involving only $D(W)$ by contracting Eq. (21) with $T_{\mu';}{}^{\mu J}(W)(\boldsymbol{k}+W)$ and $T^{\nu}{}_{;\nu'}{}^J(W)(\boldsymbol{k}+W)$. We then contract μ_1' with ν_1', etc., to obtain

$$(W-M)D(W)(l+1)(\boldsymbol{k}+W)-(l+1)(\boldsymbol{k}+W)$$
$$=i(-1)^l g^2\left(\frac{2l+1}{c_l}\right)\int\frac{d^4Q}{(2\pi)^4}\frac{Q^{\lambda_1}\cdots Q^{\lambda_l}T_{\lambda;\sigma}{}^J Q^{\sigma_1}\cdots Q^{\sigma_l}(\boldsymbol{k}+W)(\boldsymbol{Q}+m)(\boldsymbol{k}+W)D(W)}{[Q^2-m^2][(Q-k)^2-\mu^2]}.\quad(28)$$

Equation (28) is reduced to an algebraic equation for $D(W)$ by carrying out the indicated contractions and taking the trace in spinor indices. The result is

$$D(W)=\frac{1}{W-M-g^2A(W)}, \qquad (29)$$

where

$$A(W)=i\int\frac{d^4Q}{(2\pi)^4}\frac{\mathbf{Q}^{2l}(Q_0+m)}{(Q^2-m^2+i\epsilon)[(Q-k)^2-\mu^2+i\epsilon]} \qquad (30a)$$

and

$$A(-W)=i\int\frac{d^4Q}{(2\pi)^4}\times\frac{\mathbf{Q}^{2l}(-Q_0+m)}{(Q^2-m^2+i\epsilon)[(Q-k)^2-\mu^2+i\epsilon]}, \qquad (30b)$$

where $Q^\mu=(Q_0,\mathbf{Q})$ and $k^\mu=(W,\mathbf{O})$ in the center-of-mass system, and $Q_0=(Q\cdot k)/W$.

We can obtain two simultaneous linear equations for $F(W)$ and $G(W)$ with a suitable choice of projection operators. One equation is obtained by applying $\gamma^{\mu l'}\Gamma_{\mu';}{}^{\mu l}$ and $\Gamma^{\nu}{}_{;\nu'}{}^{l}(\boldsymbol{k}-W)\gamma^{\nu l'}$ to Eq. (21). By carrying out the expansion in Eq. (23) to spin $l-\frac{3}{2}$, we see that these operators are indeed orthogonal to all but the spin-$(l-\frac{1}{2})$ part of $T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)$ and hence of $\Delta_{\mu;\nu}{}^J(W)$. Proceeding as before, we get

$$[(2l+1)M-g^2C(W)]G(W)=-g^2B(W)F(W), \qquad (31)$$

where

$$B(W)=i(2l+1)\int\frac{d^4Q}{(2\pi)^4}\times\frac{\mathbf{Q}^{2l-2}Q_0^2(Q_0+m)}{(Q^2-m^2+i\epsilon)[(Q-k)^2-\mu^2+i\epsilon]}, \qquad (32)$$

$$C(W)=i(2l+1)\int\frac{d^4Q}{(2\pi)^4}\times\frac{\mathbf{Q}^{2l}Q_0}{(Q^2-m^2+i\epsilon)[(Q-k)^2-\mu^2+i\epsilon]}. \qquad (33)$$

As before, we get $B(-W)$ and $C(-W)$ by $Q_0\to -Q_0$. In particular,

$$C(-W)=-C(W). \qquad (34)$$

We get a second equation for $F(W)$ and $G(W)$ by applying $\Gamma^{\nu}{}_{;\nu'}{}^{l}(W^2)(\boldsymbol{k}-W)\gamma^{\nu l'}$ and $S_{\mu';}{}^{\mu J-1}(W)(\boldsymbol{k}+W)$ to Eq. (21) and proceeding as before:

$$\left[M-\frac{W+M}{M}\frac{2l}{2l+1}g^2B(W)-g^2\frac{C(W)}{2l+1}\right]F(W)+\left[\frac{W+M}{M}\frac{2l}{2l+1}g^2C(W)-g^2A(-W)\right]G(W)=1. \qquad (35)$$

Solving Eqs. (31) and (35) for $F(W)$ and $G(W)$ gives

$$F(W)=[(2l+1)M-g^2C(W)]/H(W), \qquad (36)$$

$$G(W)=-g^2B(W)/H(W), \qquad (37)$$

where

$$H(W)=(2l+1)M^2-2l(W+M)g^2B(W)-2Mg^2C(W)+g^4C(W)^2/(2l+1)+g^4B(W)A(-W). \qquad (38)$$

Finally, we get an equation involving $E(W)$ and $F(W)$ by applying $\gamma^{\mu l'}\Gamma_{\mu';}{}^{\mu l}(\boldsymbol{k}+W)$ and $S^{\nu}{}_{;\nu'}{}^{J-1}(\boldsymbol{k}+W)$ to (21) and proceeding as before:

$$[M-g^2C(W)/(2l+1)]F(W)-g^2B(W)E(W)=1. \qquad (39)$$

Combining Eq. (39) with Eq. (36), we find

$$E(W)=[2l(W+M)-g^2A(-W)]/H(W). \qquad (40)$$

In Sec. II, we calculated the matrix element for the exchange of a particle of spin $J=l+\frac{1}{2}$ in the u channel using the bare propagator. We get a model that satisfies two-particle unitarity in the u channel by replacing the bare propagator in (11a) with the full propagator that we have just calculated. This gives

$$\mathfrak{M}_J{}^{(+)}=g_+{}^2[(2l+1)/c_l]\bar{u}(p')p'^{\mu_1}\cdots p'^{\mu_l}\times\Delta_{\mu;\nu}{}^J(W)p^{\nu_1}\cdots p^{\nu_l}u(p). \qquad (41)$$

This matrix element can be calculated by using the expansion given in (25) and the results of the Appendix. The result is

$$\mathfrak{M}_J{}^{(+)}=A_1{}^J(W)(\boldsymbol{k}+M)+A_2{}^J(W)(\boldsymbol{k}-W), \qquad (42a)$$

where

$$\begin{aligned}A_1{}^J(W)=g^2(1/2W)\{&p_u{}^{2l}P_{l+1}'(z)D(W)\\&+m^2p_u{}^{2l-2}P_l'(z)[D(-W)-G(W)]\\&-2mE_up_u{}^{2l-2}P_l'(z)[D(-W)-f(W)]\\&+E_u{}^2p_u{}^{2l-2}P_l'(z)[D(-W)-h(W)]\\&+(\text{spin}\leq J-2)\}, \qquad (42b)\end{aligned}$$

$$A_2{}^J(W)=A_1{}^J(-W), \qquad (42c)$$

$$\begin{aligned}f(W)&=G(W)-F(W)\\&=-[(2l+1)M-g^2C(W)+g^2B(W)]/H(W), \qquad (42d)\end{aligned}$$

$$\begin{aligned}h(W)&=G(W)-2F(W)-(2l+1)E(W)\\&=-[2(l+1)(2l+1)M+2l(2l+1)W\\&\quad-2g^2C(W)-(2l+1)g^2A(-W)\\&\quad+g^2B(W)]/H(W), \qquad (42e)\end{aligned}$$

$$E_u{}^2=[W^2+m^2-\mu^2]/2W,$$

and $H(W)$ is given by (38), $G(W)$ by (37), and $D(W)$ by (29).

