Closed-Loop Calculations Using a Chiral-Invariant Lagrangian

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Using Feynman rules derived from the chiral-invariant Lagrangian for pions, explicit evaluation is made of all contributions to all Feynman amplitudes in the limit of zero external 4-momenta, through order f_π^{-6}. These calculations are formal in that they involve divergent integrals. For all except the tree-graph diagrams, the results depend explicitly upon the pion "gauge," or Weinberg's function $f(\boldsymbol{\pi}^2)$. However, we show that through order f_π^{-6} there is a unique choice of gauge for which, notwithstanding the divergences, the amplitudes vanish in the soft-pion limit in accord with the soft-pion theorems. We conjecture concerning further application of this special gauge for the removal of divergences.

I. INTRODUCTION

As was shown by Weinberg,[1] the effective Lagrangian for massless pions with $SU(2)\otimes SU(2)$ invariance may be written as

$$L=\tfrac{1}{2}(D_\mu\pi_a)(D_\mu\pi_a). \tag{1.1}$$

Here D_μ denotes the covariant derivative of the pion fields π_a ($a=1,2,3$), and may be written as

$$D_\mu\pi_a=d_{ab}(\boldsymbol{\pi})\partial_\mu\pi_b. \tag{1.2}$$

The matrix function $d_{ab}(\boldsymbol{\pi})$ depends on the manner in which the pion fields transform under the chiral generators X_a of $SU(2)\otimes SU(2)$. Following Weinberg, we write

$$[X_a,\pi_b]=-if_{ab}(\boldsymbol{\pi}), \tag{1.3}$$

with

$$f_{ab}(\boldsymbol{\pi})=\delta_{ab}f(\boldsymbol{\pi}^2)+\pi_a\pi_b g(\boldsymbol{\pi}^2). \tag{1.4}$$

Then although $f(\boldsymbol{\pi}^2)$ is arbitrary once we have ensured that $f(0)=f_\pi$, the pion decay constant, the requirement that the fields π_a transform under a nonlinear realization of $SU(2)\otimes SU(2)$ enforces the specification of $g(\boldsymbol{\pi}^2)$ by

$$g(\boldsymbol{\pi}^2)=\frac{1+2f(\boldsymbol{\pi}^2)f'(\boldsymbol{\pi}^2)}{f(\boldsymbol{\pi}^2)-2\boldsymbol{\pi}^2f'(\boldsymbol{\pi}^2)}. \tag{1.5}$$

Three choices of $f(\boldsymbol{\pi}^2)$ occur with some frequency in the literature. Weinberg himself often uses

$$f(\boldsymbol{\pi}^2)=f_\pi(1-\boldsymbol{\pi}^2/4f_\pi^2). \tag{1.6}$$

Other authors, for example, Bardeen and Lee,[2] make a choice which leads essentially to the σ model,

$$f(\boldsymbol{\pi}^2)=(f_\pi^2-\boldsymbol{\pi}^2)^{1/2}. \tag{1.7}$$

The transformation law used, for example, by Callan *et al.*,[3] by Isham,[4] and by the present author in another paper,[5] corresponds to the choice

$$f(\boldsymbol{\pi}^2)=(\boldsymbol{\pi}^2)^{1/2}\cot[(\boldsymbol{\pi}^2)^{1/2}/f_\pi]. \tag{1.8}$$

The matrix d_{ab} which enters into the definition of $D_\mu\pi_a$ is given in terms of this arbitrary function $f(\boldsymbol{\pi}^2)$ by

$$d_{ab}(\boldsymbol{\pi})=\delta_{ab}\frac{f_\pi}{[(f(\boldsymbol{\pi}^2))^2+\boldsymbol{\pi}^2]^{1/2}}+\frac{\pi_a\pi_b}{\boldsymbol{\pi}^2}\times\frac{f_\pi\{f(\boldsymbol{\pi}^2)-[(f(\boldsymbol{\pi}^2))^2+\boldsymbol{\pi}^2]^{1/2}-2\boldsymbol{\pi}^2f'(\boldsymbol{\pi}^2)\}}{(f(\boldsymbol{\pi}^2))^2+\boldsymbol{\pi}^2}, \tag{1.9}$$

from which it follows that the Lagrangian is

$$L=\tfrac{1}{2}f_\pi^2\{\delta_{ab}[f^2+\boldsymbol{\pi}^2]^{-1}+\pi_a\pi_b[f^2+\boldsymbol{\pi}^2]^{-2}\times[4\boldsymbol{\pi}^2(f')^2-4ff'-1]\}(\partial_\mu\pi_a)(\partial_\mu\pi_b). \tag{1.10}$$

The equivalence theorem[6] ensures that on-shell T-matrix elements derived from this Lagrangian are independent of the choice made for the function $f(\boldsymbol{\pi}^2)$. However, the Lagrangian is not renormalizable in the conventional sense, and is usually used only as an effective Lagrangian, which is to say that only the tree-graph contributions to a given matrix element are utilized. The corollary of the equivalence theorem proved by Coleman *et al.*[3] ensures that these too will be independent of $f(\boldsymbol{\pi}^2)$.

Let us write

$$f(\boldsymbol{\pi}^2)=f_\pi\sum_{n=0}^{\infty}a_n\left(\frac{\boldsymbol{\pi}^2}{f_\pi^2}\right)^n, \tag{1.11}$$

where $a_0=1$ and the remaining coefficients a_n are arbitrary real numbers.

