internal lines. Clearly the contribution of this particular singularity to the s-channel absorptive part is $T_{2\to n}\times T_{n\to 2}$; that is, precisely the n-body intermediate state contribution to the unitarity relation.

Thus it seems that the required singularities are present; what we cannot yet show is that *only* these are present.

Finally, if the situation regarding unitarity can be satisfactorily cleared up, we have defined a theory with all the desired properties of a true bootstrap theory. We have a well-defined set of equations, (3) and (4), for a set of vertex functions and propagators. They incorporate crossing and analyticity, and if they have solutions we have a prescription for calculating any n-legged amplitude. We then have here a set of exact bootstrap equations written in closed form, with which one can study questions of existence and uniqueness of solutions, and which present a basis for systematic approximations.

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Broken Scale Invariance in Scalar Field Theory*

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We use scalar-field perturbation theory as a laboratory to study broken scale invariance. We pay particular attention to scaling laws (Ward identities for the scale current) and find that they have unusual anomalies whose presence might have been guessed from renormalization-group arguments. The scaling laws also appear to provide a relatively simple way of computing the renormalized amplitudes of the theory, which sidesteps the overlapping-divergence problem.

INTRODUCTION

THE perturbation theory of a self-interacting scalar field is about the simplest available model field theory, and a convenient laboratory for testing new ideas in strong-interaction physics. In this paper we shall be concerned with studying the concept of broken scale invariance within such a framework. We shall find that the model calls for some unexpected modifications of our ideas on broken scale invariance. At the same time, the approach suggested by broken scale invariance leads to an interesting, and simple, new approach to renormalization. We hope that this mutual illumination of two interesting questions justifies yet another paper on scalar field theory.

In Sec. I we shall review the general properties of scale invariance as a broken symmetry, leading up to the idea of a scaling law (the analog for scale invariance of PCAC low-energy theorems). In Sec. II we shall see how the general structure of renormalized perturbation theory constrains the allowable form of the scaling law and forces it to differ from naive expectations. In Sec. III we shall show how the existence of the scaling law leads to a simple prescription for computing the renormalized Green's functions of the theory. Finally, in Sec. IV, we shall demonstrate an interesting connection between the scaling law and the predictions of the renormalization group.

I. BROKEN SCALE INVARIANCE

In simple canonical field theories it is possible to introduce an acceptable energy-momentum tensor[1,2] $\Theta_{\mu\nu}$ having the following properties: (a) $\Theta=\Theta_\mu{}^\mu$ is proportional to those terms in the Lagrangian having dimensional coupling constants (such as mass terms); (b) the charge, $D=\int d^3x\, S_0$, formed from the current $S_\mu=\Theta_{\mu\nu}x^\nu$, acts as the generator of scale transformations,

$$[D(x_0),\phi(x)]=-i(d+x\cdot\partial)\phi(x)\,, \tag{1}$$

where d is the dimension of the field; (c) the current S_μ satisfies $\partial^\mu S_\mu=\Theta$ so that it is conserved when there are no dimensional coupling constants in the Lagrangian. With the help of the current S_μ and its equal-time commutation relations with fields, given above, one is able

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[1] C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N. Y.) **59**, 42 (1970).

[2] M. Gell-Mann, University of Hawaii Summer School lectures, 1969 (unpublished).

to derive a standard sort of Ward identity[1]

$$\left[n(d-4)+4-\sum p_i\cdot\frac{\partial}{\partial p_i}\right]G(p_1\cdots p_{n-1}) = -iF(0,p_1\cdots p_{n-1})\,, \quad (2)$$

where

$$(2\pi)^4\delta(\sum_{i=1}^{n} p_i)G(p_1\cdots p_{n-1}) = \int dx_1\cdots dx_n e^{i\Sigma p_i x_i}\langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle\,, \quad (3a)$$

$$(2\pi)^4\delta(q+\sum_{i=1}^{n} p_i)F(q,q_1\cdots p_{n-1}) = \int dy dx_1\cdots dx_n e^{iq\cdot y+i\Sigma p_i x_i} \times\langle 0|T(\Theta(y)\phi(x_1)\cdots\phi(x_n))|0\rangle\,. \quad (3b)$$

The significance of such an equation is clear: If $\Theta=0$, so that $F=0$, the particle Green's functions satisfy $SG=0$, where S is the linear operator appearing in square brackets in Eq. (2). One can always find the general solution of such an equation, and it turns out to imply that, apart from explicit kinematic factors, G depends only on dimensionless ratios of momentum variables. This is precisely what one expects from naive dimensional reasoning in the event that no dimensional coupling constants are present in the theory. Therefore, the scaling law, Eq. (2), says that the matrix elements F of Θ act as the source of violations of simple dimensional scaling in the particle matrix elements G. It also appears to be of general validity, not depending on the details of the theory, providing a general framework for the study of broken scale invariance. We want to ask whether such a relation, which we shall refer to as a scaling law, actually holds in a simple renormalizable field theory.

