change $\int d^3x$ and $\sum_n$, we get after some algebra

$$C_A^{(m)}=(2\pi)^3(-2i)(-i)^m\sum_n \delta(\mathbf{P}_n-(E_n-M)\hat{q})\times(E_n-M)^m \sin\epsilon(E_n-M)\times\{|M_{pn,A}|^2\pm(-)^m|M_{np,A}|^2\},$$

$$A=\begin{pmatrix}\text{boson}\\ \text{fermion}\end{pmatrix},\quad \epsilon\to 0+$$

where the $|M_{pn,A}|^2$ and $|M_{np,A}|^2$ are spin-averaged absolute squares of matrix elements of source currents $[j_{\pi^\alpha}(0)$ for $A=\alpha$, $\bar{u}(q)f_p(0)$ for $A=p]$ between Ψ_p and intermediate state Ψ_n, Ψ_n and Ψ_p, respectively.

We find immediately that $C_A^{(m)}$ is pure imaginary for m even, pure real for m odd, as used in (18) and (23).

Note that $E_n-M\geqslant 0$ for $A=\text{boson}$, while $(E_n-M)^m \times\sin\epsilon(E_n-M)$ is even in E_n-M for m odd. Thus every summand of $\sum_n$ is positive for $A=\text{boson}$, m even, or $A=\text{fermion}$, M odd. Putting $m=0$, $A=\text{boson}$, $m=1$, $A=\text{proton}$, this proves that $\text{Im}C_\alpha^{(0)}<0$ and also $\text{Re}C_{p1}^{(1)}<0$.

VII. REMARKS

(a) Equation (21) shows that the asymptotic behavior depends critically on the smoothness of the relevant current commutator across the light cone. For example, if $F(r,t)$ is continuous in t,

$$\int_{r-\epsilon}^{r+\epsilon} F(r,t)=2\epsilon F(r,t(r)),\quad r-\epsilon\leqslant t(r)\leqslant r+\epsilon$$

by the theorem of the mean. Then $C^{(0)}=0$ if $\int dr\, rF[r,t(r)]<\infty$. On the other hand, if $F(r,t)=g(r)\delta(r-t)/r$ with $\int dr\, g(r)<\infty$ and $\neq 0$, then $C^{(0)}\neq 0$.

(b) The necessity of the frame-dependent cutoff for constant nonzero asymptotic σ is striking. For from (18) and its pp analog, if $|g(q)|^2$ were replaced by unity, $\sigma(\pi p)\sim O(1/\omega^2)$ and $\sigma(pp)\sim O(1/\omega)$ at most. However, it may well be that the assumption of the analyticity everywhere of $f_\pm(\xi)$ is not justified in local QFT. Models have been examined in another work,[2] which suggests that it is not, or perhaps better said, that the question has not much meaning in local QFT because of its divergent and ill-defined nature. Tanaka[17] gives examples of light-cone behavior of source current commutators $[\propto \partial^m\delta(-x^2)/(\partial t)^m,\ m=0,1,2,\ldots]$ which can yield $\text{Im}T\sim O(\omega)$ or even a higher power and thus constant asymptotic σ.

(c) Recent Serpukhov data[18] on total cross sections for π^-p, K^-p, $\bar{p}p$ up to $\omega=65$ GeV show some waveiness at these very high energies. This behavior can be fitted by power series in $1/\omega$ as given by this theory.

The theoretical values of the constants $C_A^{(m)}$, in particular, the values $a(AB)$ of the asymptotic cross sections, will have to await a reliable way to calculate the current comutators on the light cone. But it is seen from (19) and (23) that their scale is given by the square of the cutoff length λ.

[17] K. Tanaka, Phys. Rev. **164**, 1800 (1967).
[18] *Proceedings of the Lund International Conference on Elementary Particles, 1969*, edited by G. von Dardel (Berlingska, Lund, Sweden, 1969).

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Renormalization Constants, Wave Functions, and Energy Shifts in the Coulomb and Lorentz Gauges

KURT HALLER*
University of Connecticut, Storrs, Connecticut 06268
AND
LEON F. LANDOVITZ
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033
(Received 15 December 1969)

The scattering wave functions for quantum electrodynamics are examined in the Coulomb gauge, in the conventional Lorentz gauge, and in a reformulated version of the Lorentz gauge. It is shown that when the Lorentz gauge is formulated so that Maxwell's equations hold even when the Green's-function pole is displaced off the real axis, Z_2 is identical in the Coulomb gauge and in the Lorentz gauge. It is also shown that the unrenormalized asymptotic states in the reformulated Lorentz gauge include a partial dressing of the bare electrons by longitudinal and timelike photons sufficient to generate the electron's static electric field. It is proven that for true bound states the radiative energy shifts in the reformulated Lorentz gauge and in the conventional formulation agree.

I. INTRODUCTION

In earlier work[1–3] we introduced a reformulation of quantum electrodynamics in the Lorentz gauge, in which the physical states $|\bar{\varphi}\rangle$ are defined by

$$\Omega^{(+)}(\mathbf{x})|\bar{\varphi}\rangle=0, \tag{1a}$$

* Supported in part by the University of Connecticut Research Foundation.

[1] K. Haller and L. F. Landovitz, Phys. Rev. **171**, 1749 (1968).
[2] K. Haller and L. F. Landovitz, Phys Rev. Letters **22**, 245 (1969).
[3] K. Haller and L. F. Landovitz, Phys. Rev. **182**, 1922 (1969).

where $\Omega^{(+)}(\mathbf{x})$ is given by

$$\Omega^{(+)}(\mathbf{x})=i\sum_{\mathbf{k}} k^{1/2}\Omega^{(+)}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (1b)$$

and $\Omega^{(+)}(\mathbf{k})$ is given by[4]

$$\Omega^{(+)}(\mathbf{k})=a_{\mathbf{k},Q}+(2k^{3/2})^{-1}\rho(\mathbf{k}). \quad (1c)$$

In this formulation of the Lorentz gauge, Eq. (1a) takes the place of the subsidiary condition $\chi^{(+)}(\mathbf{x})|\varphi\rangle=0$, on which the usual formulation[5] of the Lorentz gauge is based.

In Ref. 1 it was demonstrated that Eq. (1a) is the proper subsidiary condition to use in defining the Lorentz gauge. This is because the scalar operator $\Omega^{(+)}(\mathbf{x})$ represents the invariant positive-frequency part of $\partial_\mu A_\mu$ in the presence of interactions; the equations of motion therefore imply the continued validity of Eq. (1a), once it is postulated to hold at any one time, and the permanent validity of Maxwell's equations (for the expectation values of electromagnetic fields) is guaranteed. The consequences of substituting $\chi^{(+)}(\mathbf{x})|\varphi\rangle=0$ for Eq. (1a) are discussed in Refs. 1 and 2. One conclusion that was drawn in Ref. 1 was that the form of the S matrix for quantum electrodynamics (QED) remains wholly unaltered when the proper formulation of the Lorentz gauge is replaced by the conventional one, in spite of the internal inconsistencies of the latter. In Ref. 3 it was shown that the new, proper formulation of the Lorentz gauge leads to a theory identical to QED in the Coulomb gauge, except for processes that involve unphysical photons.

These results make it particularly desirable to understand why the reported values of the renormalization constant Z_2 differ in the Lorentz and Coulomb gauges.[6] It is this question to which we address ourselves in Sec. II and in Appendix B in this paper. The conclusion that we draw is that the identity of the S matrix in the two versions of the Lorentz gauge implies only that the corresponding transition amplitudes are identical on the energy shells and in the limit in which all Green's-function poles have been taken on the real axis [i.e., $\lim_{(\epsilon\to0)}(E-H_0+i\epsilon)^{-1}$]. The definition of Z_2, however, involves the knowledge of the transition amplitudes with the Green's-function poles away from the real axis. Careful study of the transition amplitudes in this latter case reveals that Z_2 differs in the old and the new formulation of the Lorentz gauge. In the new, correct formulation, Z_2 in the Lorentz gauge agrees with the value obtained in the Coulomb gauge.

In Sec. III we note that the zero-time scattering wave function for the electron-photon system in the old and the new formulations of the Lorentz gauge differ only to order $i\epsilon$ (the displacement of the Green's-function poles off the real axis); and that these $i\epsilon$ discrepancies, though important in singular transition amplitudes, such as are involved in wave-function renormalization graphs, do not contribute in ordinary scattering amplitudes. Nevertheless, the noninteracting electron-photon states, which the wave function is assumed to approach as $t\to\pm\infty$, profoundly differ in the old and new formulations of the Lorentz gauge, even though, in both cases, the same Hamiltonian is invoked as the time-displacement operator.

In Sec. III we carefully examine all contributions to the asymptotic wave function that arise when the zero-time scattering wave function has the form dictated by the conventional version of the Lorentz gauge. We demonstrate that there are terms, which superficially appear to vanish due to the operation of the Riemann-Lebesgue lemma as $t\to\pm\infty$, which in fact persist to form the asymptotic scattering states assumed in the new, correct formulation of the Lorentz gauge. In Sec. III we also show that the old formulation of the Lorentz gauge is not wholly consistent with Maxwell's equations.

In Sec. IV we note that, unlike the case of the energy continuum, the bound-state wave functions are in no way affected by the substitution of the old for the new formulation of the Lorentz gauge. The energy shifts, too, are identical for these two cases.