To use the Lagrangian L to calculate the T-matrix element for π-π scattering requires only the isolation of the quartic term in L, which involves the parameter a_1. Explicitly, we have

$$L_4=-f_\pi^{-2}\,\tfrac{1}{2}[(1+2a_1)\boldsymbol{\pi}^2(\partial_\mu\boldsymbol{\pi})^2+(1+4a_1)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2]. \tag{1.12}$$

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[1] S. Weinberg, Phys. Rev. **166**, 1568 (1968).
[2] W. A. Bardeen and B. W. Lee, Phys. Rev. **177**, 2389 (1969).
[3] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, Jr., S. Coleman, J. Wess, and B. Zumino, *ibid.* **177**, 2247 (1969).
[4] C. J. Isham, Nuovo Cimento **59A**, 356 (1969).
[5] J. M. Charap, Can. J. Phys. (to be published).
[6] J. S. R. Chisholm, Nucl. Phys. **26**, 469 (1961); S. Kamefuchi, L. O'Raifeartaigh, and A. Salam, *ibid.* **28**, 529 (1961).

The three examples given in Eqs. (1.6)-(1.8) for $f(\boldsymbol{\pi}^2)$ have, respectively,

$$a_1=-\tfrac{1}{4}, \quad a_1=-\tfrac{1}{2}, \quad a_1=-\tfrac{1}{3}. \tag{1.13}$$

In the general case, if we calculate the tree-graph contribution to the T-matrix element for π-π scattering illustrated in Fig. 1, we obtain

$$\begin{aligned}-f_\pi^{-2}\{&\delta_{ab}\delta_{cd}[(1+2a_1)(s_{ab}+s_{ac}+s_{ad})-s_{ab}]\\ &+\delta_{ac}\delta_{bd}[(1+2a_1)(s_{ab}+s_{ac}+s_{ad})-s_{ac}]\\ &+\delta_{ad}\delta_{bc}[(1+2a_1)(s_{ab}+s_{ac}+s_{ad})-s_{ad}]\}.\end{aligned} \tag{1.14}$$

Here we have introduced

$$\begin{aligned}s_{ab}&=(p_a+p_b)^2,\\ s_{ac}&=(p_a+p_c)^2,\\ s_{ad}&=(p_a+p_d)^2,\end{aligned} \tag{1.15}$$

and, of course,

$$s_{ab}+s_{ac}+s_{ad}=p_a^2+p_d^2+p_b^2+p_d^2. \tag{1.16}$$

The on-shell matrix element obtained by setting $p_a^2=p_b^2=p_c^2=p_d^2=0$ (these are zero-mass pions) is, as guaranteed by the equivalence theorem, independent of a_1. The Adler condition[7] which requires the on-shell matrix element to vanish when $s_{ab}=s_{ac}=s_{ad}=0$ is also clearly satisfied.

It is clearly a simple and straightforward matter to compute matrix elements in the tree-graph approximation for more complicated processes involving large numbers of pions. For each such process, there will be a soft-pion theorem (Adler condition) which we are sure will be satisfied by the tree-graph contribution to the amplitude alone, as well as by the full amplitude. However, we cannot (yet) calculate the full amplitude from our Lagrangian, because it is nonrenormalizable. We are then in a rather frustrating situation; we have a Lagrangian which might be utilizable to obtain the full S matrix in this model world of chiral-symmetric interactions between massless pions. We know that if we were able to calculate the full S matrix, all the consequences of chiral algebra, in particular, the soft-pion theorems, would be satisfied, for we have built chiral algebra into the Lagrangian. We can calculate only the tree-graph contributions, and there see how the soft-pion theorems are indeed satisfied. What is crucial is the presence of derivatives in the couplings, which give factors proportional to momenta in the matrix elements, which then vanish in the soft-pion limit. Now in higher orders than the tree-graph contributions, these derivatives can sometimes act on *internal* lines, and then it is not at all clear how the soft-pion theorems come to be satisfied. But precisely because of the derivatives acting on internal lines, the loop integrations are so badly divergent as to render the theory nonrenormalizable, so that we cannot really believe any result of a perturbation calculation anyway.

[7] S. L. Adler, Phys. Rev. 137, B1022 (1965).

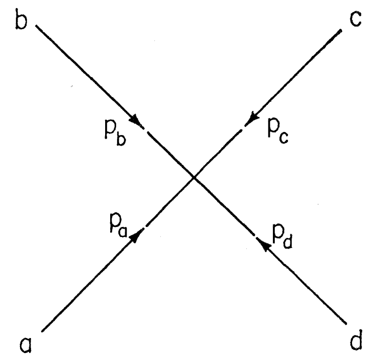

FIG. 1. Tree-graph contribution to the π-π scattering amplitude.

Even so, there may be something to be learned from the attempt to calculate higher-order terms in perturbation theory. We may not believe the results, but may still get some insight into how even in higher orders the soft-pion theorems are satisfied, and may even get a clue towards solving the problem of making meaningful calculations beyond the tree-graph approximation. So, fully cognizant of the dubious significance of the calculations, we will examine the soft-pion theorem to higher orders for π-π scattering. We need also to consider the soft-pion theorem for an even simpler process—the propagator.