It is convenient to define the scaling law for "one-particle-irreducible" Green's functions rather than the full Green's functions defined in Eq. (3). The one-particle-irreducible Green's functions, which we shall denote by $\bar{G}$ and $\bar{F}$, are obtained from the full Green's functions by first throwing away all diagrammatic contributions which fall into disjoint pieces when one internal line is cut, and then dividing out of the remainder one factor of the propagator for each external leg. This simply turns full vertices into proper vertices. The same formal arguments which led to the Ward identity for G allow one to derive a Ward identity for $\bar{G}$:

$$\left[4-nd-\sum p_i\cdot\frac{\partial}{\partial p_i}\right]\bar{G}(p_1\cdots p_{n-1}) = -i\bar{F}(0,p_1\cdots p_{n-1})\,. \quad (4)$$

The difference between the two Ward identities, the factor of $4-nd$ rather than $n(d-4)+4$, arises entirely from the different dimensions of G and $\bar{G}$.

We shall eventually be dealing with a simple theory in which the only dimensional parameter is the particle mass μ. In such a case, ordinary dimensional reasoning requires that $\bar{G}$ have the following dependence on μ:

$$\bar{G}(p_1\cdots p_{n-1})=\mu^{4-n}\Phi(p_1/\mu,\ldots,p_{n-1}/\mu)\,.$$

(It is perhaps worth inspecting a Feynman diagram or two to convince oneself that the dimension of $\bar{G}$ in power of mass is just $4-n$.) Then we have the identity

$$\left[4-n-\sum p_i\cdot\frac{\partial}{\partial p_i}\right]\bar{G}(p_1\cdots p_{n-1})=\mu\frac{\partial}{\partial\mu}\bar{G}(p_1\cdots p_{n-1})\,,$$

which leads to an equivalent form of Eq. (4):

$$[\mu\partial/\partial\mu+n\delta]\bar{G}(p_1\cdots p_{n-1})=-i\bar{F}(0,p_1\cdots p_{n-1})\,, \quad (5)$$

where

$$\delta=1-d\,.$$

From now on, we shall refer to the operator in square brackets as S.

We remarked earlier that d was the dimension of the field. For a scalar field the dimension in powers of mass is unity, so that one expects $d-1=0$. However, as Wilson[3] has pointed out, when there are interactions it is not guaranteed that the naive dimension and the dimension defined by the commutator of the generator of scale transformations with the field [as in Eq. (1)] are the same. Therefore, we shall let the term $(d-1)n$ in Eq. (5) stand. The question now is whether the scaling law of Eq. (5) actually holds in renormalized perturbation theory, and if not, whether there is a simple equation which replaces it.

The question seems interesting since the scaling law is the only obvious direct, and model-independent, expression of how specific forms of scale-invariance breaking affect particle scattering amplitudes. Also it brings in the dimension of the field in a way which may be of phenomenological significance.

II. GENERAL CONSTRAINTS ON SCALING LAW

To settle the questions raised in Sec. I, we shall study perturbation theory for a massive scalar field interacting through $\lambda\phi^4$:

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\tfrac{1}{2}\mu^2\phi^2-(\lambda/4!)\phi^4.$$

Since the only term in the Lagrangian with a dimensional coupling constant is the mass term, we expect $\Theta=\mu^2\phi^2$ to be the source for violations of scale invariance.

Since this is a renormalizable theory, there are a finite number of amplitudes which require subtractions.

[3] K. Wilson, Phys. Rev. 179, 1499 (1969).

The standard lore tells us that $\bar{G}^{(2)}$ requires two subtractions, $\bar{G}^{(4)}$ requires one, and $\bar{F}^{(2)}$ requires one, with all other amplitudes requiring no subtractions. That $\bar{F}^{(2)}$ is the only matrix element of Θ requiring a subtraction follows, via the usual power-counting arguments, from our requirement that Θ be proportional to ϕ^2. As a consequence of these subtractions, $\bar{F}^{(2)}$ and $\bar{G}^{(4)}$ are determined up to an arbitrary constant, while $\bar{G}^{(2)}$ is determined up to an arbitrary first-order polynomial in p^2. This arbitrariness means that we can choose $\bar{G}^{(4)}(0)$, $\bar{G}^{(2)}(0)$, $(d/dp^2)\bar{G}^{(2)}(p^2)|_0$, and $\bar{F}^{(2)}(0)$ at will. These parameters, which we call $-i\lambda$, $-i\mu^2$, iZ, and $2m^2$, respectively, play the role of arbitrary parameters in terms of which all the Green's functions of the theory are determined.[4]

The Green's functions of the theory which do not need a subtraction ($\bar{G}^{(n)}$, $n>4$; $\bar{F}^{(n)}$, $n>2$) have the useful property of being directly expressible via the skeleton expansion in terms of $\bar{G}^{(2)}$, $\bar{G}^{(4)}$, and $\bar{F}^{(2)}$. A skeleton diagram for $\bar{G}^{(n)}$ is a Feynman diagram containing no subgraphs identifiable as a contribution to $\bar{G}^{(2)}$ or $\bar{G}^{(4)}$, while a skeleton for $\bar{F}^{(n)}$ is a Feynman diagram containing no subgraphs identifiable as $\bar{G}^{(2)}$, $\bar{G}^{(4)}$, or $\bar{F}^{(2)}$. The skeleton expansion for a given Green's function is obtained by taking all skeleton graphs for that Green's function and replacing all point four-particle vertices by $\bar{G}^{(4)}$, all internal lines by $[-\bar{G}^{(2)}]^{-1}$,[5] and all point insertions of Θ by $\bar{F}^{(2)}$. Since the higher Green's functions are determined once the fundamental Green's functions ($\bar{G}^{(2)}$, $\bar{G}^{(4)}$, and $\bar{F}^{(2)}$) are known, presumably the scaling operator S should be such that $S\bar{G}^{(n)}=-i\bar{F}^{(n)}$ is automatically true for $n>4$ once it is true for $n<4$. The required property of S is that it be "distributive" in the following sense:

Let us consider a particular diagram in the skeleton expansion of $\bar{G}^{(n)}$. If it is a diagram with i vertices and j internal lines, its contribution to $\bar{G}^{(n)}$ is gotten by replacing each vertex by $\bar{G}^{(4)}$ and each internal line by $[-\bar{G}^{(2)}]^{-1}$ and doing the integrations over loop momenta:

$$\bar{G}^{(n)}\sim\int d\,(\text{loop momenta})\,(\bar{G}^{(4)}\cdots\bar{G}^{(4)})_{i\text{ factors}}\times([-\bar{G}^{(2)}]^{-1}\cdots[-\bar{G}^{(2)}]^{-1})_{j\text{ factors}}.$$

What happens when we act on this integral with the scaling operator found in Sec. I, $S=\mu\partial/\partial\mu+n\delta$? By the chain rule for differentiation, $\mu\partial/\partial\mu$ gives a sum of terms in which it acts independently on each factor in the integral for $\bar{G}^{(n)}$:

$$\bar{G}^{(4)}\to\mu\frac{\partial}{\partial\mu}\bar{G}^{(4)},$$

and

$$[-\bar{G}^{(2)}]^{-1}\to[-\bar{G}^{(2)}]^{-1}\left[\mu\frac{\partial}{\partial\mu}\bar{G}^{(2)}\right][-\bar{G}^{(2)}]^{-1}.$$

By virtue of the trivial topological relation $n=4i-2j$, the term $n\delta$ has the same behavior. So, if $S=\mu\partial/\partial\mu+n\delta$,

$$\begin{aligned}S\int d\,(\text{loop momenta})\,\bar{G}^{(4)}(1)\cdots[-\bar{G}^{(2)}(j)]^{-1}\\ =\int d\,(\text{loop momenta})\,\{\{[S\bar{G}^{(4)}(1)]\bar{G}^{(4)}(2)\cdots\bar{G}^{(4)}(i)+(i-1\text{ similar terms})\}[-\bar{G}^{(2)}(1)]^{-1}\cdots[-\bar{G}^{(2)}(j)]^{-1}\\ +\bar{G}^{(4)}(1)\cdots\bar{G}^{(4)}(i)\{[-\bar{G}^{(2)}(1)]^{-1}[S\bar{G}^{(2)}(1)][-\bar{G}^{(2)}(1)]^{-1}[-\bar{G}^{(2)}(2)]^{-1}\cdots[-\bar{G}^{(2)}(j)]^{-1}\\ +(j-1\text{ similar terms})\}\},\end{aligned}$$

which is to say that S acts in succession on each vertex and inserts $S\bar{G}^{(2)}$ in succession on each internal line (see Fig. 1). Any S with this property will be called distributive. Obviously, we can add to S any kind of differentiation operation, such as differentiation with respect to coupling constant, without changing its distributive nature.

Now if $S\bar{G}^{(2)}=-i\bar{F}^{(2)}$ and $S\bar{G}^{(4)}=-i\bar{F}^{(4)}$, the above recipe for acting with S on a given skeleton for $\bar{G}^{(n)}$ is as follows: Insert $\bar{F}^{(2)}$ in succession on each internal line and replace each vertex $\bar{G}^{(4)}(p_1p_2p_3)$ in succession by $\bar{F}^{(4)}(0,p_1p_2p_3)$. Inserting $\bar{F}^{(2)}$ on an internal line gives directly a skeleton for $\bar{F}^{(n)}$. Since $\bar{F}^{(4)}$ is a convergent amplitude, it has itself a skeleton expansion. Therefore, the action of replacing a vertex by $\bar{F}^{(4)}$ gives a *sum* of sleketons for $\bar{F}^{(n)}$ in which the vertex in question is replaced in turn by all the skeletons in the expansion for $\bar{F}^{(4)}$. Therefore, S turns a single skeleton for $\bar{G}^{(n)}$ into a sum of skeletons for $\bar{F}^{(n)}$. If, when we sum all skeletons for $\bar{G}^{(n)}$, we get all skeletons for $\bar{F}^{(n)}$, we have the desired result $S\bar{G}^{(n)}\equiv-i\bar{F}^{(n)}$.

Let us show that the above recipe for turning G skeletons into F skeletons exhausts all possibilities. Consider a particular skeleton, S^F for $\bar{F}^{(n)}$. When we

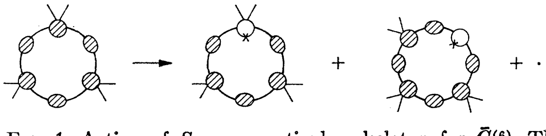

FIG. 1. Action of S on a particular skeleton for $\bar{G}^{(6)}$. The cross stands for insertion of Θ.