We can show that $\Delta_{\mu;\nu}{}^J(W)$, and hence $\mathfrak{M}_J{}^{(+)}$, has no singularities by expanding the self-energy functions in Taylor series about $W=0$. To do this, we must first expand $A(W)$, $B(W)$, and $C(W)$ about $W=0$. Equations (30), (32), and (33) express $A(W)$, $B(W)$, and $C(W)$, respectively, as four-dimensional divergent integrals. As mentioned earlier, we make these finite by using a cutoff. This is a rather crude model, but it has all of the important features of a more exact calculation. We use these integrals to express $B(0)$ and $C(0)$ in terms of $A(0)$ and to express $B'(0)$ and $C'(0)$ in terms of $A'(0)$. These relations reflect the fact that $\Sigma_J{}^{\lambda;\sigma}$, as given by Eq. (22), has no singularities in W at $W=0$. The results obtained in this manner are model independent since they depend on the $O(4)$ invariance of the Feynman integral at $W=0$ rather than the details of a model. We find that the functions $A(W)$, $B(W)$, and $C(W)$ can be expanded as[13]

$$A(W)=A(0)+WA'(0)+\cdots\,, \tag{43}$$

$$B(W)=-A(0)-3WA'(0)+\cdots\,, \tag{44}$$

$$C(W)=(2l+1)WA'(0)+\cdots\,. \tag{45}$$

The self-energy functions $A_l(W)$, $B_l(W)$, and $C_l(W)$ have cuts in the W plane from $m+\mu$ to $+\infty$ and from $-(m+\mu)$ to $-\infty$. We use Cutkosky's rules to calculate the discontinuity across the right-hand cut:

$$\begin{aligned}\mathrm{Im}A_l(W)&=-\frac{1}{8\pi^2}\int d^4Q\,\delta_+(Q^2-m^2)\\&\quad\times\delta_+((Q-k)^2-\mu^2)\mathbf{Q}^{2l}(Q_0+m) &(46)\\&=-p(W)^{2l+1}[E(W)+m]/8\pi W\,. &(47)\end{aligned}$$

Similarly, we get that

$$\mathrm{Im}B_l(W)=-(2l+1)p(W)^{2l-1}(E(W)+m)/8\pi W\,, \tag{48}$$

$$\mathrm{Im}C_l(W)=-(2l+1)p(W)^{2l+1}E(W)/8\pi W\,, \tag{49}$$

where

$$p^2(W)=[W^2-(m+\mu)^2][W^2-(m-\mu)^2]/4W^2\,,$$
$$E(W)=(W^2+m^2-\mu^2)/2W\,.$$

We write dispersion relations for $A_l(W)$, $B_l(W)$, and $C_l(W)$ using these approximations of their imaginary parts. While the number of subtractions required is ambiguous, we use the relations at $W=0$ [Eqs. (43)–(45)] as a guide. Once $A_l(W)$ is given, the first two terms in the Taylor expansion of $B_l(W)$ and $C_l(W)$ are determined by the analyticity of $\Delta_{\mu;\nu}{}^J(W)$ at $W=0$. A calculation of the second daughter trajectory would introduce further self-energy functions and more relations at $W=0$. Once $A(W)$ is given, $B(W)$ and $C(W)$ must satisfy dispersion relations twice subtracted at $W=0$ to satisfy Eqs. (43)–(45). Further self-energy functions will presumably be subtracted even more times at $W=0$. Since $D_l(W)$ has a simple pole at the renormalized mass with a residue given by the renormalized coupling constant, one could also introduce two subtractions in the dispersion relation for $A_l(W)$.

[13] Equations (43)–(45) can be obtained by evaluating $A(0)$, etc., by performing a Wick rotation and introducting polar coordinates and a cutoff Λ. In this way, one finds, for example, that

$$A(0)=\frac{2m}{(2\pi)^3}(\sqrt{\pi})\frac{\Gamma(l+\frac{3}{2})}{\Gamma(l+2)}\int_0^\Lambda dR\,\frac{R^{2l+3}}{(R^2+m^2)(R^2+\mu^2)},$$

$$A'(0)=\frac{4}{(2\pi)^3}\,\frac{\Gamma(l+\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(l+3)}\int_0^\Lambda dR\,\frac{R^{2l+5}}{(R^2+m^2)(R^2+\mu^2)^2}.$$

Note that after Reggeization we can let $\Lambda\to\infty$, since $\alpha(0)<\frac{1}{2}$ $(J=l+\frac{1}{2})$.

Since $\mathrm{Im}A_l(W)<0$, we would expect that $A_l(0)<0$. This is confirmed for $J=\frac{1}{2}$ by the Källén-Lehmann representation of the full propagator. The self-energy functions $B_l(W)$ and $C_l(W)$ can be written as

$$\begin{aligned}B_l(W)&=-A_l(0)-3WA_l{}'(0)\\&\quad+\frac{W^2}{\pi}\int_{(m+\mu)}^{\infty}dW'\,\frac{\mathrm{Im}B_l(W')}{W'^2(W'-W-i\epsilon)}\\&\quad+\frac{W^2}{\pi}\int_{-\infty}^{-(m+\mu)}dW'\,\frac{\mathrm{Im}B_l(W')}{W'^2(W'-W-i\epsilon)}\,, &(50a)\end{aligned}$$

$$\begin{aligned}C_l(W)&=W(2l+1)A_l{}'(0)\\&\quad+\frac{W^2}{\pi}\int_{(m+\mu)}^{\infty}dW'\,\frac{\mathrm{Im}C_l(W')}{W'^2(W'-W-i\epsilon)}\\&\quad+\frac{W^2}{\pi}\int_{-\infty}^{-(m+\mu)}dW'\,\frac{\mathrm{Im}C_l(W')}{W'^2(W'-W-i\epsilon)}\,. &(50b)\end{aligned}$$

Using Eqs. (43)–(45), we can expand the self-energy functions in Taylor series. From Eq. (30), we get

$$D(W)=\frac{-1}{M+g^2A(0)}-\frac{W[1-g^2A'(0)]}{[W+g^2A(0)]^2}+\cdots\,. \tag{51}$$