II. FEYNMAN RULES

The Lagrangian is of the form

$$L=\tfrac{1}{2}(\partial_\mu\boldsymbol{\pi})^2+\sum_{n=1}^{\infty}\tfrac{1}{2}f_\pi^{-2n}[\alpha_n(\boldsymbol{\pi}^2)^n(\partial_\mu\boldsymbol{\pi})^2+\beta_n(\boldsymbol{\pi}^2)^{n-1}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2], \tag{2.1}$$

in which the coefficients α_n and β_n depend on the a_n occuring in the expansion (1.11) of $f(\boldsymbol{\pi}^2)$. We have already used

$$\alpha_1=-(1+2a_1), \quad \beta_1=-(1+4a_1), \tag{2.2}$$

and will also use

$$\begin{aligned}\alpha_2&=1+4a_1+3a_1^2-2a_2,\\ \beta_2&=2+12a_1+16a_1^2-8a_2,\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\alpha_3&=2a_1^2+4a_1^3+4a_2+6a_1a_2-2a_3,\\ \beta_3&=-3-24a_1-58a_1^2-40a_1^3+20a_2+52a_1a_2-12a_3.\end{aligned} \tag{2.4}$$

In general, α_n and β_n depend upon $a_1, a_2, \ldots, a_n$.

The terms in the sum in the expression for the Lagrangian give rise to vertices in Feynman diagrams at which $2n$ lines meet, $n=2, 3, \ldots$. If at such a vertex the pion lines carry isospin indices $a_1, a_2, \ldots, a_{2n}$ and momenta (directed in towards the vertex), $p_1, p_2, \ldots, p_{2n}$, then the vertex factor in the computation of a Feynman diagram contribution to the invariant

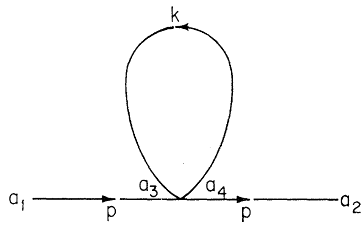

FIG. 2. Contribution of order f_π^{-2} to pion self-mass.

amplitude is

$$i f_\pi^{-2(n-1)} 2^{n-2}(n-1)!\bigg\{\delta_{a_1a_2}\delta_{a_3a_4}\cdots\delta_{a_{2n-1}a_{2n}}$$

$$\times\bigg[\alpha_{n-1}(p_1^2+p_2^2+\cdots+p_{2n}^2)+\bigg(\frac{\beta_{n-1}}{2(n-1)}-\alpha_{n-1}\bigg)$$

$$\times(s_{12}+s_{34}+\cdots+s_{2n-1,2n})\bigg)\bigg]+\cdots\bigg\}. \qquad (2.5)$$

There are $(2n)!/n!2^n$ terms within the braces { }, of which we have given just one, these corresponding to the distinct pairings of the $2n$ lines. We have used s_{ij} for $(p_i+p_j)^2$.

The meson propagator is

$$i\delta_{ab}(p^2+i\epsilon)^{-1}. \qquad (2.6)$$

The computation of a Feynman-diagram contribution to the invariant amplitude proceeds in the usual fashion, with vertex and propagator contributions being multiplied together and integrated over all loop momenta. There is an over-all phase of i. Finally, one must divide the resulting integral by a symmetry factor, appropriate to the particular diagram being considered. This factor arises because when many identical lines meet at a vertex a permutation of the lines may leave the diagram unchanged. In more familiar cases the combinations look after themselves, and there is no double counting. But in a case like the present there is double counting, unless we divide out by the appropriate factor. This symmetry factor is the number of ways the same contraction of internal fields may be made to yield the given diagram.

III. TERMS OF ORDER f_π^{-2}

If we work only to order f_π^{-2}, the only vertex to arise is the one at which four lines meet, given by setting

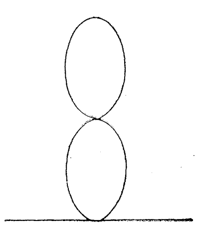

FIG. 3. Vanishing contribution of order f_π^{-4} to pion self-mass.

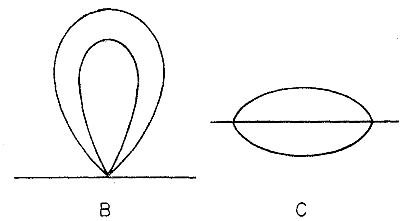

FIG. 4. Further contributions of order f_π^{-4} to pion self-mass.

$n=2$ in (2.5). It depends on a_1. The only amplitudes which acquire contributions to this order are the 2-pion to 2-pion scattering amplitude and the propagator. We have already written down the first of these in (1.14). Since it involves no loop integrations, it is finite, no matter what value is given to a_1 and, independently of the value of a_1, it vanishes when any of the external 4-momenta is set equal to zero.