II. COLLISION PROBLEM IN QED

A. Wave Functions and Transition Amplitudes

In this section we address ourselves to the scattering problem for the following cases: Case I is the scattering problem in the conventional Lorentz gauge. In this case we have

$$\lim_{\epsilon\to0}(H_0'+H_1'-E_i)|\psi_i^{(\epsilon)}\rangle=0, \quad (2a)$$

with[7]

$$|\psi_i^{(\epsilon)}\rangle=|\varphi_i\rangle+(E_i-H_0'+i\epsilon)^{-1}H_1'|\psi_i^{(\epsilon)}\rangle, \quad (2b)$$

where

$$\chi^{(+)}(\mathbf{x})|\varphi_i\rangle=0 \quad\text{and}\quad (H_0'-E_i)|\varphi_i\rangle=0.$$

Case II is the scattering problem in the reformulated Lorentz gauge. In this case we have

$$\lim_{\epsilon\to0}(\mathcal{H}_0'+\mathcal{H}_1'-E_i)|\bar\psi_i^{(\epsilon)}\rangle=0 \quad (3a)$$

and

$$|\bar\psi_i^{(\epsilon)}\rangle=|\bar\varphi_i\rangle+(E_i-\mathcal{H}_0'+i\epsilon)^{-1}\mathcal{H}_1'|\bar\psi_i^{(\epsilon)}\rangle, \quad (3b)$$

[4] $\Omega^{(+)}$ is the operator that appears in K. Bleuler, Helv. Phys. Acta **23**, 567 (1950); Gupta has recently commented [see S. N. Gupta, Phys. Rev. **180**, 1601 (1969)] to the effect that the operator which defines the subsidiary condition in S. N. Gupta, Proc. Phys. Soc. (London) **63**, 681 (1950), is the same as the one in Bleuler's later paper.

[5] See Ref. 1, footnote 1.

[6] See, for example, K. Johnson and B. Zumino, Phys. Rev. Letters **3**, 351 (1959); B. Zumino, *ibid.* **3**, 351 (1959); J. Math. Phys. **1**, 1 (1960); C. R. Hagen, Phys. Rev. **130**, 813 (1963).

[7] The symbols denoting electrodynamic Hamiltonians are used as defined in Refs. 1 and 3. The primed Hamiltonians refer to the level-shifted forms that correspond to the respective unprimed variety. Thus for example, $H_0'=H_0+\Delta E$, $H_1'=H_1-\Delta E$, $\mathcal{H}_0'=\mathcal{H}_0+\Delta E$, and $\mathcal{H}_1'=\mathcal{H}_1-\Delta E$, so that the continuum spectra of H_0 and H and of $\mathcal{H}_0$ and H coincide. Since the spectra of H_0 and $\mathcal{H}_0$ coincide, the level shift is identical in cases I–III.

where $\Omega^{(+)}(\mathbf{x})|\bar{\varphi}_i\rangle=0$ and $(\mathcal{H}_0'-E_i)|\bar{\varphi}_i\rangle=0$; $|\bar{\varphi}_i\rangle$ is given by $|\bar{\varphi}_i\rangle=e^{-D}|\varphi_i\rangle$. Case III is obtained from case II by a pseudounitary transformation; in case III we have

$$\lim_{\epsilon\to 0}(H_0'+\hat{H}_1'-E_i)|\hat{\psi}_i^{(\epsilon)}\rangle=0 \tag{4a}$$

and

$$|\hat{\psi}_i^{(\epsilon)}\rangle=|\varphi_i\rangle+(E_i-H_0'+i\epsilon)^{-1}\hat{H}_1'|\hat{\psi}_i^{(\epsilon)}\rangle. \tag{4b}$$

$\hat{H}_1'$ is given by

$$\hat{H}_1'=H_1'+[D,H]=H_1'+[D,\hat{H}] \tag{5a}$$

and also by

$$\hat{H}_1'=H_{1,T}+H_C+H_{Q,R}-\Delta E, \tag{5b}$$

where

$$H_{1,T}=-\sum_{\mathbf{k}}(2k)^{-1/2}[a_{\mathbf{k},\epsilon(i)}\mathbf{J}(-\mathbf{k})\cdot\boldsymbol{\varepsilon}(i)+a^\dagger_{\mathbf{k},\epsilon(i)}\mathbf{J}(\mathbf{k})\cdot\boldsymbol{\varepsilon}(i)],$$

$$H_C=\int d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})\rho(\mathbf{y})(8\pi|\mathbf{x}-\mathbf{y}|)^{-1},$$

$$H_{Q,R}=-\sum\tfrac{1}{2}(k)^{-1/2}\{a_{\mathbf{k},Q}[\rho(-\mathbf{k})+\mathbf{k}\cdot\mathbf{J}(-\mathbf{k})/|\mathbf{k}|]+a^\dagger_{\mathbf{k},R}[\rho(\mathbf{k})+\mathbf{k}\cdot\mathbf{J}(\mathbf{k})/|\mathbf{k}|]\}.$$

The discrepancy between Eqs. (2a) and (4a), and between (2b) and (4b), respectively, accounts for the dynamical differences between the conventional and the new formulation of QED in the Lorentz gauge. As was previously pointed out,[8] cases I and II are *not* related by a pseudounitary transformation, although cases II and III are.

From $\hat{H}=H_0'+\hat{H}_1'=e^DHe^{-D}$, we can show that

$$\hat{H}_1'=H_1'e^{-D}+(1-e^{-D})\hat{H}-H_0'(1-e^{-D}). \tag{6}$$

This, in turn, allows us to rewrite Eq. (4b),

$$\begin{aligned}|\hat{\psi}_i^{(\epsilon)}\rangle=&|\varphi_i\rangle+(E_i-H_0'+i\epsilon)^{-1}\\&\times[H_1'e^{-D}-(H_0'-E_i)(1-e^{-D})]|\hat{\psi}_i^{(\epsilon)}\rangle\\&+i\epsilon(E_i-H_0'+i\epsilon)^{-1}(1-e^{-D})\\&\times(E_i-\hat{H}+i\epsilon)^{-1}\hat{H}_1'|\varphi_i\rangle.\end{aligned} \tag{7}$$

In the limit as $\epsilon\to 0$, we know that Eq. (4a) holds, but it is not convenient to consider Eq. (7) in that limit yet. Rather we write Eq. (4b) as

$$|\hat{\psi}_i^{(\epsilon)}\rangle=|\varphi_i\rangle+(E_i-\hat{H}+i\epsilon)^{-1}\hat{H}_1'|\varphi_i\rangle \tag{4c}$$

and find that

$$(\hat{H}-E_i)|\hat{\psi}_i^{(\epsilon)}\rangle=i\epsilon(E_i-\hat{H}+i\epsilon)^{-1}\hat{H}_1'|\varphi_i\rangle. \tag{8}$$

We note that in the derivation of Eq. (8) the use of the level-shifted Hamiltonian is essential, since only in that case are $(H_0'-E_i)|\varphi_i\rangle=0$ and $\lim_{\epsilon\to 0}(\hat{H}-E_i)|\hat{\psi}_i^{(\epsilon)}\rangle=0$ both correct.

Equations (7) and (8) lead to

$$\begin{aligned}e^{-D}|\hat{\psi}_i^{(\epsilon)}\rangle=&|\varphi_i\rangle+(E_i-H_0'+i\epsilon)^{-1}H_1'e^{-D}|\hat{\psi}_i^{(\epsilon)}\rangle\\&-i\epsilon(E_0-H_0'+i\epsilon)^{-1}(1-e^{-D})|\varphi_i\rangle,\end{aligned} \tag{9}$$

[8] Ref. 1, footnote 6.

and inversion of the Green's function in Eq. (9) allows us to formally solve this equation, as follows:

$$\begin{aligned}|\bar{\psi}_i^{(\epsilon)}\rangle=e^{-D}|\hat{\psi}_i^{(\epsilon)}\rangle=&|\psi_i^{(\epsilon)}\rangle\\&-i\epsilon(E_i-H+i\epsilon)^{-1}(1-e^{-D})|\varphi_i\rangle.\end{aligned} \tag{10}$$

The transition amplitude between an initial state (i) and a final state (f) in the new Lorentz gauge formulation is given by $\bar{T}_{f,i}$, where $\bar{T}_{f,i}=\lim_{\epsilon\to 0}\bar{T}_{f,i}^{(\epsilon)}$ and where

$$\bar{T}_{f,i}^{(\epsilon)}=\langle\bar{\varphi}_f^\star|\mathcal{H}_1'|\bar{\psi}_i^{(\epsilon)}\rangle, \tag{11a}$$

or, equivalently,

$$\bar{T}_{f,i}^{(\epsilon)}=\langle\varphi_f^\star|\hat{H}_1'|\hat{\psi}_i^{(\epsilon)}\rangle; \tag{11b}$$

and, from Eq. (6), we have that

$$\begin{aligned}\bar{T}_{f,i}^{(\epsilon)}=&T_{f,i}^{(\epsilon)}+\langle\varphi_f^\star|(1-e^{-D})|\hat{\psi}_i^{(\epsilon)}\rangle(E_i-E_f)\\&+i\epsilon\{\langle\varphi_f^\star|(1-e^{-D})(E_i-\hat{H}+i\epsilon)^{-1}\hat{H}_1'|\varphi_i\rangle\\&-\langle\varphi_f^\star|H_1'(E_i-H+i\epsilon)^{-1}(1-e^{-D})|\varphi_i\rangle\},\end{aligned} \tag{12}$$

where $T_{f,i}^{(\epsilon)}=\langle\varphi_f^\star|H_1'|\psi_i^{(\epsilon)}\rangle$. The symbol $\langle n^\star|$ denotes the adjoint in the indefinite metric space, which is discussed in Ref. 1, Sec. II.