[4] The subtraction can be made at any fixed value of momentum. Since we subtract at zero four-momentum, the parameter μ is not identical to the physical mass.

[5] The amplitude which we called $G^{(2)}$ is identical to the propagator. The passage to $\bar{G}^{(2)}$ involves dividing out two factors of the propagator from $G^{(2)}$, so that $\bar{G}^{(2)}\propto[G^{(2)}]^{-1}$.

remove from it the insertion of $\bar{F}^{(2)}$, it becomes a graph $\mathcal{G}$ for $\bar{G}^{(n)}$, which may or may not be a skeleton. Suppose it *is* a skeleton, S_1^G. Then S^F is created by inserting $\bar{F}^{(2)}$ on some internal line of S_1^G. Suppose $\mathcal{G}$ is *not* a skeleton. Then it contains a subgraph identifiable as a $\bar{G}^{(2)}$ or a $\bar{G}^{(4)}$ within which is contained the line from which $\bar{F}^{(2)}$ was removed to create $\mathcal{G}$ in the first place. Actually, this subgraph cannot be a $\bar{G}^{(2)}$, since that would mean, on putting back the $\bar{F}^{(2)}$ insertion, that the original graph S^F contained an insertion identifiable as an $\bar{F}^{(2)}$, which is not allowed for an F skeleton. So, $\mathcal{G}$ contains a subgraph identifiable as a $\bar{G}^{(4)}$, which, if shrunk to a point vertex, turns $\mathcal{G}$ into a skeleton S_2^G for $\bar{G}^{(n)}$. Also, if we put the $\bar{F}^{(2)}$ insertion back into this subgraph, the subgraph becomes a skeleton for $\bar{F}^{(4)}$. So, in this case, S^F is obtained from S_2^G, a skeleton for $G^{(n)}$, by replacing one of the vertices by a skeleton for $\bar{F}^{(4)}$. Putting these two cases together, we see that all the sleketons for $\bar{F}^{(n)}$ are obtained by acting on all the skeletons for $\bar{G}^{(n)}$ in precisely the manner described in the previous paragraph. Therefore, a distributive S is guaranteed to satisfy the scaling law $S\bar{G}^{(n)}=-i\bar{F}^{(n)}$ for all n if it is true for $n=2$ and 4.

The scaling operator $S=\mu\partial/\partial\mu+n\delta$ suggested by formal arguments on broken scale invariance is of course distributive. It remains so if we add to it differentiation with respect to any parameter. The only parameter in the theory apart from the mass μ is the coupling constant λ. This suggests a more general form for S:

$$S=\mu\partial/\partial\mu+n\delta(\lambda)+f(\lambda)\partial/\partial\lambda. \tag{6}$$

The question is whether one has to make use of this freedom. One would be happier with a scaling law *not* involving differentiation with respect to the coupling constant, since that would leave open some hope of direct phenomenological application.

The simplest way to answer this question is to study the scaling-law constraints on the fundamental constants of the theory. One easily finds that

$$S\bar{G}^{(2)}(0)=-i\bar{F}^{(2)}(0) \Rightarrow (1+\delta)\mu^2=m^2,$$

$$S\bar{G}^{(2)}(0)'=-i\bar{F}^{(2)}(0)' \Rightarrow 2\delta Z+f\frac{\partial Z}{\partial\lambda}=\alpha=-\left.\frac{\partial}{\partial p^2}\bar{F}^{(2)}(0,p)\right|_{p=0},$$

$$S\bar{G}^{(4)}(0)=-i\bar{F}^{(4)}(0) \Rightarrow 4\delta\lambda+f=\beta=\bar{F}^{(4)}(0).$$

The quantities α and β correspond to Green's functions not requiring subtractions, and so can be calculated in terms of the basic parameters of the theory to any desired order. If we make the conventional choice $Z=1$,[6] we can explicitly solve these equations for the parameters δ, f, and m^2:

$$\delta=\tfrac{1}{2}\alpha,$$
$$f=\beta-2\alpha\lambda,$$
$$m^2=\mu^2(1+\tfrac{1}{2}\alpha).$$

If we look at the lowest-order contributions to α and β, we find that they are both $O(\lambda^2)$. This has the immediate consequence that $f\neq0$, except possibly for some specific value of λ.[7] Therefore, we have to live with the general form of the scaling law, Eq. (6). We return to the question of interpretation later on. It should also be noted that the scaling law determines, to any desired order in perturbation, the funny constants δ and f appearing in S.