The diagram of Fig. 2 gives a contribution of order f_π^{-2} to the self-mass of the pion. The symmetry factor is 2, and direct application of the Feynman rules of Sec. II gives for its contribution A to the invariant amplitude

$$A=-i\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\delta_{a_3a_4}}{k^2+i\epsilon}f_\pi^{-2}\{\delta_{a_1a_2}\delta_{a_3a_4}[\alpha_1(2p^2+2k^2)]$$

$$+\delta_{a_1a_3}\delta_{a_2a_4}[\alpha_1(2p^2+2k^2)+(\tfrac{1}{2}\beta_1-\alpha_1)2(p+k)^2]$$

$$+\delta_{a_1a_4}\delta_{a_2a_3}[\alpha_1(2p^2+2k^2)+(\tfrac{1}{2}\beta_1-\alpha_1)2(p-k)^2]\}$$

$$=-i\delta_{a_1a_2}f_\pi^{-2}(2\alpha_1+\beta_1)\int\frac{d^4k}{(2\pi)^4}\frac{p^2+k^2}{k^2+i\epsilon}. \qquad (3.1)$$

Of course the integral diverges. But we shall require that the "soft-pion theorem for the propagator" be satisfied to order f_π^{-2}. And this theorem says that, to this order, the mass of the pion stays zero. Since Fig. 2 is the only contributing diagram of order f_π^{-2}, the requirement is that A vanishes at $p^2=0$. This *can* be satisfied if and only if

$$3\alpha_1+\beta_1=0. \qquad (3.2)$$

Note that the requirement is that the most divergent part of the integral vanishes; in this simple case, when this requirement is satisfied the whole integral vanishes, but this will not always be so. However, we shall always find that the soft-pion theorems relate to the most divergent parts of the integrals, as may be seen on dimensional grounds.

The condition determines uniquely

$$a_1=-\tfrac{2}{5}, \qquad (3.3)$$

which, taken together with Eqs. (2.2), gives

$$\alpha_1=-\tfrac{1}{5},\quad \beta_1=\tfrac{3}{5}. \qquad (3.4)$$

IV. TERMS OF ORDER f_π^{-4}

Working to the next higher order, i.e., to f_π^{-4}, we encounter diagrams with two closed loops contributing to the self-mass, diagrams with one closed loop contributing to the 2-pion to 2-pion amplitude, and tree-graph diagrams for 3-pion to 3-pion (or 2 to 4) processes, which we shall not evaluate explicitly.

Consider first the self-mass contributions. The diagram of Fig. 3 gives zero if (3.3) is satisfied, since it contains A as a factor. However, this is not true for the contributions B and C of the two diagrams of Fig. 4. The symmetry factors for these diagrams are 8 and 6, respectively.

Direct evaluation gives

$$B=\delta_{a_1a_2}f_\pi^{-4}5(3\alpha_2+\beta_2)\times\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4l}{(2\pi)^4}\frac{p^2+k^2+l^2}{k^2l^2},\quad(4.1)$$

or, setting $p^2=0$,

$$B_0=\delta_{a_1a_2}f_\pi^{-4}10(3\alpha_2+\beta_2)\mathcal{W},\quad(4.2)$$

where $\mathcal{W}$ is the divergent integral

$$\mathcal{W}\equiv\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2}.\quad(4.3)$$

Similarly, at $p^2=0$ the contribution C_0 of the second diagram of Fig. 4 is

$$C_0=-\delta_{a_1a_2}f_\pi^{-4}2(3\alpha_1^2+2\alpha_1\beta_1+2\beta_1^2)\mathcal{W}.\quad(4.4)$$

Here, and subsequently, we have performed only permissible manipulations of divergent integrals, viz., symmetric integration, which sets to zero expressions like

$$\int d^4k\ f(k^2)k,\quad(4.5)$$

as well as, of course, having utilized δ functions. We do *not* make displacements of the origin of integration.

Just as the condition $A_0=0$ for the vanishing of the self-mass to order f_π^{-2} led to Eq. (3.2) and hence to the determination of a_1 as in Eq. (3.3), so now the condition that the self-mass vanishes also to order f_π^{-4} leads to a determination of a_2. We have in this case the requirement

$$B_0+C_0=0,\quad(4.6)$$

which can be formally satisfied despite the divergence

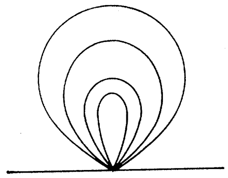

FIG. 5. Representative contribution of a certain class to pion self-mass; see text.

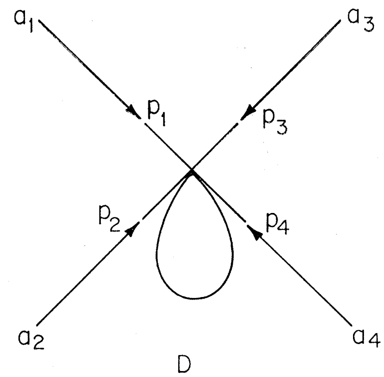

FIG. 6. Contribution D to π-π scattering amplitude.

of $\mathcal{W}$ if and only if

$$10(3\alpha_2+\beta_2)-2(3\alpha_1^2+2\alpha_1\beta_1+2\beta_1^2)=0.\quad(4.7)$$

Using Eqs. (3.4) and (2.3), this leads uniquely to

$$a_2=-9/175,\quad\alpha_2=-3/175,\quad\beta_2=30/175.\quad(4.8)$$

It is not hard to see that the condition that the self-mass vanishes at all orders through f_π^{-2r} will determine the coefficients $a_1, a_2, \ldots, a_r$ (as well as the corresponding α_i and β_i); for to order f_π^{-2r}, there is one and only one diagram which involves α_r and β_r (always in the combination $3\alpha_r+\beta_r$), namely, the diagram like that of Fig. 5 with r closed loops. All other contributions of this order involve coefficients of lower index i. Then by complete induction on r, our statement is proved, since a determination of $3\alpha_r+\beta_r$ gives a determination of a_r. Thus formally there is a *unique* choice of $f(\boldsymbol{\pi}^2)$ for which this kind of naive calculation will give a vanishing self-mass to the pion to all orders in f_π^{-2}, i.e., for which the soft-pion theorem for the propagator is satisfied. What is of crucial interest is the question whether the *same* choice of $f(\boldsymbol{\pi}^2)$ works also for other processes. We do not know how to prove the conjecture that it does, but we will show in the rest of this paper that it is satisfied to low orders in the expansion in f_π^{-2}.