The self-energy functions $f(W)$ and $h(W)$ can be expanded as

$$f(W)=\frac{-1}{M+g^2A(0)}+\cdots\,, \tag{52}$$

$$h(W)=\frac{-2(l+1)}{M+g^2A(0)}-\frac{W\times 2l[1-g^2A'(0)]}{[M+g^2A'(0)]^2}+\cdots\,. \tag{53}$$

From (30) we see that the pole in the W plane is displaced from $W=M(J)$ to the zero of $W-M-g^2\times A(W)=0$. This is the well-known phenomenon of mass renormalization. The full propagator $\Delta_{\mu;\nu}{}^J(W)$ has a simple pole at $W=m^*(J)$, the physical (renormalized) mass, with a residue given by the renormalized coupling constant $G^2(J)$. These effects come only from $D(W)$ and are given by

$$m^*(J)=M(J)+g^2(J)A(m^*(J))\,, \tag{54a}$$

$$G^2(J)=g^2(J)/[1+g^2(J)A'(m^*(J))]\,. \tag{54b}$$

In this calculation we will not rewrite the self-energy functions in terms of the renormalized mass and coupling constant because we are only interested in the qualitative features of the amplitude after Reggeization.

As we mentioned at the beginning of this section, we can get the results for abnormal vertices from the normal ones by $M \to -M$. Since we want Regge poles corresponding to $\frac{1}{2}^+$ particles rather than $\frac{1}{2}^-$ particles, we make the substitution $M \to -M$ before Reggeizing the amplitude.

Applying the Sommerfeld-Watson transform to Eq. (42), one has

$$R=\sum_l [A_1{}^J(W)(\boldsymbol{k}+W)+A_2{}^J(W)(\boldsymbol{k}-W)] \\ =R_1(W)(\boldsymbol{k}+W)+R_2(W)(\boldsymbol{k}-W), \quad (55)$$

where $R_2(W)=R_1(-W)$ and

$$\begin{aligned} R_1(W)=-\tfrac{1}{2}i\int_C \frac{dJ}{\cos\pi J}\frac{1}{2W}p_u{}^{2(J-\frac{1}{2})}P_{J+\frac{1}{2}}{}'(-z_u)\Bigg\{&\frac{g^2(J)}{W+M(J)-g^2(J)A_J(W)} \\ &-(m-E_u)^2\frac{g^2(J+1)}{W-M(J+1)+g^2(J+1)A_{J+1}(-W)}-m^2g^4(J+1)B_{J+1}(W)/\tilde{H}_{J+1}(W) \\ &-2mE_u[-2(J+1)g^2(J+1)M(J+1)+g^4(J+1)(B_{J+1}(W)-C_{J+1}(W))]/\tilde{H}_{J+1}(W) \\ &+E_u{}^2[(2J+3)g^2(J+1)(-2(J+1)M(J+1)+2JW)-2g^4C_{J+1}(W) \\ &-2(J+1)g^4A_{J+1}(-W)+g^4B_{J+1}(W)]/\tilde{H}_{J+1}(W)+(\text{lower-spin terms})\Bigg\}, \end{aligned} \quad (56)$$

where $\tilde{H}$ means that we have substituted $M \to -M$ in (38).

When we open up the contour C, we pick up the leading Regge pole of $J=\alpha_0(W)$, where

$$W-M(\alpha_0(W))+g^2(\alpha_0(W))A_{\alpha_0(W)}(-W)=0. \quad (57)$$

We have defined $\alpha_0(W)$ so that $A_1(W)\sim s^{\alpha_0(-W)-1/2}$ in accordance with the usual conventions. In principle, we could compute $J=\alpha_0(W)$ from (57) once $M(J)$ and g^2 were given. Since the self-energy function $A_J(W)$ has cuts from $W=m+\mu$ to $+\infty$ and from $W=-(m+\mu)$ to $-\infty$, the resulting trajectory would properly become complex above threshold, $|W|>m+\mu$. Here we just take $\alpha_0(W)$ as given and assume that $g^2(J)$ is sufficiently analytic to deform the contour.

In this calculation we are primarily interested in the poles arising from

$$\begin{aligned} \tilde{H}_{J+1}(-W)=&2(J+1)M^2(J+1) \\ &+2g^2(J+1)M(J+1)C_{J+1}(W) \\ &-g^2(2J+1)(W-M(J+1))B_{J+1}(W) \\ &+g^4[C_{J+1}{}^2(W)/2(J+1)+B_{J+1}(W)A_{J+1}(-W)] \\ &=0. \end{aligned} \quad (58)$$

While it is essentially impossible to solve (58) analytically, we can get the qualitative features of the solutions. In particular, we can look at the small-W and small-g^2 limits.

From Eq. (52), we see that $\tilde{H}_{J+1}(0)=0$ has at least two roots:

$$\begin{aligned} \tilde{H}_{J+1}(0)=&[2(J+1)M(J+1)+g^2(J+1)A_{J+1}(0)] \\ &\times[M(J+1)-g^2(J+1)A_{J+1}(0)]=0. \end{aligned} \quad (59)$$

Setting the second factor equal to zero gives

$$J=\alpha_1(0)=\alpha_0(0)-1. \quad (60)$$

Setting the first factor equal to zero gives another root, $J=\alpha_A(0)$, that is not related to the leading trajectory in any simple way. However, the expansions given in (52) and (53) show that $J=\alpha_A(0)$ does not have a singular residue at $W=0$, so that we have not introduced any additional singularities into the amplitude. This is reminiscent of the "aunt" trajectories found in Bethe-Salpeter equation calculations[10]; however, we shall call them companion trajectories.

We can use the theory of implicit functions[14] to obtain integral representations of $\alpha_A(W)$ and $\alpha_1(W)$ for small $|W|$. The first derivative of the leading trajectory at $W=0$ is given by

$$\alpha_0{}'(0)=\frac{-1}{2\pi i}\int_{\gamma_0} dJ\frac{g^2(J)A_J{}'(0)-1}{M(J)-g^2(J)A_J(0)}. \quad (61)$$

Similarly,

$$\alpha_{1,A}{}'(0)=\frac{1}{2\pi i}\int_{\gamma_{1,A}} dJ\frac{-(2J-1)g^2M(J+1)A_{J+1}{}'(0)+(2J+1)g^2A_{J+1}(0)-2g^4A_{J+1}(0)A_{J+1}{}'(0)}{[2(J+1)M(J+1)+g^2A_{J+1}(0)][M(J+1)-g^2A_{J+1}(0)]}, \quad (62)$$

[14] E. Hille, *Analytic Function Theory* (Ginn, Boston, 1959), Vol. I. The theory of inverse and implicit functions is discussed in Sec. 9.4, pp. 265–275.

where γ_1 is a contour that includes only the $J=\alpha_1(0) =\alpha_0(0)-1$ root and γ_A includes only the root $J=\alpha_A(0)$. For the first daughter trajectory, we get that

$$\alpha_1'(0)=\frac{\alpha_0(0)-\frac{1}{2}}{\alpha_0(0)+\frac{1}{2}}\alpha_0'(0), \tag{63}$$

as required by analyticity.[15,16]

For small g^2, $\alpha_A'(0)$ is given by

$$\alpha_A'(0)=-\alpha_1'(0)+O(g^2). \tag{64}$$

For $-\frac{1}{2}<\alpha_0(0)<\frac{1}{2}$, $\alpha_1'(0)<0$ and $\alpha_A'(0)>0$. This is useful in getting a qualitative picture of the first daughter and companion trajectories as a function of W.