III. COMPUTING FUNDAMENTAL GREEN'S FUNCTIONS

We have yet to show, of course, that the scaling law is satisfied for the full Green's functions $\bar{G}^{(2)}$ and $\bar{G}^{(4)}$. This is rendered somewhat difficult by the fact that $\bar{G}^{(2)}$ and $\bar{G}^{(4)}$ are beset by overlapping divergences and not easy to calculate by standard techniques. What we can do, however, is to use the scaling law to *compute* $\bar{G}^{(2)}$ and $\bar{G}^{(4)}$ in a systematic way which not only guarantees that they satisfy the scaling law, but automatically solves the overlapping divergence problem. To see how this goes, let us consider the scaling law for $\bar{G}^{(4)}$, slightly rearranged:

$$\mu\frac{\partial}{\partial\mu}\bar{G}^{(4)}(p_1p_2p_3)=-\left(4\delta+f\frac{\partial}{\partial\lambda}\right)\bar{G}^{(4)}(p_1p_2p_3)-i\bar{F}^{(4)}(0;\,p_1p_2p_3)$$
$$=\Phi(p_1p_2p_3;\,\bar{G}^{(2)},\bar{G}^{(4)},\bar{F}^{(2)}). \tag{7}$$

This is a differential equation for $G^{(4)}$ which can easily be integrated:

$$\bar{G}^{(4)}(p_1p_2p_3)=-i\lambda-\int_0^1\frac{d\alpha}{\alpha}\Phi(\alpha p_1,\alpha p_2,\alpha p_3;\bar{G}^{(2)},\bar{G}^{(4)},\bar{F}^{(2)})$$

[the convergence of the integral is guaranteed by our choice of δ and f, which implies that $\Phi(0,0,0)=0$]. The contribution to this integral of a particular order in λ involves in the integrand the fundamental Green's functions only to *lower* orders in λ [this is because both δ and f are $O(\lambda^2)$ and $\bar{F}^{(4)}$ has a skeleton expansion]. Therefore, we have a systematic scheme for determining $\bar{G}^{(4)}$: If the fundamental Green's functions are known to $O(\lambda^{n-1})$, the requirement that $\bar{G}^{(4)}$ satisfy the scaling law uniquely determines $\bar{G}^{(4)}$ to $O(\lambda^n)$ and no convergence difficulties arise. It should be remembered that the condition $\Phi(0,0,0)=0$ is what is used to determine

[6] This choice of Z does not leave the propagator correctly normalized at the physical particle pole, meaning that a finite rescaling eventually has to be carried out. This does not affect any of our arguments.

[7] It is worth pointing out that we may *impose* $\delta=0$ at the price of fixing Z to be a function of λ such that $\partial Z/\partial\lambda=\alpha/\beta$.

the nth-order contributions to δ and f, thus completing the induction. If we had a scheme for computing $\bar{F}^{(2)}$ in terms of lower-order fundamental Green's functions, we could use the scaling law $S\bar{G}^{(2)}=-i\bar{F}^{(2)}$ to compute $\bar{G}^{(2)}$ systematically in the same way.

$\bar{F}^{(2)}$ is beset with overlapping divergences, as are $\bar{G}^{(2)}$ and $\bar{G}^{(4)}$, and is not easy to handle directly. We can sidestep this problem by studying the scaling law satisfied by $\bar{F}^{(n)}$ rather than $\bar{G}^{(n)}$. The Green's function $\bar{F}^{(n)}(q,p_1\cdots p_n)$ is really quite analogous to $\bar{G}^{(n+1)}$—instead of being a matrix element of $n+1$ identical fields, it is a matrix element of n identical fields plus another field identifiable as the trace of the energy-momentum tensor. Therefore, we expect that when it is operated on by the appropriate scaling operator $\hat{S}$, it yields the matrix element $\bar{H}^{(n)}$ of n identical fields plus *two* traces, one of which carries zero four-momentum:

$$\hat{S}\bar{F}^{(n)}(q,p_1\cdots p_{n-1})=-i\bar{H}^{(n)}(0q;\,p_1\cdots p_{n-1})\,.$$

Such an equation is easy to derive heuristically by standard Ward-identity methods, and one finds

$$\hat{S}=\mu\partial/\partial\mu+n\delta+\bar{\delta}-2\,,$$

where the $\bar{\delta}$ accounts for the difference between the real and naive dimension of Θ in the same way that δ takes care of the possible anomaly in the dimension of the field ϕ. The virtue of this scaling law is that the usual power-counting arguments tell us that all of the matrix elements $\bar{H}^{(n)}$ are primitively convergent, even for $n=2$.[8] Therefore, all $\bar{H}^{(n)}$ have skeleton expansions in terms of the fundamental Green's functions $\bar{G}^{(2)}$, $\bar{G}^{(4)}$, and $\bar{F}^{(2)}$.

The $\bar{F}^{(n)}$ with $n>2$ are all primitively convergent and possess skeleton expansions in terms of $\bar{G}^{(2)}$, $\bar{G}^{(4)}$, and $\bar{F}^{(2)}$. Therefore, we want the corresponding equations $\hat{S}\bar{F}^{(n)}=-i\bar{H}^{(n)}$ to be automatically valid once the fundamental equations $S\bar{G}^{(2)}=-i\bar{F}^{(2)}$, $S\bar{G}^{(4)}=-i\bar{F}^{(4)}$, and $\hat{S}\bar{F}^{(2)}=-i\bar{H}^{(2)}$ are assumed valid. Precisely the same type of argument that led to the requirement that S be distributive then implies that $\hat{S}=S+\bar{\delta}-2$. The quantity $\bar{\delta}$ is then determined by the equation $\hat{S}\bar{F}^{(2)}(0)=-i\bar{H}^{(2)}(0)$ in a perfectly straightforward way, and is found to be $O(\lambda)$.