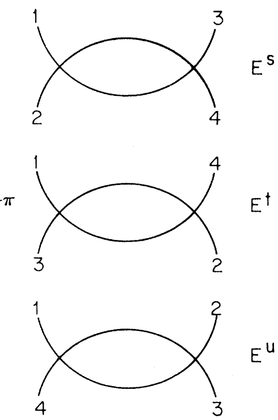

FIG. 7. Contributions E to π-π scattering amplitude.

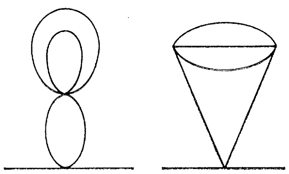

FIG. 8. Vanishing contributions of order f_π^{-6} to pion self-mass.

Still working to order f_π^{-4}, we turn now to the 2-pion to 2-pion scattering amplitude, where we encounter one-closed-loop contributions. Apart from self-mass insertions on the legs of the tree diagram, which vanish because $A=0$, these are the contributions of the diagrams of Figs. 6 and 7. The contribution D from Fig. 6 (which has symmetry factor 2) is

$$D=-if_\pi^{-4}\int\frac{d^4k}{(2\pi)^4}$$

$$\times\{\delta_{a_1a_2}\delta_{a_3a_4}[4(3\alpha_2+\beta_2)k^2+5(\beta_2-4\beta_2)s_{12}]$$

$$+\delta_{a_1a_3}\delta_{a_2a_4}[4(3\alpha_2+\beta_2)k^2+5(\beta_2-4\alpha_2)s_{13}]$$

$$+\delta_{a_1a_4}\delta_{a_2a_3}[4(3\alpha_2+\beta_2)k^2+5(\beta_2-4\beta_2)s_{14}]\}. \quad (4.9)$$

where $s_{ij}=(p_i+p_j)^2$ are the Mandelstam invariants, which are to be set to zero in the soft-pion limit. This gives

$$D_0=-i(\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3})$$
$$\times f_\pi^{-4}4(3\alpha_2+\beta_2)\mathfrak{X}, \quad (4.10)$$

where we have introduced $\mathfrak{X}$ for the divergent integral

$$\mathfrak{X}\equiv\int\frac{d^4k}{(2\pi)^4}. \quad (4.11)$$

In a similar fashion one derives, for the soft-pion value ($s_{12}=s_{13}=s_{14}=0$), E_0^s of the contribution of the first diagram in Fig. 7 (which has symmetry factor 2),

$$E_0^s=if_\pi^{-4}[\delta_{a_1a_2}\delta_{a_3a_4}(6\alpha_1^2+4\alpha_1\beta_1)$$
$$+(\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3})\beta_1^2]\mathfrak{X}. \quad (4.12)$$

The other two diagrams give contributions E_0^t and E_0^u, which may be obtained from E_0^s by cyclic permutation of the coefficients of the three products of Kronecker δ symbols. The sum of these three diagrams then gives

$$E_0=E_0^s+E_0^t+E_0^u$$
$$=i(\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3})$$
$$\times f_\pi^{-4}2(3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2)\mathfrak{X}. \quad (4.13)$$

In order that the Adler condition be met to order f_π^{-4}, we then require

$$D_0+E_0=0, \quad (4.14)$$

which is possible only if

$$-4(3\alpha_2+\beta_2)+2(3\alpha_1^2+2\alpha_1\beta_1+\beta_1^2)=0. \quad (4.15)$$

Provided that (3.2) is satisfied, this yields the *same* results (4.8) as did the requirement on the self-mass.

Thus encouraged, we turn to the next higher order.

V. TERMS OF ORDER f_π^{-6}

In order f_π^{-6}, we must consider diagrams with three closed loops in the self-mass, diagrams with two closed loops in the 2-pion to 2-pion scattering amplitude, diagrams with one closed loop in the 3-pion to 3-pion (or 2 to 4) amplitude, and the tree graph for processes like 4-pion to 4-pion scattering.

The self-mass acquires contributions from a number of diagrams like that of Fig. 8, which being proportional to $3\alpha_1+\beta_1$ vanish without further constraint. The nontrivial diagrams are those of Fig. 9; their symmetry factors are, respectively, 48, 12, 4, and 24. Explicit evaluation of their contributions at $p^2=0$, using the Feynman rules, gives

$$F_0=i\delta_{a_1a_2}f_\pi^{-6}105(3\alpha_3+\beta_3)\mathfrak{Y},$$
$$G_0=-i\delta_{a_1a_2}f_\pi^{-6}10[\beta_1(\beta_2-4\alpha_2)$$
$$+3(\alpha_1+\beta_1)(3\alpha_2+\beta_2)]\mathfrak{Y},$$
$$H_0=i\delta_{a_1a_2}f_\pi^{-6}[(42\alpha_1^3+72\alpha_1^2\beta_1+94\alpha_1\beta_1^2+26\beta_1^3)\mathfrak{Y},$$
$$+\tfrac{1}{2}(\beta_1-2\alpha_1)(3\beta_1^2-2\alpha_1\beta_1-3\alpha_1^2)\mathfrak{Z}], \quad (5.1)$$
$$I_0=-i\delta_{a_1a_2}f_\pi^{-6}5[4(\beta_1\beta_2+2\alpha_1\beta_2+3\beta_1\alpha_2-\alpha_1\alpha_2)\mathfrak{Y}$$
$$+\tfrac{1}{2}(\beta_1-2\alpha_1)(\beta_2-4\alpha_2)\mathfrak{Z}].$$