Equation (12) affords us substantial insights into the relation between the old and the new formulations of the Lorentz gauge. When the transition amplitude is nonvanishing on the energy shell as $\epsilon\to 0$ (as, for example, in a scattering transition amplitude involving absorption and emission of photons by electrons as in $\gamma_q+e_{\mathbf{k}}\to\gamma_{q'}+e_{\mathbf{k}'}$), the quantity proportional to $i\epsilon$ vanishes with respect to $T_{f,i}^{(\epsilon)}$ in the limit $\epsilon\to 0$, and the transition amplitudes $\bar{T}_{f,i}$ and $T_{f,i}$ are identical on the energy shell. Off the energy shell they differ by $\langle\varphi_f^\star|(1-e^{-D})|\hat{\psi}_i^{(\epsilon)}\rangle(E_i-E_f)$; this latter quantity never affects any adiabatic processes (like scattering phenomena) but is involved in nonadiabatic events (see, for example, Ref. 2).

When, on the other hand, $\bar{T}_{f,i}$ and $T_{f,i}$ display (0/0) singularities on the energy shell, as in the case of wave-function renormalization terms, then we need to consider the coefficients of the $i\epsilon$ terms carefully. When such singularities occur, $\bar{T}_{f,i}$ and $T_{f,i}$ will differ (as will also Z_2 for the two different formulations of the Lorentz gauge). Since we have shown in Ref. 3 that the transition amplitude $\bar{T}_{f,i}$ is identical to the one that results from the Coulomb gauge, on and off the energy shell and independently of any restrictions on any limiting processes, the renormalization constant Z_2 in the reformulated Lorentz gauge and in the Coulomb gauge *must agree*.

B. Gauge Invariance of Z_2

In this section we compare Z_2 in the two formulations of the Lorentz gauge. We first define the operators

$$T^{(\epsilon)}(E_i)=H_1'+H_1'(E_i-H+i\epsilon)^{-1}H_1',$$
$$\bar{T}^{(\epsilon)}(E_i)=\mathcal{H}_1'+\mathcal{H}_1'(E_i-H+i\epsilon)^{-1}\mathcal{H}_1',$$

and

$$\hat{T}^{(\epsilon)}(E_i)=\hat{H}_1'+\hat{H}_1'(E_i-\hat{H}+i\epsilon)^{-1}\hat{H}_1'.$$

We then use Eq. (12) to write

$$\langle e_p|\hat{T}^{(\epsilon)}(E_p)|e_p\rangle=\langle e_p|T^{(\epsilon)}(E_p)|e_p\rangle +i\epsilon[\langle e_p|(1-e^{-D})(E_p-\hat{H}+i\epsilon)^{-1}\hat{H}_1'|e_p\rangle -\langle e_p|H_1'(E_p-H+i\epsilon)^{-1}(1-e^{-D})|e_p\rangle], \quad (13)$$

with $(H_0'-E_p)|e_p\rangle=0$. The level shift for this case is $\Delta E=\delta E_p$, the self-energy of the electron[9] of momentum **p**.

It is possible to relate[10] Z_2 to the expectation values of the $T^{(\epsilon)}$ operators defined above; in the case of the old formulation of the Lorentz gauge this leads to[11]

$$\langle e_p|T^{(\epsilon)}(E_p)|e_p\rangle=i\epsilon[(Z_2)_L-1]; \quad (14a)$$

for the case of the new formulation we have

$$\langle \bar{e}_p|\tilde{T}^{(\epsilon)}(E_p)|\bar{e}_p\rangle=i\epsilon[(\bar{Z}_2)_L-1]. \quad (14b)$$

For the case of computational simplicity, we make use of

$$\langle \bar{e}_p|\tilde{T}^{(\epsilon)}(E_p)|\bar{e}_p\rangle=\langle e_p|\hat{T}^{(\epsilon)}(E_p)|e_p\rangle. \quad (15a)$$

Equation (15a) demonstrates that the expression for $(\bar{Z}_2)_L$ is entirely independent of whether it is evaluated in the formalism of case II or III. We can define $(\bar{Z}_2)_L$ by

$$\lim_{\epsilon\to 0}[\langle e_p|(E_p-\hat{H}+i\epsilon)^{-1}|e_p\rangle -(\bar{Z}_2)_L\langle e_p|(E_p-H_0'+i\epsilon)^{-1}|e_p\rangle]=0 \quad (15b)$$

or, equivalently, by

$$\lim_{\epsilon\to 0}[\langle \bar{e}_p|(E_p-H+i\epsilon)^{-1}|\bar{e}_p\rangle -(\bar{Z}_2)_L\langle \bar{e}_p|(E_p-\mathcal{H}_0'+i\epsilon)^{-1}|\bar{e}_p\rangle]=0 \quad (15c)$$

using the representations of cases III and II, respectively.

Equations (14) and (15a) lead to

$$(Z_2)_L=(\bar{Z}_2)_L+\lim_{\epsilon\to 0}\langle e_p|[H_1'(E_p-H+i\epsilon)^{-1}(1-e^{-D}) -(1-e^{-D})(E_p-\hat{H}+i\epsilon)^{-1}\hat{H}_1']|e_p\rangle. \quad (16)$$

Note that there are no further $(i\epsilon)^{-1}$ singularities in the expectation value on the right-hand side of Eq. (16); this is a corroboration of the result that the electron self-energy is identical in cases I–III.

To lowest order, Eq. (16) yields

$$(\mathfrak{z}_2)_L=(\bar{\mathfrak{z}}_2)_L+\lim_{\epsilon\to 0}\langle e_p|[H_1'(E_p-H_0'+i\epsilon)^{-1}D -D(E_p-H_0'+i\epsilon)^{-1}H_1']|e_p\rangle, \quad (17a)$$

which can be rewritten as

$$(\mathfrak{z}_2)_L=(\bar{\mathfrak{z}}_2)_L-\lim_{\epsilon\to 0}\sum_{n,\mathbf{k}}|\langle e_p|\rho(\mathbf{k})|n\rangle|^2(2k^3)^{-1} \times(E_p-E_n+k)(E_p-E_n-k+i\epsilon)^{-1}, \quad (17b)$$

where $\sum_n$ signifies summation over the eigenstates of the noninteracting fermion Hamiltonian.

For corroboration we relate this result to the discrepancy between $(Z_2)_C$ (in the Coulomb gauge) and $(Z_2)_L$ (as evaluated in the old Lorentz gauge formulation), remembering that $(Z_2)_C$ and $(\bar{Z}_2)_L$ are identical. From

$$(Z_2)_L=\lim_{\epsilon\to 0}i\epsilon\langle e_p|(E_p-H+i\epsilon)^{-1}|e_p\rangle, \quad (15d)$$

we obtain, in lowest order,

$$(\mathfrak{z}_2)_L=1+\lim_{\epsilon\to 0}(i\epsilon)^{-1}[\sigma^{(\epsilon)}(E_p)-\delta E_p^{(2)}], \quad (18)$$

where $\sigma^{(\epsilon)}(E_p)=\sum_n|\langle e_p|H_1|n\rangle|^2(E_p-E_n+i\epsilon)^{-1}$. H_1 can be written[12]

$$H_1=\sum_{\mathbf{k}}(2k^{1/2})^{-1}\{-\sum_i[a_{\mathbf{k},\boldsymbol{\epsilon}(i)}\mathbf{J}(-\mathbf{k})\cdot\boldsymbol{\varepsilon}(i) +a^\dagger_{\mathbf{k},\boldsymbol{\epsilon}(i)}\mathbf{J}(\mathbf{k})\cdot\boldsymbol{\varepsilon}(i)]\sqrt{2} +a_{\mathbf{k},R}[\rho(-\mathbf{k})-\mathbf{J}(-\mathbf{k})\cdot\mathbf{k}/|\mathbf{k}|] -a_{\mathbf{k},Q}[\rho(-\mathbf{k})+\mathbf{J}(-\mathbf{k})\cdot\mathbf{k}/|\mathbf{k}|] -a^\dagger_{\mathbf{k},R}[\rho(\mathbf{k})+\mathbf{J}(\mathbf{k})\cdot\mathbf{k}/|\mathbf{k}|] +a^\dagger_{\mathbf{k},Q}[\rho(\mathbf{k})-\mathbf{J}(\mathbf{k})\cdot\mathbf{k}/|\mathbf{k}|]\}, \quad (19)$$

and it is convenient to write

$$\sigma^{(\epsilon)}(E_p)=\sigma_t^{(\epsilon)}(E_p)+\sigma_{nt}^{(\epsilon)}(E_p),$$

where σ_t and σ_{nt} are the parts of σ that originate from transverse photons and from nontransverse (longitudinal and timelike) photons, respectively.