The scaling law for $F^{(2)}$ can now be rewritten as

$$\begin{aligned}(\mu\partial/\partial\mu-2)\bar{F}^{(2)}(q;\,p)&=(-2\delta-\bar{\delta}-f\partial/\partial\lambda)\bar{F}^{(2)}(q,p)-i\bar{H}^{(2)}(qq;\,p)\\&=X(q,p;\,\bar{G}^{(2)},\bar{G}^{(4)},\bar{F}^{(2)})\,.\end{aligned}$$

Upon integration, this gives

$$\bar{F}^{(2)}(q,p)f=\bar{F}^{(2)}(0,0)-\int_0^1\frac{d\alpha}{\alpha}X(\alpha q,\alpha p,\bar{G}^{(2)},\bar{G}^{(4)},\bar{F}^{(2)})\,.$$

[8] Each insertion of ϕ^2 (remember that $\Theta\propto\phi^2$ in this theory) reduces the degree of divergence by 2. The only divergent matrix element with one insertion of Θ is $\bar{F}^{(2)}$, and its degree of divergence is 0. Therefore, $\bar{H}^{(2)}$ has degree of divergence -2.

The convergence of the integral is guaranteed because $\bar{\delta}$ is so chosen that $X(0,0)=0$. Also, this equation determines the nth-order contribution to $\bar{\delta}$ in terms of lower-order quantities. Just as in the analysis of Eq. (7), we see that to calculate the right-hand side of this equation to $O(\lambda^n)$ requires the knowledge of $\bar{G}^{(2)}$, $\bar{G}^{(4)}$, and $\bar{F}^{(2)}$ only up to $O(\lambda^{n-1})$. Therefore, taken together with the corresponding equations for $\bar{G}^{(2)}$ and $\bar{G}^{(4)}$, this equation provides a systematic scheme for computing the fundamental Green's functions of the theory to successively higher orders in perturbation theory in a way automatically consistent with the scaling law. This method also completely avoids divergence difficulties since the calculations always start with amplitudes possessing a skeleton expansion.

The amplitudes $\bar{G}^{(2)}$ and $\bar{G}^{(4)}$ so computed are not obviously identical to those one would determine by the usual methods of renormalized perturbation theory. We therefore are obligated to verify that they possess the usual properties of analyticity and unitarity. If they do, then they must be perfectly acceptable amplitudes, and in fact identical to the amplitudes computed in the usual way.

Let us first consider the question of analyticity. The amplitude $\bar{G}^{(4)}$ is computed from Eq. (7) by successive approximations: The fundamental Green's functions correct to order $n-1$ are inserted on the right-hand side, and the equation is then integrated to get $G^{(4)}$ correct to order n:

$$\bar{G}^{(4)}(p_1p_2p_3)=-i\lambda-\int_0^1\frac{d\alpha}{\alpha}\Phi(\alpha p_1,\alpha p_2,\alpha p_3;\,\bar{G}^{(4)},\bar{G}^{(2)},\bar{F}^{(2)})\,.$$

If $\Phi(p_1p_2p_3)$ has a cut of the usual form, say, along the line $(2\mu)^2<p_1{}^2<\infty$, then of course so does

$$\int_0^1\frac{d\alpha}{\alpha}\Phi(\alpha p_1,\alpha p_2,\alpha p_3)\,.$$

In fact, so long as the cuts in Φ are of the usual sort in s, t, u, and masses, then $\bar{G}^{(4)}$ has precisely the same cuts, with different discontinuities across the cuts. But the cuts of Φ to a given order are just the cuts of $\bar{G}^{(4)}$ to lower order and the cuts of $\bar{F}^{(4)}(0,p_1p_3p_2)$ to the same order. Since $\bar{F}^{(4)}(0;\,p_1p_2p_3)$ is constructed via a standard skeleton expansion, it will have the usual cuts which, because the momentum arguments are the same, coincide with those of $\bar{G}^{(4)}$. Since we are proceeding by induction, we assume that $\bar{G}^{(4)}$ to lower order has the proper cuts. Then Φ will have exactly the same singularity structure as we would expect for $\bar{G}^{(4)}$, which means that the next approximation to $\bar{G}^{(4)}$, gotten by integrating Φ, has the right singularity structure. The same sort of arguments, of course, apply to $\bar{G}^{(2)}$ since it is computed in much the same way.

The question of unitarity is less trivial but is attacked in a similar manner. Let us for simplicity consider the

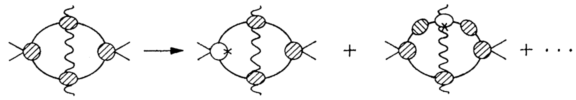

FIG. 2. Action of S on the two-body unitarity integral. The cross stands for insertion of Θ and the wavy line stands for replacing the affected propagators by their discontinuity across the single-particle pole.