One can also compute the residues $\beta(W)$ using the integral-representation approach. It is found that $\beta_1(W)\sim 1/W^2$ and $\beta_A(W)\sim$const for small W, and that the $1/W^2$ and $1/W$ terms in $\beta_1(W)$ have the form required by analyticity.[16]

Next let us consider the small-g^2 limit. As $g^2\to 0$, $\tilde{H}_{J+1}(-W)\to M^2(J+1)$. Let $J=a$ be the value of J such that

$$M(a)=0.$$

Then we can make perturbative expansions about $g^2=0$. If we let $\alpha_{1,A}(0)=a-1+g^2x_{1,A}+\cdots$, then, assuming $A_a(0)<0$, we have

$$x_1<0, \tag{65a}$$

$$x_A>0 \quad \text{for} \quad 0<a<\tfrac{1}{2} \tag{65b}$$

$$<x_1<0 \quad \text{for} \quad -\tfrac{1}{2}<a<0. \tag{65c}$$

The above relations tell us that $\alpha_1(0)<\alpha_A(0)$ for $0<a<\frac{1}{2}$ and $\alpha_A(0)<\alpha_1(0)$ for $-\frac{1}{2}<a<0$. The point $a=0$ is a singular point of the system and must be excluded from the present discussion.

We can calculate the large-W behavior by a perturbative expansion of $J=\alpha(W,g)$ about $g=0$ by setting

$$J=a-1+gb(W)+\tfrac{1}{2}g^2c(W)+\cdots \tag{66}$$

and solving for the unknown functions $b(W)$, $c(W)$, $\ldots$.

Since the equation $H_{J+1}(-W)=0$ is quadratic in $M(J+1)$, it can be factored into two factors linear in $M(J+1)$:

$$\begin{aligned}\tilde{H}_{J+1}(-W)=2(J+1)\bigg(&M(J+1)+\frac{g^2\Phi(J+1,\,W)}{4(J+1)}\\&-\frac{g}{4(J+1)}[g^2\Phi(J+1,\,W)-8(J+1)(W\Theta+g^2\Psi)]^{1/2}\bigg)\\ \times\bigg(&M(J+1)+\frac{g^2\Phi(J+1,\,W)}{4(J+1)}+\frac{g}{4(J+1)}\\&\times[g^2\Phi-8(J+1)(W\Theta+g^2\Psi)]^{1/2}\bigg),\end{aligned} \tag{67}$$

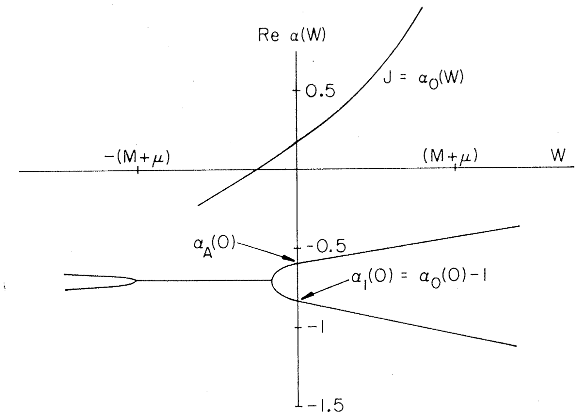

FIG. 4. Qualitative behavior of the Regge trajectories in the small-g^2 limit for $0<\alpha_0(0)<\frac{1}{2}$.

where

$$\begin{aligned}\Phi(J,W)&=2C_J(-W)+(2J-1)B_J(-W),\\ \Theta(J,W)&=(2J-1)B_J(-W),\\ \Psi(J,W)&=C_J(-W)/2J+B_J(-W)A_J(W).\end{aligned}$$

When $g^2\Phi-8(J+1)(W\Theta+g^2\Psi)=0$, the two factors are equal and we have a branch point of $J=\alpha(W)$. This means that there is at least one point at which the two roots $\alpha_A(W)$ and $\alpha_1(W)$ coalesce. When $g^2=0$, this point is at $W=0$. As $g^2\to 0$, this point moves toward the Reα axis in a plot of Reα versus W.

When

$$g^2|\Phi^2-8(J+1)\Psi|<|8(J+1)W\Theta|,$$

we can expand the square root in a power series in g^2 using the binomial theorem. This expansion is valid for large W, but not for the region around $W=0$. If we set one of the two factors in (67) equal to zero, expand the square root, and use Eq. (66), we get, upon setting the terms of order g equal to zero,

$$b(W)=\mp\alpha_0'(0)[-(2a-1)WB_a(-W)/2a]^{1/2}. \tag{68}$$

Notice that $J=\alpha(W;g)$ is not an analytic function of g^2.

Since $B_a(W)$ has cuts from $m+\mu$ to $+\infty$ and from $-(m+\mu)$ to $-\infty$, $b(W)$ has a nonzero real part for $|W|>m+\mu$. For $|W|\to\infty$, $b(W)\sim\pm W$.

Figure 4 shows how the parent and first daughter trajectories look for $0<a<\frac{1}{2}$. For this case, $\alpha_1(0)<a-1<\alpha_A(0)$. We also know that $\alpha_1'(0)<0$ and that $\alpha_A'(0)\simeq-\alpha_1'(0)$. This means that the branch point occurs for W small and negative. When $W<-(m+\mu)$, $b(W)$ becomes complex and $J=\alpha(W)$ again has a nonzero real part.

Figure 5 shows how the trajectories behave as $g^2\to 0$. The first daughter and companion trajectories coalesce and form a fixed double pole as the two branches approach each other. The branch point moves towards $W=0$ as $g^2\to 0$. For W less than the branch point, $\alpha(W)$ is imaginary. As $g^2\to 0$, the two imaginary branches approach the real plane and become part of

[15] G. Domokos and P. Suranyi, Nuovo Cimento **56A**, 445 (1968); **57A**, 813 (1968); N. W. MacFadyen, Phys. Rev. **171**, 1691 (1968).

[16] P. K. Kuo and J. Walker, Phys. Rev. **175**, 1794 (1968); D. Steele and J. D. Sullivan, *ibid.* **166**, 1515 (1968).