We find that $\sigma_{nt}^{(\epsilon)}(E_p)$ is given by

$$\sigma_{nt}^{(\epsilon)}(E_p)=-\sum_{n,\mathbf{k}}\frac{|\langle e_p|\rho(\mathbf{k})|n\rangle|^2}{2k^3} \times\frac{(E_p-E_n+k)(E_n-E_p+k)}{E_p-E_n-k+i\epsilon}, \quad (20a)$$

[9] Since the Coulomb gauge is not manifestly covariant, we have had recourse to a noncovariant formulation of the subtraction procedure in order to give a unified treatment of the renormalization problem for cases I–III. The values of the renormalization constants may depend on the details of the subtraction and limiting procedures that are adopted. However, gauge invariance of the renormalization constants is true, independently of these procedures, provided that the same procedures are invoked in the different gauges.

[10] B. S. DeWitt, UCRL Report No. UCRL-2884, 1955, Eq. (10.84) (unpublished).

[11] $(Z_2)_L$ will be used to designate the electron wave-function renormalization constant in the conventional Lorentz gauge, $(\bar{Z}_2)_L$ in the reformulated Lorentz gauge, and $(Z_2)_C$ in the Coulomb gauge. $(\mathfrak{z}_2)_L$, $(\bar{\mathfrak{z}}_2)_L$, and $(\mathfrak{z}_2)_C$ will be used to denote the iterative series of the corresponding electron wave-function renormalization constant up to first order in α.

[12] Equation (19) incorporates correction of a sign error in Eq. (7) of Ref. 1.

and we can represent this by

$$\sigma_{nt}^{(\epsilon)}(E_p)=\sigma_{nt}^{(0)}(E_p)+i\epsilon\tau_{nt}^{(\epsilon)}(E_p),$$

where

$$\sigma_{nt}^{(0)}=\sum_{n,\mathbf{k}}|\langle e_p|\rho(\mathbf{k})|n\rangle|^2(E_p-E_n+k)(2k^3)^{-1};$$

since $[H_0,\rho]$ and ρ commute, σ_{nt} can be shown to be given by

$$\sigma_{nt}^{(0)}=\sum_{n,\mathbf{k}}|\langle e_p|\rho(\mathbf{k})|n\rangle|^2(2k^2)^{-1}. \qquad (20b)$$

$\tau_{nt}^{(\epsilon)}(E_p)$ is given by

$$\tau_{nt}^{(\epsilon)}(E_p)=-\sum_{n,\mathbf{k}}|\langle e_p|\rho(\mathbf{k})|n\rangle|^2(2k^3)^{-1} \times(E_p-E_n+k)(E_p-E_n-k+i\epsilon)^{-1}. \qquad (20c)$$

Up to second order in the electric charge, we therefore find that[13]

$$(\mathfrak{z}_2)_L=\lim_{\epsilon\to 0}[1+\tau_t^{(\epsilon)}(E_p)+\tau_{nt}^{(\epsilon)}(E_p)].$$

In the Coulomb gauge, $(\mathfrak{z}_2)_C$ to the same order is given by

$$(\mathfrak{z}_2)_C=\lim_{\epsilon\to 0}[1+\tau_t^{(\epsilon)}(E_p)],$$

since there are no nontransverse photons in that gauge, and since H_C can only contribute to δE_p but not to $(\mathfrak{z}_2)_C$ to second order in the electric charge.

We therefore find that

$$(\mathfrak{z}_2)_L=(\mathfrak{z}_2)_C-\lim_{\epsilon\to 0}\sum_{n,\mathbf{k}}|\langle e_p|\rho(\mathbf{k})|n\rangle|^2(2k^3)^{-1} \times(E_p-E_n+k)(E_p-E_n-k+i\epsilon)^{-1},$$

which verifies Eq. (17b).

III. ASYMPTOTIC BEHAVIOR OF SCATTERING WAVE FUNCTIONS

A. General Kinematics

In this section we review a scattering formalism that allows us to identify the asymptotic forms of scattering wave functions from the Hamiltonian governing the behavior of the colliding system. For this purpose it is important to allow the dynamics of the system to determine the temporal evolution of the wave functions without dictating the explicit forms of the asymptotic states by adiabatically switching the interaction terms or by imposing artificial temporal limits on wave functions.

The treatment to which we have recourse, and which for completeness we review here, is due to Moses.[14] Following Moses, we will treat a Hamiltonian H given by $H=H_0+H_1$, where H_0 and H_1 are both strictly time independent. We will study the behavior of a wave function $|\psi(t)\rangle$ which obeys

$$(H_0+H_1)|\psi(t)\rangle=i(\partial/\partial t)|\psi(t)\rangle, \qquad (21)$$

and, in particular, we will examine the asymptotic limits to which such a wave function can tend.[15]

Let us consider a set of states $|u(E_q,b)\rangle$ and $|\omega(E_q,\alpha)\rangle$ so that

$$(H_0-E_q)|u(E_q,b)\rangle=0 \qquad (22a)$$

and

$$(H-E_q)|\omega(E_q,\alpha)\rangle=0; \qquad (22b)$$

b and α refer to eigenvalues (discrete or continuous) of operators P and Q, respectively, for which $[H_0,P]=0$ and $[H,Q]=0$, although $[H_0,Q]$ need not vanish. We assume that P and Q remove all degeneracy left by H_0 H, respectively. We moreover assume that the energy continua of H_0 and H coincide.

We can represent the wave function $\psi(0)$ by

$$|\psi(0)\rangle=\sum_b\int dE_q\, f(E_q,b)|u(E_q,b)\rangle \qquad (23a)$$

and, at all other times, $|\psi(t)\rangle$ by

$$|\psi(t)\rangle=\sum_b\int dE_q\, f(E_q,b;t)|u(E_q,b)\rangle. \qquad (23b)$$

Similarly, since the ω and the u span the same space, we can also represent $\psi(0)$ by

$$|\psi(0)\rangle=\sum_\alpha\int dE_k\, g(E_k,\alpha)|\omega(E_k,\alpha)\rangle \qquad (24a)$$

and $|\psi(t)\rangle$ by

$$|\psi(t)\rangle=\sum_\alpha\int dE_k\, g(E_k,\alpha)|\omega(E_k,\alpha)\rangle\exp(-iE_kt); \qquad (24b)$$

in Eqs. (23a) and (24a), $f(E_q,b)$ and $g(E_k,\alpha)$ are smooth and reasonably narrow spectral packet functions.

We can expand $|\omega(E_k,\alpha)\rangle$ in terms of the eigenstates of H_0 and P according to

$$|\omega(E_k,\alpha)\rangle=\sum_b\int dE_q\,\chi(E_q,b;E_k,\alpha)|u(E_q,b)\rangle. \qquad (25)$$

Equations (22a) and (22b) then lead to

$$(E_k-E_q)\chi(E_q,b;E_k,\alpha) =\sum_{b'}\int dE_{q'}\,V(E_q,b;E_{q'},b')\chi(E_q',b';E_k,\alpha), \qquad (26a)$$

[13] See Ref. 11.

[14] H. E. Moses, Nuovo Cimento 1, 103 (1955).

[15] The Hamiltonians H, H_0, and H_1, the wave functions $\psi(t)$ and $u(E,i)$, and the amplitudes $T^{(\epsilon)}(a,b)$ and $\chi(a,b)$ refer to a general scattering problem here. These symbols, as used in Sec. III A, have no relation to the quantities represented by the same symbols in the rest of the paper, where they refer to electrodynamic quantities.

where

$$V(E_q,b;\,E_{q'},b')=\langle u(E_q,b)|H_1|u(E_{q'}b')\rangle. \tag{26b}$$

According to Eq. (26a), $\chi(E_q,b;\,E_k,\alpha)$ is presumably singular; if the integral on the right-hand side of Eq. (26a) is bounded, $\chi(E_q,b;\,E_k,\alpha)$ has a pole at $E_k=E_q$. We therefore take the multiple-valuedness of $(E_k-E_q)\times\chi(E_q,b;\,E_k,\alpha)$ into account by defining

$$\tilde{T}^{(\epsilon)}(E_q,b;\,E_k,\alpha)=(E_k-E_q)\chi(E_q,b;\,E_k,\alpha)\,, \tag{27a}$$

where $\tilde{T}^{(\epsilon)}(E_q,b;\,E_k,\alpha)$ obeys

$$\tilde{T}^{(\epsilon)}(E_q,b;\,E_k,\alpha)=\sum_{b'}\int dE_{q'}\,V(E_q,b;\,E_{q'},b')\times\{\tilde{T}^{(\epsilon)}(E_{q'},b';\,E_k,\alpha)[E_k-E_{q'}+i\epsilon]^{-1}+\lambda(E_{q'};\,b',\alpha)\delta(E_k-E_{q'})\}. \tag{27b}$$

Here we have invoked the wisdom of hindsight to choose $\tilde{T}^{(\epsilon)}(E_q,b;\,E_k,\alpha)$ to have a pole slightly displaced into the lower half-plane; the $\lambda(E_{q'};\,b',\alpha)\delta(E_k-E_{q'})$ has been added to compensate for the arbitrariness of the displacement of the pole in the integral and to add an arbitrary amount of homogeneous solution to $|\psi(t)\rangle$.

We can now write Eq. (24b) as

$$|\psi(t)\rangle=\sum_{b,\alpha}\int dE_q dE_k\,g(E_k,\alpha)\{\tilde{T}^{(\epsilon)}(E_q,b;\,E_k,\alpha)\times[E_k-E_q+i\epsilon]^{-1}+\lambda(E_q;\,b,\alpha)\delta(E_k-E_q)\}|u(E_q,b)\rangle\times\exp(-iE_kt); \tag{28a}$$

here orders of integration over E_q and E_k have been interchanged. We choose $\lambda(E_q;\,b,\alpha)$ so that $\sum_\alpha g(E_k,\alpha)\times\lambda(E_k;\,b,\alpha)=f(E_k,b)$, in order to specify the incident wave packet in terms of eigenstates of Q.