These expressions involve the divergent integrals

$$\mathfrak{Y}\equiv\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4l}{(2\pi)^4}\int\frac{d^4m}{(2\pi)^4}\frac{1}{k^2l^2} \quad (5.2)$$

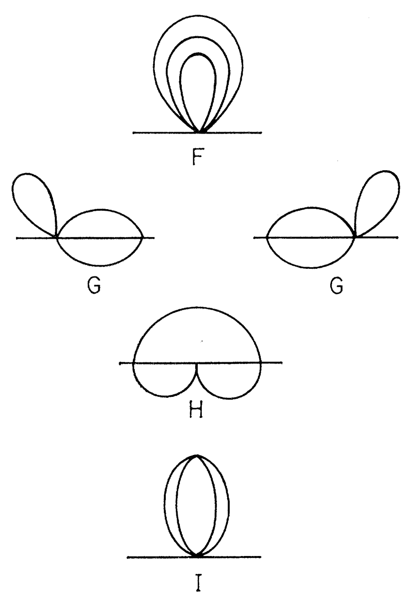

FIG. 9. Further contributions of order f_π^{-6} to pion self-mass.

and

$$\mathfrak{z}\equiv\int\frac{d^4k}{(2\pi)^4}\times\left[\int\frac{d^4l}{(2\pi)^4}\int\frac{d^4m}{(2\pi)^4}\frac{k^2}{l^2m^2}(2\pi)^4\delta^{(4)}(k+l+m)\right]^2. \quad (5.3)$$

These integrals have the same degree of divergence, but are of different structure. For the f_π^{-6} contribution to the self-mass to vanish, we require

$$F_0+2G_0+H_0+I_0=0. \quad (5.4)$$

This reduces to

$$[105(3\alpha_3+\beta_3)-20(\alpha_1+\beta_1)(8\alpha_2+5\beta_2)+(42\alpha_1^3+72\alpha_1^2\beta_1+94\alpha_1\beta_1^2+26\beta_1^3)]\mathcal{Y}+\tfrac{1}{2}(\beta_1-2\alpha_1)\times[(3\beta_1^2-2\alpha_1\beta_1-3\alpha_1^2)-5(\beta_2-4\alpha_2)]\mathfrak{z}=0. \quad (5.5)$$

If the conditions (4.8) and (3.4) are satisfied, the coefficient of $\mathfrak{z}$ vanishes without further ado. Then (5.4) reduces to the requirement that the coefficient of $\mathcal{Y}$ in (5.5) should be zero, which determines

$$\begin{aligned}a_3&=-163/2\times3^2\times5^3\times7,\\ \alpha_3&=19/3^2\times5^3\times7,\\ \beta_3&=119/3\times5^3\times7.\end{aligned} \quad (5.6)$$

Turning now to the two-closed-loop contributions to the 2-pion to 2-pion scattering amplitude, we consider (apart from vanishing diagrams corresponding to self-energy insertions) the diagrams of Fig. 10, as well as crossed variants of them. The symmetry factors for J and for each of the variants of K, L, M, and N are, respectively, 8, 4, 4, 2, and 6. When evaluated in the soft-pion limit (external momenta all set to zero), the contributions are

$$J_0=(\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3})\times f_\pi^{-6}84(3\alpha_3+\beta_3)\mathcal{W}, \quad (5.7)$$

$$K_0{}^s=K_0{}'^s=-f_\pi^{-6}(12\alpha_2+11\beta_2)[2\alpha_1\delta_{a_1a_2}\delta_{a_3a_4}+\beta_1(\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3})]\mathcal{W}, \quad (5.8)$$

$$L_0{}^s=f_\pi^{-6}[12\alpha_1(3\alpha_1^2+3\alpha_1\beta_1+\beta_1^2)\delta_{a_1a_2}\delta_{a_3a_4}+2\beta_1^3(\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3})]\mathcal{W}, \quad (5.9)$$

$$M_0{}^s=M_0{}'^s=f_\pi^{-6}6(\alpha_1+\beta_1)^2[2\alpha_1\delta_{a_1a_2}\delta_{a_3a_4}+\beta_1(\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3})]\mathcal{W}, \quad (5.10)$$

$$N_0{}^1=-f_\pi^{-6}4(6\alpha_1\alpha_2+2\alpha_1\beta_2+2\alpha_2\beta_1+3\beta_1\beta_2)\times(\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3})\mathcal{W}. \quad (5.11)$$

A slight simplification has already been achieved by using (3.2). The t- and u-channel crossed versions of K, K', L, M, and M' are obtained from the s-channel results given above by cyclic permutation of the coefficients of the products of Kronecker-δ symbols.

There are also three further diagrams similar to that which yields $N_0{}^1$, obtained by singling out successively the other legs of the diagram; each of these diagrams then gives an equal contribution.