amplitude $\bar{G}^{(4)}$ in a kinematic region where only a two-particle cut in the variable $(p_1+p_2)^2$ exists. The normal unitarity prediction for the discontinuity across this cut is

$$Du\bar{G}^{(4)}(p_1p_2p_3p_4) = \int \frac{d^4l}{(2\pi)^4}\frac{d^4l'}{(2\pi)^4}(2\pi)^4\delta(p_1+p_2+l+l') \times\{\bar{G}^{(4)}(p_1p_2ll')\Delta(l)\Delta(l')\bar{G}^{(4)}(ll'p_3p_4)\},$$

where $\Delta(l)$ is the discontinuity of the propagator across its single-particle pole [if the physical mass is $\bar{\mu}^2$ and the propatagor is conventionally normalized, $\Delta(l) = 2\pi\delta(l^2-\bar{\mu}^2)\theta(l_0)$]. If we act on this equation with the scaling operator S, the distributive property of S means that we get four terms, corresponding to S acting in succession on each of the four terms in curly brackets:

$$SD_\mu\bar{G}^{(4)}(p_1p_2p_3p_4) = \int \frac{d^4l}{(2\pi)^4}\frac{d^4l'}{(2\pi)^4}(2\pi)^4\delta(p_1+p_2+l+l') \times\{[S\bar{G}^{(4)}(p_1p_2ll')]\Delta(l)\Delta(l')\bar{G}^{(4)}(ll'p_3p_4) +\bar{G}^{(4)}(p_1p_2ll')[S\Delta(l')]\Delta(l)\bar{G}^{(4)}(ll'p_3p_4)+\cdots\}.$$

Since S acting on $\bar{G}^{(4)}$ gives $\bar{F}^{(4)}$ and since S acting on the propagator gives the insertion of $\bar{F}^{(2)}$ on the propagator,[9] these four terms can be represented graphically as in Fig. 2. These terms are immediately recognized as the standard unitarity contributions to the two-particle discontinuity of $-i\bar{F}^{(4)}(0;\, p_1p_2p_3p_4)$, which, because it is computed from a skeleton expansion, is known to satisfy normal unitarity. Therefore, if we let D stand for the operation of taking the discontinuity, we have, schematically,

$$S[D_u\bar{G}^{(4)}]=D[-\bar{F}^{(4)}]=D[S\bar{G}^{(4)}]=S[D\bar{G}^{(4)}],$$

where the last equation follows because the operations S and taking the discontinuity commute. Therefore, $S[D\bar{G}^{(4)}-D_uG^{(4)}]=0$. We gave arguments for this equation on the two-particle cut only, but it obviously works for any cut and we can take it to be true in general. If we assume that $\bar{G}^{(4)}$ satisfies normal unitarity up to order λ^{n-1}, then arguments of the kind we have often used [see Eq. (7) *et seq.*] imply that to order λ^n, $\bar{G}^{(4)}$ satisfies

$$\mu\frac{\partial}{\partial\mu}[D\bar{G}^{(4)}-Du\bar{G}^{(4)}]=0.$$

This allows the solution

$$D\bar{G}^{(4)}-D_u\bar{G}^{(4)}=\text{momentum-independent constant}.$$

Since there is a kinematic region where both $D\bar{G}^{(4)}$ and $D_u\bar{G}^{(4)}$ vanish (below the two-particle threshold), the constant on the right-hand side of this equation must in fact vanish. Therefore, we can show inductively that $\bar{G}^{(4)}$ satisfies the usual unitarity relation. Similar arguments, of course, apply to $\bar{G}^{(2)}$.

These arguments for analyticity and unitarity are somewhat sketchy, but presumably could be made rigorous.[10] They seem, however, sufficiently convincing to make us believe the proposed scheme for computing $\bar{G}^{(2)}$ and $\bar{G}^{(4)}$.

To summarize, we have done two things by this rather long argument. First of all, we have shown that the particle amplitudes in this theory satisfy a scaling law, albeit one which differs in a profound way from the one suggested by naive broken-scale-invariance requirements. Secondly, we have shown how this scaling law is used to compute the amplitudes of the theory in a way which automatically avoids all questions of divergence, overlapping or otherwise. Finally, it should be noted that these arguments will generalize in an obvious way to any renormalizable field theory, and might even be of some use in making simpler calculations of higher-order quantities.

IV. INTERPRETATION AND CONNECTION WITH RENORMALIZATION GROUP

At this point we should make some effort to understand why the scaling law takes on the form it does. If we could somehow "turn off" the explicit scale-invariance-breaking terms in the Lagrangian—in this case, the mass term—then the particle amplitudes would satisfy $S\bar{G}^{(n)}=0$. If S were simply $\mu\partial/\partial\mu+n\delta$, this would imply that the functions $\bar{G}^{(n)}$ are homogeneous functions of their momentum arguments of degree $4-nd$, with $d=1-\delta$.[11] This is what one might call naive scaling, appropriately modified for the anomalous dimensions of the fields. "Turning off" the mass terms can actually be achieved in practice by taking appropriate asymptotic limits of momenta, and one would expect the Green's functions to satisfy naive scaling in such limits. In fact, $S=\mu\partial/\partial\mu+n\delta+f\partial/\partial\lambda$, so that even though we can achieve $S\bar{G}^{(n)}=0$ in appropriate asymptotic regions, this does not mean that the $\bar{G}^{(n)}$ satisfy naive scaling in the same limit. It appears that naive scaling is replaced by some restriction on the joint de-

[9] We assume, and can show directly, that S acting on the *discontinuity* Δ of the propagator gives the *discontinuity* of the insertion of $\bar{F}^{(2)}$ on the propagator.