We also let

$$\sum_\alpha g(E_k,\alpha)\tilde{T}^{(\epsilon)}(E_q,b;\,E_k,\alpha)=\sum_a f(E_k,a)T^{(\epsilon)}(E_q,b;\,E_k,a).$$

Equation (28a) then reads

$$|\psi(t)\rangle=\sum_a\int dE_q\,f(E_q,a)|u(E_q,a)\rangle\times\exp(-iE_qt)+\sum_{a,b}\int dE_q dE_k\,f(E_k,a)T^{(\epsilon)}(E_q,b;\,E_k,a)\times[E_k-E_q+i\epsilon]^{-1}|u(E_q,b)\rangle\exp(-iE_kt). \tag{28b}$$

If we now wish to evaluate the probability amplitude of finding the system $|\psi(t)\rangle$ in the state defined by $|u(E_q,b)\rangle$, we compute the amplitude $\phi(t)=\langle u(E_q,b)|\psi(t)\rangle$. We can represent this quantity by

$$\phi(t)=\phi_i(t)+\phi_s(t)+\phi_v(t)\,, \tag{29a}$$

where $\phi_i(t)$, $\phi_s(t)$, and $\phi_v(t)$ are given by

$$\phi_i(t)=f(E_q,b)\exp(-iE_qt)\,,$$

$$\phi_s(t)=\sum_a f(E_q,a)T^{(\epsilon)}(E_q,b;\,E_q,a)\times\int dE_k\exp(-iE_kt)[E_k-E_q+i\epsilon]^{-1}, \tag{29b}$$

which may be rewritten as

$$\phi_s(t)=-2\pi i\sum_a f(E_q,a)T^{(\epsilon)}(E_q,b;\,E_q,a)\times\exp(-iE_qt)\theta(t) \tag{29c}$$

[where $\theta(t)$ is the step function $\theta(t)=1$ for $t>0$, $\theta(t)=0$ for $t<0$], and

$$\phi_v(t)=\sum_a\int dE_k[f(E_k,a)T^{(\epsilon)}(E_q,b;\,E_k,a)-f(E_q,a)T^{(\epsilon)}(E_q,b;\,E_q,a)]\exp(-iE_kt)\times[E_k-E_q+i\epsilon]^{-1}. \tag{29d}$$

Note that for the choice of $\lambda(E_q;\,b,a)$ we have made, Eq. (27b) becomes

$$T^{(\epsilon)}(E_q,b;\,E_k,a)=V(E_q,b;\,E_k,a)+\sum_{b'}\int dE_{q'}\,V(E_q,b;\,E_{q'},b')\times T^{(\epsilon)}(E_{q'},b';\,E_k,a)[E_k-E_{q'}+i\epsilon]^{-1}. \tag{27c}$$

If $\lim_{\epsilon\to0}T^{(\epsilon)}(E_{q'},b';\,E_k,a)$ is bounded and has a bounded derivative on the energy shell, in the interval over which the integral in Eq. (29d) extends, $\phi_v(t)$ vanishes in the limit $t\to\pm\infty$. The asymptotic limits of $\phi(t)$ then are $\phi_i(t)$ as $t\to-\infty$, and $\phi_i(t)+\phi_s(t)$ as $t\to+\infty$; thus $|\psi(t)\rangle$ describes scattering from (E_q,a) to the various (E_q,b) states. If $T^{(\epsilon)}(E_q,b;\,E_k,a)$ does not obey the previously specified requirements, however, then $\phi_v(t)$ does not vanish asymptotically, and the provisional surmise that we made, that the eigenstates of H_0 form the asymptotic states of the system, was mistaken. In this case, the application of artificial "switching" formalisms, that compel the system to approach the eigenstate of H_0 asymptotically, would be inconsistent with the dynamical laws as given by the time-independent Hamiltonian. It is simple to show that for potential scattering from well-behaved short-range potentials, the vanishing of $\phi_v(t)$ as $t\to\pm\infty$ is guaranteed. In field theories the transition amplitudes commonly violate the conditions required for the vanishing of $\phi_v(t)$; the wave function $|\psi(t)\rangle$ develops components other than the incident part, which persist at all times, even as $t\to-\infty$. These components augment the incident wave to "renormalize" it, i.e., to provide it with a cloud of virtual particles generated by the self-interactions of its constituent members. In Appendix A we will illustrate this effect in detail. In the following, Sec. III B, we will discuss the application of this procedure to QED.

B. Asymptotic States in QED in Different Gauges

In QED the complication inherent in disconnected vacuum-polarization bubbles prevents us from simply collecting the renormalized "incident wave" components from the time-dependent wave function (as can be done in the case of a static model). Nevertheless, we can invoke this procedure to clarify the relation between the asymptotic states of QED in the conventional formulation of the Lorentz gauge on one hand, and the reformulated Lorentz gauge on the other.

The interaction Hamiltonian H_1 in the conventional formulation of the Lorentz gauge may be written $H_1 = H_{1,T} + H_{Q,R} + [H_0, D]$.

If we consider a collision, for example, one between an electron and a photon, then among other matrix elements in the transition amplitude we find a sequence of terms in which $[H_0,D]$ is always the operative part of the interaction Hamiltonian. This sequence of terms gives rise to a contribution to the wave function which is given by

$$|\psi_D(t)\rangle = \int dE_i\, f(E_i)\exp(-iEit)\sum_n |n\rangle(E_i-E_n+i\epsilon)^{-1} \times\{\langle n|[H_0',D]|i\rangle+\langle n|[H_0',D][E_i-H_0'+i\epsilon]^{-1} \times[H_0',D]|i\rangle+\cdots+\langle n|[H_0',D][E_i-H_0'+i\epsilon]^{-1}\cdots [E_i-H_0'+i\epsilon]^{-1}[H_0',D]|i\rangle\}. \quad (30)$$

Since $[H_0',D]$ and D commute, and ignoring all terms proportional to $(i\epsilon)^n$, the transition amplitude to a state $|l\rangle$ is given by

$$\phi_l(t) = \int dE_i\, f(E_i)\exp(-iE_it)\langle l^\star|[-D+(2!)^{-1}D^2 -(3!)^{-1}D^3+\cdots+(-1)^n(n!)^{-1}D^n]|i\rangle. \quad (31)$$

Here, the various matrix elements $\langle l^\star|D|i\rangle$, $\langle l^\star|D^n|i\rangle$ contain a sufficient number of δ functions in momenta to eliminate the integration over the energy variable.[16] For example, if $|i\rangle$ is a state of an electron and a transverse photon $|e_{\mathbf{k}}\gamma_{\mathbf{p},t}\rangle$, and $|l\rangle$ is a state of an electron, a transverse and an "R"-type photon $|e_{k'}\gamma_{\mathbf{p}',t}\gamma_{\mathbf{q},R}\rangle$, then $\langle l^\star|D|i\rangle$ is given by

$$\langle l^\star|D|i\rangle = \bar{u}_{\mathbf{k}'}\gamma_4 u_{\mathbf{k}}\delta_{(\mathbf{k}-\mathbf{k}'),\mathbf{q}}\delta_{\mathbf{p}',\mathbf{p}}(2q^{3/2})^{-1};$$

the $\delta_{(\mathbf{k}-\mathbf{k}'),\mathbf{q}}\delta_{\mathbf{p}',\mathbf{p}}$ term absorbs the integration $\int dE_i$, and there is no further integration to which the Riemann-Lebesque lemma can apply. A similar condition obtains in the case of $\langle l^\star|D^n|i\rangle$. The effect of these δ functions is that the amplitude $\phi_l(t)$ persists at all times; in particular, it does not disappear even at $t\to-\infty$, and it combines with the original incident state to give the state

$$|\bar{\varphi}_i(t)\rangle = \sum_j e^{-D}|j\rangle f(E_j)\exp(-iE_jt). \quad (32)$$

There are, of course, many other contributions to the transition amplitude, among them many additional terms contributing to wave-function renormalization[17] (i.e., those terms which do not vanish as $t\to-\infty$). Among them are terms containing transverse photons, and other terms in which the Q photons produced by the $[H_0,D]$ part of H_1 are annihilated by $H_{Q,R}$ to give rise to wave-function renormalization terms which contribute Dalitz pairs to the dressed electron wave function; in addition, there are terms in which disconnected vacuum-polarization bubbles accompany other diagrams, and there are the previously ignored $O((i\epsilon)^n)$ terms.

All these contributions are dictated by the form of the Hamiltonian which is in this way "correcting" or renormalizing the assumed incident (asymptotic) wave function.

In the new formulation of the Lorentz gauge, that subset of terms contributing to the renormalization of the incident wave, in which $[H_0,D]$ is the only operative part of the Hamiltonian, has been collected in closed form at the very outset of the formulation of the scattering problem; the asymptotic states thus are $e^{-D}|i\rangle$.