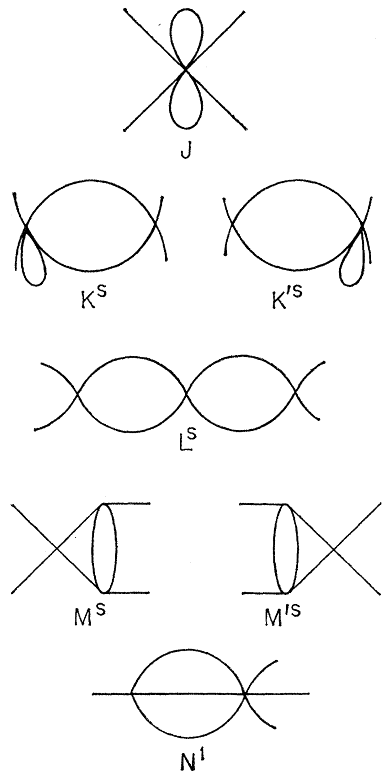

FIG. 10. Contributions of order f_π^{-6} to π-π scattering.

If we denote by X the expression

$$X\equiv(\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3})f_\pi^{-6}\mathcal{W}, \quad (5.12)$$

we obtain

$$J_0=84(3\alpha_3+\beta_3)X, \quad (5.13)$$

$$K_0{}^s+K_0{}^t+K_0{}^u+K_0{}'^s+K_0{}'^t+K_0{}'^u=-4(\alpha_1+\beta_1)(12\alpha_2+11\beta_2)X, \quad (5.14)$$

$$L_0{}^s+L_0{}^t+L_0{}^u=4(9\alpha_1^3+9\alpha_1^2\beta_1+3\alpha_1\beta_1^2+\beta_1^3)X, \quad (5.15)$$

$$M_0{}^s+M_0{}^t+M_0{}^u+M_0{}'^s+M_0{}'^t+M_0{}'^u=24(\alpha_1+\beta_1)^3X, \quad (5.16)$$

$$N_0{}^1+N_0{}^2+N_0{}^3+N_0{}^4=-16(6\alpha_1\alpha_2+2\alpha_1\beta_2+2\alpha_2\beta_1+3\beta_1\beta_2)X. \quad (5.17)$$

The total contribution of order f_π^{-6} to the 2-pion to 2-pion scattering amplitude in the soft-pion limit is obtained by adding the expressions in (5.13)–(5.17). The soft-pion theorem requires that this total should be zero, and this gives an equation between the α's and β's. Using (3.4) and (4.8) this in turn becomes an equation for $3\alpha_3+\beta_3$, and again gives a determination of a_3, α_3, and β_3. The result is the *same* as that obtained from consideration of the self-mass, namely, that given in (5.6).

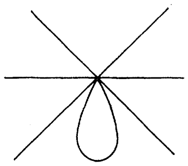

FIG. 11. Contribution O to 3π to 3π amplitude.

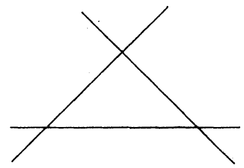

FIG. 13. Contribution Q to 3π to 3π amplitude.

As a final check, we have also looked at the 3-pion to 3-pion amplitude in the limit in which all six external momenta are set to zero. Three sets of diagrams contribute nontrivially. There is the single diagram of Fig. 11. The symmetry factor is 2, and its contribution in this limit is O_0,

$$O_0=-i(\delta_{a_1a}\delta_{2a_3a_4}\delta_{a_5a_6}+\text{permutations})\times f_\pi^{-6}24(3\alpha_3+\beta_3)\mathfrak{X}. \quad (5.18)$$

There are 15 diagrams with the topology of Fig. 12, each with symmetry factor 2. Their sum contributes P_0,

$$P_0=i(\delta_{a_1a_2}\delta_{a_3a_4}\delta_{a_5a_6}+\text{permutations})f_\pi^{-6}\times 24(3\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1+\beta_1\beta_2)\mathfrak{X}. \quad (5.19)$$

Finally, there are 15 diagrams with the topology of Fig. 13, symmetry factor 1, and their sum contributes Q_0,

$$Q_0=-(\delta_{a_1a_2}\delta_{a_3a_4}\delta_{a_5a_6}+\text{permutations})f_\pi^{-6}\times 8(3\alpha_1^3+3\alpha_1^2\beta_1+3\alpha_1\beta_1^2+\beta_1^3)\mathfrak{X}. \quad (5.20)$$

The condition

$$O_0+P_0+Q_0=0 \quad (5.21)$$

once again is satisfied with the values for the α's and β's already determined.

VI. DISCUSSION

The chiral-invariant Lagrangian for pions, given in (1.1), is unique once the function $f(\boldsymbol{\pi}^2)$ is specified. This function is in some ways analogous to the gauge of electrodynamics. Physically significant quantities, like on-shell amplitudes, should be "gauge" invariant. And because the Lagrangian is chiral invariant, the soft-pion theorems should be satisfied by these amplitudes. Certainly both these properties—gauge invariance and the soft-pion theorems—hold for the tree-graphs which are usually all that are retained in calculations based on an "effective" or "phenomenological" Lagrangian. What we have seen is that if one calculates contributions beyond the tree-graph diagrams using conventional Feynman rules, then the resulting amplitudes are, firstly (as is of course very well known) badly divergent.

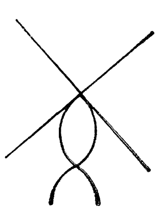

FIG. 12. Contribution P to 3π to 3π amplitude.