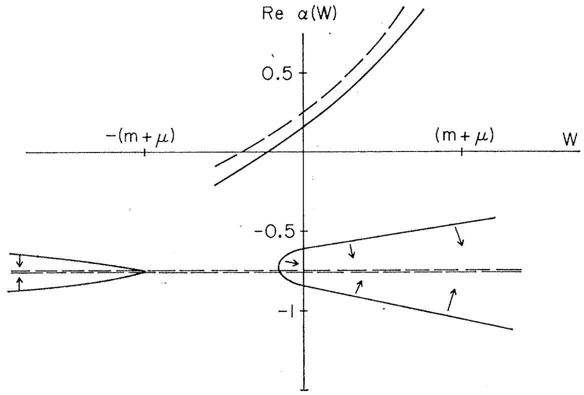

FIG. 5. Behavior of the trajectories shown in Fig. 4 as $g^2 \to 0$. The solid lines are the trajectories for $g^2 \neq 0$; the dashed lines are the trajectories for $g^2=0$. When $g^2=0$, the first daughter and companion trajectories coalesce to form a fixed double pole.

the fixed double pole. For $W<-(m+\mu)$, the two complex branches approach the fixed double pole as $g^2 \to 0$. The intercept at $W=0$, $\alpha_0(0)$, moves toward a as $g^2 \to 0$; for $g^2 \neq 0$, $\alpha_0(0) \neq a$ because of mass renormalization.

Figure 6 shows the trajectories for $-\frac{1}{2}<a<0$. In this case, $\alpha_A(0)<\alpha_1<a-1$. We also know that $\alpha_1'(0)<0$ and $\alpha_A'(0) \simeq -\alpha_1(0)$. This means that the branch point occurs for small, positive W. Above threshold, $|W|>m+\mu$, $b(W)$ becomes complex, so that $J=\alpha(W)$ again has a nonzero real part.

IV. CONCLUSIONS

The daughter and companion trajectories seem to depend quite sensitively on the details of the model and are therefore much harder to interpret than the leading trajectory. The companion trajectory $\alpha_A(0)$ must have a series of daughters at $\alpha_A(0)-1$, $\alpha_A(0)-2$, ..., to cancel the singularities that the companion trajectory introduces. Since in the simple model without self-energy bubbles the second daughter contains a fixed fourth-order pole, we speculate that when self-energy bubbles are included, the result will be a second daughter and three other trajectories, one of which will presumably be the first daughter of the companion trajectory we have just calculated. Lower daughter trajectories are progressively more complicated.

We have assumed that α is a function of W instead of $u=W^2$. It is sometimes felt that experiment suggests that the leading fermion trajectories are MacDowell degenerate, i.e., that they have the form $\alpha(W)=a+bW^2$. While it is possible to get leading trajectories of this form out of the model, it is unlikely that the daughters will be MacDowell degenerate. If $\alpha_0'(0)=0$, then analyticity at $W=0$ requires that $\alpha_n'(0)=0$,[15,16] where $\alpha_n'(0)$ is the slope of the nth daughter at $W=0$. In this model, however, $\alpha_A'(0)$ is simply related to $\alpha_1'(0)$ only in the small-g^2 limit. Equation (61) reveals that $\alpha_0'(0)=0$ requires that $g^2 A_{\alpha_0(0)}'(0)=1$, so that $\alpha_A'(0)$ will be nonzero in general. Since there will still be a branch point, there will be no sharp distinctions between the first daughter and companion trajectories away from $W=0$. The second and further daughters will be even more complicated. Such a model is indeed difficult to reconcile with a simple picture of a parent and a series of daughter trajectories of the form $\alpha(W)=a+bW^2$.

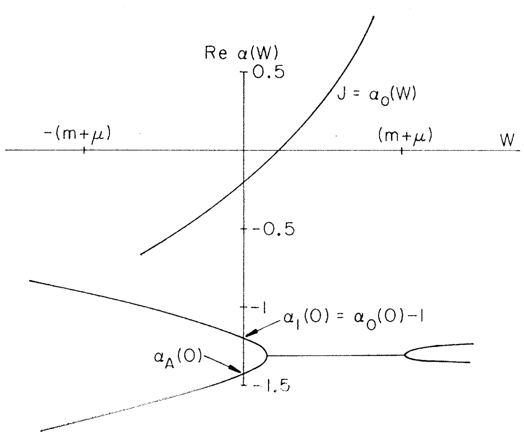

FIG. 6. Qualitative behavior of the Regge trajectories in the small-g^2 limit for $-\frac{1}{2}<\alpha_0(0)<0$.

ACKNOWLEDGMENTS

I am grateful to Professor J. D. Sullivan for suggesting this topic and for his guidance and encouragement throughout its development. I also wish to acknowledge the hospitality of the Aspen Center for Physics and the Boulder Summer Institute for Theoretical Physics where part of this manuscript was prepared.

APPENDIX

In this appendix we summarize the important properties of fermion propagators. We discuss, in order, the half-integral-spin projection operator, an expression for the half-integral-spin projection operator in terms of integral-spin operators and γ matrices, and the off-mass-shell propagator.

We write the projection operator of a state with definite spin and parity as $T_{\mu;\nu}{}^J(W^2)(\pm \boldsymbol{k}+W)/2W$, where $W^2=k^2$ and $(\pm \boldsymbol{k}+W)/2W$ is the energy projection operator. The argument W^2 of $T_{\mu;\nu}{}^J(W^2)$ distinguishes it from the numerator of the off-mass-shell propagator $T_{\mu;\nu}{}^J(M^2)$. The tensor $T_{\mu;\nu}{}^J(W^2)$ has $2l$ indices, where $J=l+\frac{1}{2}$, and is determined by the following conditions[17]:

$$\begin{aligned}
&\text{(a)} \quad T_{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_l;\,\nu}{}^J(k^2)=T_{\mu_1\cdots\mu_j\cdots\mu_i\cdots\mu_l;\,\nu}{}^J(k^2)\,,\\
&\text{(b)} \quad k^{\mu_1}T_{\mu_1\cdots\mu_l;\,\nu}{}^J(k^2)=0\,,\\
&\text{(c)} \quad \gamma^{\mu_1}T_{\mu_1\cdots\mu_l;\,\nu}{}^J(k^2)=0\,,\\
&\text{(d)} \quad T_{\mu;\,\delta}{}^J(k^2)T^{\delta;}{}_{\nu}{}^J(k^2)=T_{\mu;\,\nu}{}^J(k^2)\,.
\end{aligned} \tag{A1}$$

[17] C. Fronsdal, Nuovo Cimento Suppl. 9, 416 (1958).