It is important to understand that the form of the asymptotic states of the scattering system is not arbitrary. The Hamiltonian unambiguously determines it once the wave function at $t=0$ is known. If a wrong choice of asymptotic wave functions is made at the outset of a problem, the dynamical laws "notify" us of that fact by generating terms in the wave function which fail to vanish in the limit $t\to-\infty$ and augment, or renormalize, the "incident" wave by the so-called wave-function renormalization terms. The wave function $|\bar{\varphi}_n\rangle$ can therefore be understood in the following way: In the conventional formulation of the Lorentz gauge, the initially assumed form of asymptotic electron wave functions (i.e., eigenstates of H_0) are unaccompanied by any electromagnetic field. In the reformulated version of the Lorentz gauge, the assumed asymptotic electron wave functions (i.e., eigenstates of $\mathcal{H}_0$) are "dressed" in a cloud of longitudinal and timelike photons. This cloud does not, of course, represent the complete renormalized asymptotic electron state, since all transverse photons and the contributions of H_C are omitted from it. However, the wave functions $|\bar{\varphi}_n\rangle$ which satisfy Eq. (1a) do include a sum over a subset of "bare" electron and longitudinal and timelike

[16] See Appendix A.

[17] We will use the term "wave-function renormalization" to include all the multiparticle components of the incident state of the system.

photon states which form a part of the complete asymptotic electron states.

C. Scattering States and Maxwell's Equations

The Hamiltonian H, which generates the equations of motion of QED in the Lorentz gauge, gives rise to the wave equation $\Box A_\mu = -j_\mu$. The subsidiary condition $\partial_\mu A_\mu = 0$ makes this wave equation equivalent to the full set of Maxwell's equations. In Ref. 1 it was shown how the reformulated version of the Lorentz gauge guarantees the permanent validity of $\langle\bar{\psi}_{(t)}{}^\star|\partial_\mu A_\mu \times|\bar{\psi}_{(t)}\rangle = 0$, whereas this latter equation does not hold for $|\psi(t)\rangle$ in the conventional formulation of the Lorentz gauge. In the latter case the expectation values of the electromagnetic fields may not obey Maxwell's equations.

If we examine a scattering wave function $|\psi_i{}^{(\epsilon)}\rangle$ up to first order in electric charge, we find that, to this order,

$$|\psi_i{}^{(\epsilon)}\rangle^{(1)} = |\varphi_i\rangle + (E_i - H_0' + i\epsilon)^{-1} H_1' |\varphi_i\rangle,$$

and $\Omega^{(+)}(\mathbf{x})|\psi_i{}^{(\epsilon)}\rangle^{(1)}$ is given by

$$\Omega^{(+)}(\mathbf{x})|\psi_i{}^{(\epsilon)}\rangle^{(1)} = \Omega^{(+)}(\mathbf{x})|\varphi_i\rangle + \sum_j \Omega^{(+)}(\mathbf{x})|\varphi_j\rangle\langle\varphi_j|H_1'|\varphi_i\rangle[E_i - E_j + i\epsilon]^{-1}. \quad (33a)$$

For example, if $|\varphi_i\rangle$ is a one-electron state $|e_\mathrm{p}\rangle$ then, if we define $\xi_a = \Omega^{(+)}(\mathbf{x})|e_\mathrm{p}\rangle$, we have that

$$\xi_a = \tfrac{1}{2} i \sum_{\mathbf{k}} \frac{\rho(\mathbf{k})}{k} e^{i\mathbf{k}\cdot\mathbf{x}} |e_\mathrm{p}\rangle, \quad (33b)$$

and if we let ξ_b be given by

$$\xi_b = \sum_j \Omega^{(+)}(\mathbf{x})|\varphi_j\rangle\langle\varphi_j|H_1'|e_\mathrm{p}\rangle(E_i - E_j + i\epsilon)^{-1},$$

then we have that

$$\xi_b = -i(2k)^{-1} \sum_j |\varphi_j\rangle\langle\varphi_j|\rho(\mathbf{k})|e_p\rangle \times(\omega_p - E_j - k)(\omega_p - E_j - k + i\epsilon)^{-1}.$$

We see then that $\Omega^{(+)}(\mathbf{x})|\psi_i{}^{(\epsilon)}\rangle^{(1)}$ is given by

$$\Omega^{(+)}(\mathbf{x})|\psi_i{}^{(\epsilon)}\rangle^{(1)} = -\tfrac{1}{2}\epsilon \sum_{\mathbf{k},j} \frac{|\varphi_j\rangle\langle\varphi_j|\rho(\mathbf{k})|e_p\rangle}{k(\omega_p - E_j - k + i\epsilon)} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (33c)$$

which, to $O(\epsilon)$, does not vanish.

More generally, since we have shown that $e^{-D}|\hat{\psi}_i{}^{(\epsilon)}\rangle = |\bar{\psi}_i{}^{(\epsilon)}\rangle$, and $\Omega^{(+)}(\mathbf{x})|\bar{\psi}_i{}^{(\epsilon)}\rangle = 0$, we have, from Eq. (10), that

$$\Omega^{(+)}(\mathbf{x})|\psi_i{}^{(\epsilon)}\rangle = i\epsilon\Omega^{(+)}(\mathbf{x})(E_i - H + i\epsilon)^{-1} \times(1 - e^{-D})|\varphi_i\rangle, \quad (34a)$$

which in general does not vanish to $O(\epsilon)$.

The physical significance of the nonvanishing of $\Omega^{(+)}(\mathbf{x})|\psi_i{}^{(\epsilon)}\rangle$ can be understood in the following way: The validity of the subsidiary condition $\partial_\mu A_\mu = 0$ at $t=0$ implies

$$\langle\zeta^\star|\nabla\cdot\mathbf{A} - i\Pi_4|\zeta\rangle = 0, \quad (35a)$$

where $|\zeta\rangle$ is any state vector that satisfies Eq. (1a); this is equivalent to

$$i\sum_{\mathbf{k}} k^{1/2}\langle\zeta^\star|(\Omega^{(+)}(\mathbf{k}) - \Omega^{(+)}(-\mathbf{k})^\star)|\zeta\rangle e^{i\mathbf{k}\cdot\mathbf{x}} = 0. \quad (35b)$$

Similarly, the equation

$$\langle\zeta^\star|(\nabla\cdot\mathbf{E} - \rho)|\zeta\rangle = 0 \quad (36)$$

is equivalent to

$$\sum_{\mathbf{k}} k^{3/2}\langle\zeta^\star|(\Omega^{(+)}(\mathbf{k}) + \Omega^{(+)}(-\mathbf{k})^\star)|\zeta\rangle e^{i\mathbf{k}\cdot\mathbf{x}} = 0. \quad (34b)$$

Equation (35b) must be true for both $|\psi_i{}^{(\epsilon)}\rangle$ and $|\bar{\psi}_i{}^{(\epsilon)}\rangle$, since either subsidiary condition $\chi^{(+)}(\mathbf{x})|\varphi\rangle = 0$ or $\Omega^{(+)}(\mathbf{x})|\bar{\varphi}\rangle = 0$ guarantees it. The failure of $\Omega^{(+)}(\mathbf{x})|\psi_i{}^{(\epsilon)}\rangle = 0$, to $O(\epsilon)$, implies that, to that order, the expectation value of $\nabla\cdot\mathbf{E} - \rho = 0$ does not hold. Although in many cases (such as in the calculation of finite cross sections) such discrepancies to $O(\epsilon)$ will vanish in the limit $\epsilon \to 0$, this need not be the case for quantities for which the first order in ϵ is the leading term (as, for example, in the case of Z_2). In contrast to this, $\langle\phi^\star|(\nabla\cdot\mathbf{E} - \rho)|\phi\rangle = 0$ is always exactly obeyed by scattering states in the reformulated Lorentz gauge, even when the energy pole of the Green's function is displaced from the real axis.

It is of interest to compare the states $|\varphi_i\rangle$ and $|\bar{\varphi}_i\rangle$, the initially assumed asymptotic states (before wave-function renormalization) for the two versions of the Lorentz gauge, with respect to the expectation values of the electromagnetic fields which they imply. In the case of $|\varphi_i\rangle$ states, we have that

$$\sum_{\mathbf{k}} k^{3/2}\langle\varphi_2{}^\star|\chi^{(+)}(\mathbf{k}) + \chi^{(+)}(-\mathbf{k})^\star|\varphi_i\rangle = 0,$$

i.e., that

$$\langle\varphi_i{}^\star|\nabla\cdot\mathbf{E}|\varphi_i\rangle = 0.$$

In the case of $|\bar{\varphi}_i\rangle$ states, we have that

$$\langle\bar{\varphi}_i{}^\star|\nabla\cdot\mathbf{E} - \rho|\bar{\varphi}_i\rangle = 0.$$

In other words, in the conventional formulation of the Lorentz gauge, the $|\varphi_i\rangle$ electron states consist of electrons unaccompanied by their static field; in the new formulation, the $|\bar{\varphi}_i\rangle$ electron states consist of electrons that do carry their static electric field with them. Neither the $|\varphi_i\rangle$ nor the $|\bar{\varphi}_i\rangle$ electron states are accompanied by any magnetic fields, since in both, H_0 and $\mathcal{H}_0$, the photons are decoupled from the transverse currents. In both these cases the magnetic field "dressing" is left to subsequent wave-function renormalization by H_1 and $\mathcal{H}_1$, respectively.

IV. BOUND-STATE PROBLEM

In this section we will demonstrate that the radiative corrections to bound states in an external potential V are identical in the new formulation of the Lorentz gauge, and in the procedure that is invoked in making such calculations in the usual formulation of the Lorentz gauge,[18] provided that $[V,\rho(\mathbf{x})]=0$; this condition on V is in fact satisfied for large classes of potentials, including those equivalent to c-number potentials $[V=\int d\mathbf{x}\,\psi^{+}(\mathbf{x})\psi(\mathbf{x})V(\mathbf{x})$, as for example, a Coulomb potential$]$.