[10] See in this connection T. T. Wu, Phys. Rev. **125**, 1436 (1962).

[11] Recall that $\mu\partial/\partial\mu+n\delta\equiv 4-nd-\sum p_i\cdot\partial/\partial p_i$.

pendence of $\bar{G}^{(n)}$ on momenta *and* coupling constant. That S contains the term $f(\lambda)\partial/\partial\lambda$ is apparently equivalent to saying that even in the absence of explicit symmetry-breaking terms (mass terms), scale invariance is still broken by some mechanism. The nature of this mechanism is not hard to find: We assumed that in the Lagrangian

$$\mathfrak{L}=\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\tfrac{1}{2}\mu^2\phi^2-\frac{\lambda}{4!}\phi^4$$

only the mass term $\mu^2\phi^2$ contributes to scale-invariance breaking. In fact, any term in the Lagrangian with dimension different from four will contribute.[2] The interaction term $\lambda\phi^4$ has dimension four to lowest order in perturbation theory, but, just as ϕ has a dimension different from one when interactions are considered,[3] so will ϕ^4 have a dimension different from four. Therefore, one can expect the $\lambda\phi^4$ term to contribute to scale-invariance breaking even though it has a dimensionless coupling constant. The interesting thing about the scaling law is that it provides a simple analytic expression of the effect of this implicit sort of symmetry breaking.

We mentioned that whenever the right-hand side of the equation $SG=-iF$ could be neglected, one obtained a constraint on the joint dependence of G on momenta and coupling constant rather than naive scaling. We would like to pursue this somewhat further in order to establish a connection with the renormalization group. Consider for a moment the scaling law for the two-particle amplitude: $S\bar{G}^{(2)}(p^2)=-i\bar{F}^{(2)}(0,p)$. If we take the limit $p^2\to-\infty$, Weinberg's theorem[12] implies $\bar{F}^{(2)}\to(p^0)\times$(powers of $\ln p^2$) while $\bar{G}^{(2)}\to(p^2)\times$(powers of $\ln p^2$). If we collect together all those terms in $\bar{G}^{(2)}$ which are proportional to p^2 and denote them by $p^2\,\Phi(\ln(-p^2/\mu^2),\ \lambda)$, the scaling law clearly implies that $S\Phi=0$, a rather severe restriction on the form of Φ. In fact, since $S=\mu\partial/\partial\mu+f(\lambda)\partial/\partial\lambda+2\delta$ is a linear operator, it is quite easy to get the general solution for Φ:

$$\Phi(z,\lambda)=\hat{\Phi}(z+\eta(\lambda))\,\exp\left(-\int^{\lambda}\frac{d\lambda'}{2f(\lambda')}\delta(\lambda')\right)$$

where

$$\eta(\lambda)=\int^{\lambda}\frac{d\lambda'}{2f(\mu')},$$

and $\hat{\Phi}$ is an arbitrary function of one variable. This kind of correlation between the asymptotic dependence on momentum and the dependence on coupling constant is typical of renormalization-group arguments,[13] and it is interesting to see how easily it emerges from the scaling law. Similar considerations would apply to other amplitudes than the propagator, and presumably allow one to extract renormalization-group conclusions in an expeditious manner.

CONCLUSIONS

We undertook this investigation in the hope of finding out just how scale invariance is broken in a model field theory. We found that the source (in the sense of the scaling law) for violations of naive dimensional scaling was not simply those terms in the Lagrangian having dimensional coupling constants. The interpretation of this is relatively simple: A term in the Lagrangian will not break scale invariance only if its dimension (as defined by commutation with the dilation generator) is exactly equal to four. But the terms in the Lagrangian with dimensionless coupling constants are guaranteed to have dimension four only to lowest order in perturbation—when the effects of interactions are considered, their dimensions will change and they will contribute to scale-invariance breaking. The surprising thing we found was that these "implicit" breaking terms could be incorporated in the scaling law by a rather simple change in its form. The resulting scaling law probably cannot be used in a direct phenomenological fashion, but by studying a special asymptotic limit we were able to recover the results of the renormalization group. There are other asymptotic limits in which field-theory scattering amplitudes are supposed to have simple forms (the impact-parameter representation, for instance) and one might find useful constraints on such forms by studying their compatibility with the scaling law. Another aspect of this is that the scaling law appears to provide a particularly simple approach to renormalization—one can use it to completely sidestep questions of overlapping divergences and obtain a relatively simple prescription for computing renormalized amplitudes.

Finally, it should be said that we confined ourselves to a scalar-field theory only for the sake of simplicity. It seems quite clear that the general ideas which we have developed are applicable, with simple modifications, to any renormalizable theory. It is not clear that any of this has immediate practical importance. Nonetheless, it is always useful to see an old problem in a new light, and we hope that this, along with whatever clarification of the problems of broken scale invariance we may have achieved, justifies yet another paper on scalar field theory.

Note added in manuscript: For another, not dissimilar, approach to these questions, the reader should consult a recent paper by K. Wilson, this issue, Phys. Rev. D **2**, 1478 (1970).

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[12] S. Weinberg, Phys. Rev. **118**, 838 (1960).

[13] J. D. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Vol. II, p. 368.