Similar conditions hold for the ν indices. From conditions (a) and (c), it can be shown that $T_{\mu;\nu}{}^J(W^2)$ is also traceless:

$$g^{\mu_1\mu_2}T_{\mu;\nu}{}^J(W^2)=0. \tag{A2}$$

In many calculations it is useful to have a formula expressing $T_{\mu;\nu}{}^J(W^2)$ in terms of the integral-spin projection operator $\Gamma_{\mu;\nu}{}^l(W^2)$, γ matrices, etc. Fronsdal[17] gives the following formula:

$$T_{\mu_1\cdots\mu_l;\nu_1\cdots\nu_l}{}^J(k^2)=\frac{l+1}{2l+3}\gamma^{\mu_{l+1}}\Gamma_{\mu_1\cdots\mu_{l+1};\nu_1\cdots\nu_{l+1}}{}^{l+1}(k^2)\gamma^{\nu_{l+1}}. \tag{A3}$$

One can easily verify that (A3) satisfies the conditions of (A1). Physically, (A3) corresponds to adding a spin-$(l+1)$ operator to a spin-$\frac{1}{2}$ operator to get a total spin of $J=l+\frac{1}{2}$.

Another useful form of $T_{\mu;\nu}{}^J(k^2)$ can be derived by applying the relativistic generalization of the usual spin $l+\frac{1}{2}$ projection operator to $\Gamma_{\mu;\nu}{}^l(k^2)(\boldsymbol{k}+W)$, where $k^2=W^2$. This is a relativistic generalization of Zemach's work[18] and has the advantage of a clear physical motivation. The tensor $\Gamma_{\mu;\nu}{}^l(k^2)(\boldsymbol{k}+W)$ is a mixture of spin $l+\frac{1}{2}$ and $l-\frac{1}{2}$ since it is traceless, symmetric, and satisfies $k^{\mu_1}\Gamma_{\mu;\nu}{}^l(k^2)(\boldsymbol{k}+W)=0$. From the conditions in (A1), the part that gives zero when contracted with γ^{μ_1} is pure spin $l+\frac{1}{2}$; the other part is pure spin $l-\frac{1}{2}$, although it has $2l$ indices.

The result is

$$T_{\mu;\nu}{}^J(W^2)(\boldsymbol{k}+W)=\frac{l+1}{2l+3}\left[\Gamma_{\mu;\nu}{}^l(W^2)+\frac{i}{l+1}\sum_{n=1}^{l}\sigma_{\mu_n}{}^{\mu_n'}(W^2)\times\Gamma_{\mu_1\cdots\mu_n'\cdots\mu_l;\nu}{}^l(W^2)\right](\boldsymbol{k}+W), \tag{A4}$$

where

$$\sigma_{\mu\nu}(W^2)=\tfrac{1}{2}i[\gamma_\mu(W^2),\gamma_\nu(W^2)]$$

and

$$\gamma_\mu(W^2)=\gamma_\mu-\boldsymbol{k}k_\mu/W^2.$$

In the center-of-mass frame ($\mathbf{k}=0$),

$$\sigma_{ij}(W^2)=\sigma_{ij}=\epsilon_{ijk}\sigma_k,$$
$$\sigma_{0\nu}(W^2)=0,$$
$$\gamma_\mu(W^2)=(0,\boldsymbol{\gamma}).$$

An equivalent form of Eq. (A4) is

$$T_{\mu;\nu}{}^J(k^2)(\boldsymbol{k}+W)=\left[\Gamma_{\mu;\nu}{}^l(k^2)-\frac{1}{2l+1}\sum_{n=1}^{l}\gamma_{\mu_n}(k^2)\times\gamma^{\mu_n'}\Gamma_{\mu_1\cdots\mu_n'\cdots\mu_l;\nu}{}^l(k^2)\right](\boldsymbol{k}+W). \tag{A5}$$

Equation (A5) has an interesting interpretation. Physically, Eq. (A5) corresponds to adding spin $\frac{1}{2}$ to spin l to get total spin $J=l+\frac{1}{2}$. The projection operator for spin $l-\frac{1}{2}$ is $Q^{l-\frac{1}{2}}=1-P^{l+\frac{1}{2}}$. Hence, a spin projection for spin $l-\frac{1}{2}$ is

$$\Theta_{\mu_1\cdots\mu_l;\nu_1\cdots\nu_l}{}^{J-1}(W^2)=\frac{1}{2l+1}\sum_{n=1}^{l}\gamma_{\mu_n}(W^2)\gamma^{\mu_n'}\Gamma_{\mu_1\cdots\mu_n'\cdots\mu_l;\nu}{}^l(W^2). \tag{A6}$$

Therefore in Eq. (A6), the second term is the spin-$(l-\frac{1}{2})$ part that is being removed from $\Gamma_{\mu;\nu}{}^l(W^2)\times(\boldsymbol{k}+W)$. Although $\Theta_{\mu;\nu}{}^{J-1}(W^2)$ represents spin $l-\frac{1}{2}$, it has $2l$ indices and must satisfy a more complicated set of constraints than Eq. (A1).

There is another projection operator of spin $l-\frac{1}{2}$ that is orthogonal to both $T_{\mu;\nu}{}^J(W^2)$ and $\Theta_{\mu;\nu}{}^{J-1}(W^2)$:

$$S_{\mu;\nu}{}^{l-\frac{1}{2}}(k^2)=\frac{1}{l}\sum_{i,j}\frac{k_{\mu_i}k_{\nu_j}}{k^2}T_{\mu[\mu_i];\nu[\nu_j]}{}^{J-1}(k^2). \tag{A7}$$

From (A1), it is obvious that $g^{\mu_l\nu_l}T_{\mu_1\cdots\mu_l;\nu_1\cdots\nu_l}{}^J(k^2)$ is proportional to $T_{\mu;\nu}{}^{J-1}(k^2)$:

$$g^{\mu_l\nu_l}T_{\mu_1\cdots\mu_l;\nu_1\cdots\nu_l}{}^J(k^2)=[(l+1)/l]T_{\mu_1\cdots\mu_{l-1};\nu_1\cdots\nu_{l-1}}{}^{J-1}(k^2). \tag{A8}$$

From (A9), we get the following useful relation:

$$T_{\mu_1\cdots\mu_l;}{}^{\mu_1\cdots\mu_l J}(k^2)=l+1. \tag{A9}$$

With the kinematics defined as in Sec. II, the contracted form of the spin projection operator is

$$p'^{\mu_1}\cdots p'^{\mu_l}T_{\mu;\nu}{}^J(k^2)p^{\nu_1}\cdots p^{\nu_l}(\boldsymbol{k}+W)=\frac{(-1)^lc_l}{2l+1}[\mathbf{p}^{2l}P_{l+1}'(z)(\boldsymbol{k}+W)-(m+E)^2(\boldsymbol{k}-W)\mathbf{p}^{2l-2}P_l'(z)]. \tag{A10}$$

(See Ref. 12 for formulas with one or two indices left uncontracted.)