The bound-state problem in the reformulated Lorentz gauge can be specified by the equation

$$(\mathcal{H}_0+V+\mathcal{H}_1-\bar{E}_n)|\bar{\Psi}_n\rangle=0 \qquad (37)$$

subject to the constraint $\Omega^{(+)}(\mathbf{x})|\bar{\Psi}_n\rangle=0$. If we consider $|\bar{\Psi}_n\rangle$ to be generated iteratively, starting with $|\bar{\Phi}_n\rangle$, where the latter obey $(\mathcal{H}_0'+V-E_n^{(0)})|\bar{\Phi}_n\rangle=0$, then it is clear that the constraint imposed by the subsidiary condition $\Omega^{(+)}(\mathbf{x})|\bar{\Phi}\rangle=0$ never needs to be applied, since $\mathcal{H}_1'$ never connects states that satisfy it with states that do not.

The bound-state problem in the conventional Lorentz gauge can be specified by $(H+V-E_n)|\Psi_n\rangle=0$, subject to the constraint $\chi^{(+)}|\Psi_n\rangle=0$. In this case, if we solve the problem iteratively, starting with $|\Phi_n\rangle$ given by $(H_0'+V-E_n^{(0)})|\Phi_n\rangle=0$, we would quickly find that the higher-order corrections fail to obey the constraint imposed by this subsidiary condition. If we were to take this constraint seriously, we would have to project out the part of the iterated wave function that violates it.

We might do this by defining a projection operator α in the Fock space of noninteracting photons and electrons (moving in the external potential V) so that α projects out those n-particle states that include unphysical photons.[19] We would then solve the eigenvalue problem

$$(H_0'+V+\alpha H_1\alpha-(E_n)_\alpha)|\Psi_n\rangle_{(\alpha)}=0 \qquad (38)$$

by an iteration procedure, and if we start with $|\Phi_n\rangle$ as before, we would be assured that the constraint $\chi^{(+)}|\Psi_n\rangle_{(\alpha)}=0$ would be obeyed. The question could then be raised whether Eqs. (37) and (38) give identical solutions; and because of the fact that $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$ are constrained to occupy different parts of the indefinite metric space, we might discover that indeed they do not. Equation (38) is not, however, the one which has been used to evaluate radiative corrections to bound states in QED.[20] In actual practice, the constraint imposed by a subsidiary condition has always been ignored in these calculations, so that the wave function has always been allowed to spread into any part of the space that the dynamics of the problem $(H_0+V+H_1-E_n)|\Psi_n\rangle=0$ dictated. The usual computational practice can be illustrated by choosing the unperturbed state vector to be a one-electron state $|\mathcal{E}_{(0)}\rangle$, which solves the problem $(H_0'+V-E_0^{(0)})|\mathcal{E}_{(0)}\rangle=0$; then the first-order correction to $|\mathcal{E}_{(0)}\rangle$ is given by

$$|\mathcal{E}_{(1)}\rangle=\sum_l{}'\,|\Phi_l\rangle\langle\Phi_l|H_1'|\mathcal{E}_{(0)}\rangle(E_l^{(0)}-E_0^{(0)})^{-1} \qquad (39)$$

(where $\sum_l{}'$ indicates summation over all l except $l=0$) and the set $|\Phi_l\rangle$ consists of states containing some Q-type photons which violate $\chi^{(+)}(\mathbf{x})|\Phi_l\rangle=0$. These are not, however, projected out, but are admitted as basis vectors for the exact solution to the problem.

The consequence of the aforegoing circumstances is that we must compare the bound-state problem in the two versions of the Lorentz gauge when *no* subsidiary constraints are imposed on the wave functions: in the case of the reformulated version because the theory implies independence of the computational procedure from the constraints, and in the case of the traditional formulation because the actual practice has been to ignore them.

The expression for the iterative expansion of the bound-state wave function $|\Psi_n\rangle$ in the usual formulation of the Lorentz gauge can be written[21]

$$|\Psi_n\rangle=|\Phi_n\rangle+(1-|\Phi_n\rangle\langle\Phi_n|)\times(H_0'+V-E_n^{(0)})^{-1}(\Delta_n-H_1')|\Psi_n\rangle, \qquad (40)$$

where

$$\Delta_n=E_n-E_n^{(0)}=\langle\Phi_n|H_1'|\Psi_n\rangle(\langle\Phi_n|\Psi_n\rangle)^{-1},$$
$$(H_0+V+H_1-E_n)|\Psi_n\rangle=0,$$
$$(H_0'+V-E_n^{(0)})|\Phi_n\rangle=0.$$

In the reformulated Lorentz gauge, the corresponding wave function can be written

$$|\hat{\Psi}_n\rangle=|\Phi_n\rangle+(1-|\Phi_n\rangle\langle\Phi_n|)\times(H_0'+V-E_n^{(0)})^{-1}(\bar{\Delta}n-\hat{H}_1')|\hat{\Psi}_n\rangle, \qquad (41a)$$

where

$$\bar{\Delta}_n=\hat{E}_n-E_n^{(0)}.$$

The use of Eq. (6) leads to

$$|\hat{\Psi}_n\rangle=|\Phi_n\rangle+(1-|\Phi_n\rangle\langle\Phi_n|)\times(H_0'+V-E_n^{(0)})^{-1}(\bar{\Delta}_n-H_1')e^{-D}|\hat{\Psi}_n\rangle+(1-|\Phi_n\rangle\langle\Phi_n|)(1-e^{-D})|\hat{\Psi}_n\rangle. \qquad (41b)$$

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[18] This result supercedes an earlier report on this topic by the authors.

[19] Because the space underlying this theory is an indefinite metric space, care must be taken not to misunderstand the terminology describing the zero-norm particle states. The subsidiary condition $\chi^{(+)}(\mathbf{x})|\varphi\rangle=0$ is equivalent to $a_{\mathbf{k},Q}|\varphi\rangle=0$. State vectors violate this condition when they include any $a^\dagger_{\mathbf{k},Q}$ operators operating to their right, as for example $a^\dagger_{\mathbf{k},Q}|\varphi\rangle$ with $|\varphi\rangle$ consisting of bare electrons and transverse photons. Such "forbidden" state vectors, however, correspond to a probability amplitude for observing R-type rather than Q-type photons, since $\langle\varphi_a^\star|\varphi_b\rangle=1$ if $|\varphi_b\rangle=a^\dagger_{\mathbf{k},Q}|\varphi\rangle$ and $|\varphi_a\rangle=a^\dagger_{\mathbf{k},R}|\varphi\rangle$. The convention we adopt is that we refer to the forbidden photon as Q type, referring to the operator structure of the ket vector, rather than to the designation of the nonvanishing amplitude in the indefinite metric space.

[20] See, for example, R. W. Mills and N. M. Kroll, Phys. Rev. **98**, 1489 (1955); G. W. Erickson and D. R. Yennie, Ann. Phys. (N.Y.) **35**, 271 (1965).

[21] K. Gottfried, in *Quantum Mechanics* (Benjamin, New York, 1966), Chap. 45.

Since $\langle\Phi_n|(1-e^{-D})|\hat{\Psi}_n\rangle=0$ for $|\Phi_n\rangle$ in the physical space, Eq. (41b) can be rewritten

$$e^{-D}|\hat{\Psi}_n\rangle=|\Phi_n\rangle+(1-|\Phi_n\rangle\langle\Phi_n|)\times(H_0'+V-E_n^{(0)})^{-1}(\bar{\Delta}_n-H_1')e^{-D}|\hat{\Psi}_n\rangle. \quad (42)$$

$\bar{\Delta}_n$ can be shown to be

$$\bar{\Delta}_n=\langle\Phi_n|\hat{H}_1'|\hat{\Psi}_n\rangle(\langle\Phi_n|\hat{\Psi}_n\rangle)^{-1}=\langle\Phi_n|H_1'e^{-D}|\hat{\Psi}_n\rangle(\langle\Phi_n|e^{-D}\hat{\Psi}_n\rangle)^{-1};$$

it follows[22] therefore that for the iterative solutions $e^{-D}|\hat{\Psi}_n\rangle=|\Psi_n\rangle$ and that $\bar{\Delta}_n=\Delta_n$.

V. DISCUSSION

One conclusion that we may draw is that the subsidiary condition $\chi^{(+)}(\mathbf{x})|\varphi\rangle=0$ leads to inconsistencies, and from the logical point of view is entirely unsuitable as a basis for QED in the Lorentz gauge. Nevertheless, the theory is remarkably resistant to mistakes that might stem from the substitution of $\chi^{(+)}(\mathbf{x})|\varphi\rangle=0$ for the correct $\Omega^{(+)}(\mathbf{x})|\bar{\varphi}\rangle=0$. In Sec. IV we have shown that bound-state wave functions, which are iterated from an unperturbed state vector obeying $\chi^{(+)}(\mathbf{x})|\Phi\rangle=0$, develop corrections that violate this subsidiary condition. Within the context of the "old" Lorentz-gauge theory this would of course be a paradox, and if the subsidiary condition were taken seriously, one would be obliged to project the offending terms out of the solution in each order. When this is not done (it is, in fact, not done in actual practice) the iterative solution "repairs" itself, and instead of obeying $\chi^{(+)}(\mathbf{x})|\Psi\rangle=0$, it obeys the new subsidiary condition $\Omega^{(+)}(\mathbf{x})|\hat{\Psi}\rangle=0$. Thus, the dynamics forces the wave function and the energy shift to obey the correct subsidiary condition even in spite of our failure to compel it to do so.