But, what is not so well known, they are also, if interpreted formally so that coefficients of infinite quantities are given significance, not explicitly gauge invariant and for the general gauge do not even formally satisfy the soft-pion theorems.

What we conjecture to be true in general, and what we have verified explicitly for low orders in f_π^{-2}, is that there is a *unique* choice of gauge for which the soft-pion theorems *are* satisfied for *all* processes, even when the amplitudes are calculated using the Feynman rules obtained, as we have obtained them, from a naive treatment of the highly singular expressions represented by high powers of field variables. We conjecture further that this same choice of gauge may render finite (after *conventional* renormalization) even the hard-pion amplitudes. If these conjectures are indeed valid, we have a prescription for obtaining finite, meaningful amplitudes starting from the nonpolynomial, nonrenormalizable Lagrangian of (1.1).

We suspect that the breakdown of gauge invariance is only apparent, and has its origin in the naive interpretation of high powers of fields operators that has been adopted implicitly in deriving the Feynman rules, in particular, (2.5). For example, in order to calculate correctly the Schwinger terms in current-current commutators in electrodynamics, an interpretation of so singular a quantity as $\bar{\psi}\gamma_\mu\psi$ is needed which introduces explicitly the electromagnetic potentials A_μ and so is dependent on the gauge. Presumably some similar gauge-dependent interpretation of singular expressions like $\boldsymbol{\pi}^{2n}$ is needed before one can calculate correctly and obtain a gauge-independent result from the Lagrangian of (1.1). What we conjecture is that for our unique choice of gauge, $f(\boldsymbol{\pi}^2)$, the guage-*dependent* results of the naive theory and the gauge-*independent* result of the "correct" theory will coincide.

One of the puzzles we have not been able to solve is to guess the function $f(\boldsymbol{\pi}^2)$, the first few terms in the analytic expansion of which we have derived:

$$f(x)=f_\pi\left(1-\tfrac{2}{5}x-\frac{9}{175}x^2-\frac{163}{15750}x^3+\cdots\right). \quad (6.1)$$

Note added in proof. We have now found that this choice of the "gauge" function f leads to a matrix function $d_{ab}(\boldsymbol{\pi})$, and hence to a metric $g_{ab}(\boldsymbol{\pi})\equiv d_{ac}d_{cb}$, which to order f_π^{-6} is unimodular, i.e.,

$$\det d=\det g=1,$$

and we naturally conjecture that this is the defining property of the special gauge we have considered. It then follows that $f(\boldsymbol{\pi}^2)$ is the solution of the differential equation

$$2\boldsymbol{\pi}^2 f' = f - f_\pi^{-3}(f^2+\boldsymbol{\pi}^2)^2\,,$$

with $f(0)=f_\pi$. This has as solution

$$f=(\boldsymbol{\pi}^2)^{1/2}\cot\tfrac{1}{2}\nu\,,$$

where ν is given implicitly by

$$\nu-\sin\nu=\tfrac{4}{3}(\boldsymbol{\pi}^2/f_\pi^2)^{3/2}\,.$$

The metric g_{ab} then has the attractively simple form

$$g_{ab}=f_\pi^2(f^2+\boldsymbol{\pi}^2)^{-1}(\delta_{ab}-\pi_a\pi_b/\boldsymbol{\pi}^2) + f_\pi^{-4}(f^2+\boldsymbol{\pi}^2)^2(\pi_a\pi_b/\boldsymbol{\pi}^2)^2\,.$$

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Variational Calculation of Three-Body Breakup Amplitudes*

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A previously reported variational principle is used to compute three-body elastic and breakup scattering amplitudes for positive total energies. The calculation converges quite rapidly and requires very little computer time. The forms of the variational principle suitable for systems where some of the particles are identical bosons are also derived. The examples presented are for Yamaguchi potentials and do not represent any physical system; they were chosen to have large breakup cross sections. The results are presented as Dalitz plots and other graphs which exhibit some interesting features.

I. INTRODUCTION

In a previous article,[1] we developed a variational principle for three-body scattering amplitudes and applied it to a problem involving separable potentials. Numerical results were presented for both elastic and rearrangement scattering beneath the breakup threshold. There we saw that with only modest amounts of computer time, we could obtain highly accurate values of the T matrices for negative values of the total energy W. It was pointed out in PSW that the greatest advantage of our approach would be in calculations above the breakup threshold since the asymptotic form of the wave functions we use is the same above the breakup threshold as beneath the threshold. In fact, a calculation for $W>0$ is essentially no more difficult than a calculation beneath threshold. We have now completed calculations for processes, including breakup, for total energies greater than zero. As will be seen, the procedure is very successful; the convergence is rapid, and minimal computer time is needed.

In Sec. II we review the notation of PSW and rewrite the variational principles found there. We then specialize the breakup principle to separable potentials and present the results of a calculation using Yamaguchi potentials. For problems in which some of the particles are identical bosons, considerably fewer computations are required. The formulas for these systems are derived in Appendices A and B. In Secs. III and IV we discuss in detail two such systems and present Dalitz plots for the breakup cross sections. A considerable amount of structure is evident. Appendix C is a discussion of a class of integrals occurring in the calculations.

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[1] S. C. Pieper, L. Schlessinger, and J. Wright, Phys. Rev. D 1, 1674 (1970). We refer to this as PSW.

II. VARIATIONAL PRINCIPLE

In this section we review the conventions and the variational principle of PSW. We also specialize the