Next we come to the problem of choosing the off-mass-shell propagator. This is even more complicated than it is for bosons. The naive approach would be to substitute the corresponding expansion of $\Gamma_{\mu;\nu}{}^l(M^2)$ given in Ref. 14 into (A3) or (A5). However, this method produces extraneous terms of spin $l+\frac{1}{2}$ and lower. To illustrate this, let us substitute into (A5):

$$T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)=\frac{l+1}{2l+3}\gamma^{\mu_{l+1}}\Gamma_{\mu;\nu}{}^{l+1}(k^2)\gamma^{\nu_{l+1}}(\boldsymbol{k}+M)-\frac{k^2-M^2}{(2l+3)M^2}\times\Gamma_{\mu;\nu}{}^l(k^2)(\boldsymbol{k}+M)+(\text{lower-spin terms}). \tag{A11}$$

The second term, $[(k^2-M^2)/M^2]\Gamma_{\mu;\nu}{}^l(k^2)(\boldsymbol{k}+M)$, vanishes on the mass shell and is a mixture of spin $l+\frac{1}{2}$ and spin $l-\frac{1}{2}$. Furthermore, since its leading term is not singular at $k^2=0$, it is not needed to cancel singularities in the first term of (A11). In addition, it intro-

[18] C. Zemach, Phys. Rev. 140, B97 (1965).

duces singularities of its own that must be canceled by additional projection operators of lower spin. This is clearly an unnecessary complication. This same ambiguity is illustrated by the fact that when in the three forms of the projection operator, given in (A3), (A4), and (A5), respectively, we substitute $\boldsymbol{k}\pm W\to\boldsymbol{k}\pm M$ and $g_{\mu\nu}(k^2)\to g_{\mu\nu}(M^2)$ everywhere, as suggested by Feynman perturbation theory, the three resulting expressions are equal only on the mass shell ($k^2=M^2$); off the mass shell, all three take on different values. Physically the differences correspond to nonsingular couplings of the daughter trajectories.

There is a systematic procedure for eliminating such terms. We start by taking (A5) off the mass shell. We do this by replacing $\boldsymbol{k}\pm W\to\boldsymbol{k}\pm M$ and $g_{\mu\nu}(k^2)\to g_{\mu\nu}(M^2)$ everywhere. We start with

$$\Gamma_{\mu;\nu}{}^{l}(M^2)(\boldsymbol{k}+M)+\frac{1}{2l+1}\sum_n\gamma_{\mu_n}(M^2)(\boldsymbol{k}-M)\times\gamma^{\mu_n'}\Gamma_{\mu_1\cdots\mu_n'\cdots\mu_l;\nu}{}^{l}(M^2)\,,\quad\text{(A12)}$$

where $\gamma_\mu(M^2)=\gamma^\delta g_{\delta\mu}(M^2)$. The second term, however, still contains unwanted terms that vanish on the mass shell. They can be eliminated by expanding (A12) using Durand's[9] expansion of $\Gamma_{\mu;\nu}{}^{l}(M^2)$ and making the following substitution:

$$\left(\gamma_\mu-\frac{k_\mu\boldsymbol{k}}{M^2}\right)(\boldsymbol{k}-M)\left(\gamma_\nu-\frac{k_\nu\boldsymbol{k}}{M^2}\right)\to\left(\gamma_\mu+\frac{k_\mu}{M}\right)(\boldsymbol{k}-M)\left(\gamma_\nu+\frac{k_\nu}{M}\right).\quad\text{(A13)}$$

Since $\boldsymbol{k}(\boldsymbol{k}-M)=-M(\boldsymbol{k}-M)$ if $k^2=M^2$, the two expressions are equal on the mass shell. Unfortunately, the resulting off-mass-shell propagator is difficult to express in closed form. However, a closed form is not needed in order to expand $T_{\mu;\nu}{}^{J}(M)$ in a series of projection operators.

To complete the expansion we must expand expressions like Eq. (A13) in projection operators. The factor $\boldsymbol{k}\pm M$ is first expanded as

$$\boldsymbol{k}\pm M=\frac{W\pm M}{2W}(\boldsymbol{k}+W)+\frac{W\mp M}{2W}(\boldsymbol{k}-W)\,,\quad\text{(A14)}$$

where $k^2=W^2$. Using (A14) and the relations $\boldsymbol{k}(\boldsymbol{k}+W)=W(\boldsymbol{k}+W)$ and $\boldsymbol{k}(\boldsymbol{k}-W)=-W(\boldsymbol{k}-W)$, we get

$$\begin{aligned}(\gamma_\mu+k_\mu/M)(\boldsymbol{k}-M)(\gamma_\nu+k_\nu/M)&\\=\frac{W-M}{2W}\gamma_\mu(W^2)(\boldsymbol{k}+W)\gamma_\nu(W^2)&+\frac{W+M}{2W}\gamma_\mu(W^2)(\boldsymbol{k}-W)\gamma_\nu(W^2)+\frac{W^2-M^2}{2MW^2}k_\mu(\boldsymbol{k}+W)\gamma_\nu(W^2)\\&+\frac{W^2-M^2}{2MW^2}k_\mu(\boldsymbol{k}-W)\gamma_\nu(W^2)+\frac{W^2-M^2}{2MW^2}\gamma_\mu(W^2)(\boldsymbol{k}+W)k_\nu+\frac{W^2-M^2}{2MW^2}\gamma_\mu(W^2)(\boldsymbol{k}-W)k_\nu\\&+\frac{W^2-M^2}{M^2W^2}\frac{W+M}{2W}(\boldsymbol{k}+W)k_\mu k_\nu+\frac{W^2-M^2}{M^2W^2}\frac{W-M}{2W}(\boldsymbol{k}-W)k_\mu k_\nu\,,\end{aligned}\quad\text{(A15)}$$

where $\gamma_\mu(W^2)=\gamma^\nu g_{\mu\nu}(W^2)$.

Using the substitution given in (A13) and the expansion given in Ref. 9, we get

$$\sum_n\gamma_{\mu_n}(M^2)(\boldsymbol{k}-M)\gamma^{\mu_n'}\Gamma_{\mu_1\cdots\mu_n'\cdots\mu_l;\nu}{}^{l}(M^2)\to\sum_n\sum_{\text{perm}}(\gamma_{\mu_n}+k_{\mu_n}/M)(\boldsymbol{k}-M)(\gamma_{\nu_{i(n)}}+k_{\nu_{i(n)}}/M)g_{\mu_1\nu_{i(1)}}(M^2)\cdots[g_{\mu_n\nu_{i(n)}}(M^2)]\cdots g_{\mu_l\nu_{i(l)}}(M^2)+\cdots,\quad\text{(A16)}$$

where the expression in square brackets is to be omitted. We now substitute (A15) and

$$g_{\mu\nu}(M^2)=g_{\mu\nu}(W^2)-(W^2-M^2)k_\mu k_\nu/M^2W^2\quad\text{(A17)}$$

into Eq. (A16) and express the results as projection operators of various spins. Using (A15) and replacing all $g_{\mu\nu}(M^2)$ by $g_{\mu\nu}(W^2)$ gives the spin-$(l-\frac{1}{2})$ terms. Replacing one or more $g_{\mu\nu}(M^2)$ by $-(W^2-M^2)k_\mu k_\nu/M^2W^2$ and the rest by $g_{\mu\nu}(W^2)$ gives spin-$(l-\frac{3}{2})$ and lower terms. The final result is, keeping only $l+\frac{1}{2}$ and $l-\frac{1}{2}$ terms,

$$\begin{aligned}T_{\mu;\nu}{}^{J}(M)(\boldsymbol{k}+M)&=T_{\mu;\nu}{}^{J}(W^2)\left[\frac{W+M}{2W}(\boldsymbol{k}+W)+\frac{W-M}{2W}(\boldsymbol{k}-W)\right]\\&-\frac{1}{2l+1}\frac{W^2-M^2}{2MW^2}\Big[\sum_n k_{\mu_n}\gamma^{\mu_n'}\Gamma_{\mu_1\cdots\mu_n'\cdots\mu_l;\nu}{}^{l}(W^2)-\sum_n\Gamma_{\mu;\nu_1\cdots\nu_n'\cdots\nu_l}{}^{l}(k^2)\gamma^{\nu_n'}k_{\nu_n}\Big]\big[(\boldsymbol{k}+W)+(\boldsymbol{k}-W)\big]\\&-\frac{2l}{2l+1}\frac{W^2-M^2}{M^2}S_{\mu;\nu}{}^{J-1}(W^2)\left[\frac{W+M}{2W}(\boldsymbol{k}+M)+\frac{W-M}{2W}(\boldsymbol{k}-W)\right]+[\text{spin-}(J-2),\ \ldots,\text{ terms}].\end{aligned}\quad\text{(A18)}$$

The contracted form of the off-mass-shell propagator can be expressed in closed form or in a series corresponding to Eq. (A18). The closed form is obtained from Eq. (A10) by substituting $\bar{p}^2$ for $\mathbf{p}^2$ and $\bar{p}^2\cos\bar{\theta}$ for $\mathbf{p}^2\cos\theta$; in addition, we replace

$$\boldsymbol{k}\pm W\rightarrow\boldsymbol{k}\pm M \quad\text{and}\quad E\rightarrow\bar{E}\equiv(p\cdot k)/M.$$

Starting from the form given in Eq. (A10) is equivalent to the substitution stated in Eq. (A13). Then the off-mass-shell equivalent is

$$\begin{aligned}p'^{\mu_1}\cdots p'^{\mu_l}T_{\mu;\nu}{}^J(M)p^{\nu_1}\cdots p^{\nu_l}(\boldsymbol{k}+M)&\\=\frac{(-1)^l c_l}{2l+1}\Big[\bar{p}^{2l}P_{l+1}'(\bar{z})(\boldsymbol{k}+M)&\\-(m+\bar{E})^2\bar{p}^{2l-2}P_l'(\bar{z})(\boldsymbol{k}-M)\Big],&\qquad\text{(A19)}\end{aligned}$$

where

$$\begin{aligned}\bar{p}^2&=\bar{p}'^2=-p^\mu p^\nu g_{\mu\nu}(M^2),\\ \bar{p}^2\cos\bar{\theta}&=-p^\mu p^\nu g_{\mu\nu}(M^2),\\ \bar{E}&=p^\mu k_\mu/M.\end{aligned}$$

The contracted form of Eq. (A18) can be derived by either contracting Eq. (A18) with the initial and final momenta or by expanding Eq. (A19) using the well-known properties of Legendre polynomials. The result is

$$\begin{aligned}p'^{\mu_1}\cdots p'^{\mu_l}T_{\mu;\nu}{}^J(M)p^{\nu_1}\cdots p^{\nu_l}(\boldsymbol{k}+M)&\\=\frac{(-1)^l c_l}{2l+1}\Bigg[\mathbf{p}^{2l}P_{l+1}'(z)(\boldsymbol{k}+M)&\\+(2l+1)\left(\frac{W^2-M^2}{M^2}\right)E^2\mathbf{p}^{2l-2}P_l'(z)(\boldsymbol{k}+M)&\\-\left(m+\frac{WE}{M}\right)^2\mathbf{p}^{2l-2}P_l'(z)(\boldsymbol{k}+M)+\cdots\Bigg],&\qquad\text{(A20)}\end{aligned}$$

where $E=(p\cdot k)/W$.

Finally, we must consider the effect of parity. A fermion of spin $J=l+\frac{1}{2}$ is said to have "normal" parity if its parity is $(-1)^l$ (i.e., $J^P=\frac{1}{2}^+,\ \frac{3}{2}^-,\ \frac{5}{2}^+,\ \ldots$); a boson has "normal" parity if its parity is $(-1)^J$ (i.e., $J^P=0^+,\ 1^-,\ 2^+,\ \ldots$). A particle with the opposite parity has "abnormal" parity ($J^P=\frac{1}{2}^-,\ \frac{3}{2}^+,\ \ldots$, for fermions; $J^P=0^-,\ 1^+,\ \ldots$, for bosons). The normality of a vertex is the product of the normalities of each particle, $n_v=n_1n_2n_3$. We can go from a normal vertex to an abnormal vertex in a fermion-fermion-boson interaction by inserting a factor of $(i\gamma_5)$ at each vertex. In π-N scattering, for example, this means that exchanging a particle with $J^P=\frac{1}{2}^+,\ \frac{3}{2}^-,\ \frac{5}{2}^+$ results in an abnormal vertex and an $i\gamma_5$ factor is needed at each vertex. Inserting $i\gamma_5$'s at the two vertices is equivalent to substituting $M\rightarrow -M$ in the propagator. The off-mass-shell propagator is obtained from Eq. (A12) by expanding and using the substitution in Eq. (A13). Since

$$(i\gamma_5)(\boldsymbol{k}+M)(i\gamma_5)=\boldsymbol{k}-M,$$

$$\begin{aligned}(i\gamma_5)\left(\gamma_\mu+\frac{k_\mu}{M}\right)(\boldsymbol{k}-M)\left(\gamma_\nu+\frac{k_\nu}{M}\right)(i\gamma_5)&\\=\left(\gamma_\mu-\frac{k_\mu}{M}\right)(\boldsymbol{k}+M)\left(\gamma_\nu-\frac{k_\nu}{M}\right),&\end{aligned}$$

we can calculate $(i\gamma_5)T_{\mu;\nu}{}^J(M)(\boldsymbol{k}+M)(i\gamma_5)$ by making the substitution $M\rightarrow -M$.