The resistance of QED to mistakes resulting from the use of the wrong subsidiary condition $\chi^{(+)}(\mathbf{x})|\varphi\rangle=0$ is not, however, absolute. In the case of the energy continuum the situation is somewhat similar to the bound-state problem, but with the important exception that in this case there are discrepancies between $|\psi^{(\epsilon)}(E_i)\rangle$ and $|\bar{\psi}^{(\epsilon)}(E_i)\rangle$ of order $i\epsilon$. Here, too, the scattering wave function repairs itself to an extent after the wrong subsidiary condition $\chi^{(+)}(\mathbf{x})|\varphi\rangle=0$ has been used; but the repair is not complete and the remaining discrepancy between $|\psi^{(\epsilon)}(E_i)\rangle$ and $|\bar{\psi}^{(\epsilon)}(E_i)\rangle$ accounts for the fact that the former of these wave functions leads to an electron wave-function renormalization constant $(Z_2)_L$, different from $(Z_2)_C$. We believe that this discrepancy can be understood as a consequence of the fact that in the definition of $(Z_2)_L$, the Green's-function poles of propagators (in the old Lorentz-gauge formulation) are displaced into a region in which the theory develops inconsistencies and in which Maxwell's equations do not hold. If the Lorentz-gauge problem is properly treated, Z_2 is identical in the Lorentz and Coulomb gauges.

[22] Strictly speaking, the proof is valid for a true bound state, in which the effect of the perturbation is to displace an energy pole on the left-hand energy axis. In QED, the stable 1*s* hydrogenic state marks the onset of a cut and the 2*s* and 2*p* states are resonances in the energy continuum. We therefore must understand the perturbative procedure in the sense of F. E. Low, Phys. Rev. **88**, 53 (1952), and the theorem proven by Eq. (42) is not, strictly speaking, applicable to QED. The discrepancy between $(Z_2)_C$ and $(Z_2)_L$ should persist, for example.

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APPENDIX A

In the appendix we will clarify the procedure (used in Sec. III) by which we identified the wave-function renormalization terms from the persistent asymptotic parts of the scattering wave function; we will do this first by using a static model as an illustrative example, and then by applying the method directly to the wave functions described in Sec. III.

In the static model we will couple an isovector boson to an isospinor static source; we have that

$$H=\bar{H}_0+\bar{H}_1=H_0+H_1, \quad (\text{A1})$$

where

$$\bar{H}_0=\sum_{\mathbf{k},\alpha}a^\dagger_{\mathbf{k},\alpha}a_{\mathbf{k},\alpha}\omega_k+M_0,$$

$$\bar{H}_1=\sum_{\mathbf{k},\alpha}(a_{\mathbf{k},\alpha}V_{\mathbf{k},\alpha}+a^\dagger_{\mathbf{k},\alpha}V^\dagger_{\mathbf{k},\alpha}),$$

where $V_{\mathbf{k},\alpha}=g_0(2\omega_k)^{1-/2}\tau_\alpha U(k)$ and α designates the isospin. H_0 and H_1 are given by $H_0=\bar{H}_0+\delta M$ and $H_1=\bar{H}_1-\delta M$, where δM is the self-mass of the static source. The spectrum of eigenstates of H_0 consists of the bare source itself, and of superpositions of free bosons and the static source; we will designate these states as $|N\rangle$ and $|\mathbf{k}_1,\cdots,\mathbf{k}_i;N\rangle=(i!)^{-1/2}\times a_{\mathbf{k}(1)}{}^\dagger\cdots a^\dagger{}_{\mathbf{k}(i)}|N\rangle$, respectively; we will consistently suppress the isospin index in our notation.

For the case of boson scattering by the static source in this model, we evaluate the wave function $|\psi(t)\rangle$, given by

$$|\psi(t)\rangle=\int d\mathbf{p}\,f(\mathbf{p},\mathbf{p}_0)\exp(-i\omega_pt)|\mathbf{p};N\rangle+\int d\mathbf{k}\,f(\mathbf{k},\mathbf{p}_0)\Big\{T^{(\epsilon)}(0;\mathbf{k})(\omega_k+i\epsilon)^{-1}|N\rangle+\int d\mathbf{q}\,T^{(\epsilon)}(\mathbf{q};\mathbf{k})(\omega_k-\omega_q+i\epsilon)^{-1}|\mathbf{q};N\rangle+\int d\mathbf{q}_1d\mathbf{q}_2\,T^{(\epsilon)}(\mathbf{q}_1,\mathbf{q}_2;\mathbf{k})(\omega_k-\omega_{q(1)}-\omega_{q(2)}+i\epsilon)^{-1}\times|\mathbf{q}_1,\mathbf{q}_2;N\rangle+\cdots+\int d\mathbf{q}_1d\mathbf{q}_2\cdots d\mathbf{q}_n\,T^{(\epsilon)}(\mathbf{q}_1,\mathbf{q}_2\cdots,\mathbf{q}_n;\mathbf{k})\times(\omega_k-\omega_{q(1)}-\omega_{q(2)}-\cdots-\omega_{q(n)}+i\epsilon)^{-1}\times|\mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_n;N\rangle\Big\}\exp(-i\omega_kt). \quad (\text{A2})$$

Here $f(\mathbf{q},\mathbf{p}_0)$ is a spectral wave packet {for example, $f(\mathbf{q},\mathbf{p}_0)=\exp[(-|\mathbf{q}-\mathbf{p}_0|^2)L^2]$, where L is the coherence length of the projectile particle}; $T^{(\epsilon)}(j;\mathbf{k})$ is given by $T^{(\epsilon)}(j;\mathbf{k})=\langle j|T^{(\epsilon)}(E_k)|\mathbf{k}\rangle$, where $T^{(\epsilon)}(E_k)=H_1+H_1\times(E_k-H+i\epsilon)^{-1}H_1$. The ket $|\psi(t)\rangle$ is a special case of the wave function appearing in Eq. (28b) and is used to represent scattering from the initial state $|\mathbf{p}_0;N\rangle$ [suitably smeared out with the packet function $f(\mathbf{q},\mathbf{p}_0)$] to the set of final states $|N\rangle,\cdots,|\mathbf{k}_1,\cdots,\mathbf{k}_n;N\rangle$.

The probability amplitude for finding the system in a state at time t is given by

$$\phi(\mathbf{q}_1,\cdots,\mathbf{q}_n;t)=\langle N;\mathbf{q}_1,\cdots,\mathbf{q}_n|\psi(t)\rangle.$$

We can now systematically examine the various $\phi(\mathbf{q}_1,\cdots,\mathbf{q}_n;t)$ (characteristic of the various multiboson states) to establish what their asymptotic temporal limits are.

In the case of $\phi(0;t)=\langle N|\psi(t)\rangle$ we have that $T_{(1)}{}^{(\epsilon)}(0;\mathbf{k})$, the first-order contribution to $T^{(\epsilon)}(0;\mathbf{k})$, is given by

$$T_{(1)}{}^{(\epsilon)}(0;\mathbf{k})=\langle N|H_1|\mathbf{k};N\rangle=V_{\mathbf{k}} \tag{A3}$$

and

$$\phi_{(1)}(0;t)=\int d\mathbf{k}\, f(\mathbf{k},\mathbf{p}_0)V_{\mathbf{k}}(\omega_k+i\epsilon)^{-1}\exp(-i\omega_k t).$$

In the limit $t\to\pm\infty$, $\phi_{(1)}(0;t)$ vanishes by the Riemann-Lebesgue lemma since the coefficient of $\exp(-i\omega_k t)$ in the integrand is bounded.

In the case of $\phi(\mathbf{q};t)$, we can express $T^{(\epsilon)}(\mathbf{q};\mathbf{k})$ by separating it into a part $[T^{(\epsilon)}(\mathbf{q};\mathbf{k})]_s$ which describes scattering and a part $[T^{(\epsilon)}(\mathbf{q};\mathbf{k})]_r$ which represents a renormalization effect by writing

$$T^{(\epsilon)}(\mathbf{q};\mathbf{k})=[T^{(\epsilon)}(\mathbf{q};\mathbf{k})]_s+[T^{(\epsilon)}(\mathbf{q};\mathbf{k})]_r.$$

To lowest (second) order in g_0, the two separate parts are given by

$$[T_{(2)}{}^{(\epsilon)}(\mathbf{q};\mathbf{k})]_s=(V_{\mathbf{k}}V_{\mathbf{q}}-V_{\mathbf{q}}V_{\mathbf{k}})(\omega_k)^{-1} \tag{A4}$$

and

$$[T_{(2)}{}^{(\epsilon)}(\mathbf{q};\mathbf{k})]_r =\left\{(2\pi)^{-3}\int d\boldsymbol{\kappa}\,|V_\kappa|^2(-\omega_\kappa+i\epsilon)^{-1}-\delta M^{(2)}\right\}\delta_{\mathbf{q},\mathbf{k}}. \tag{A5}$$

We note that the contribution of $[T_{(2)}{}^{(\epsilon)}(\mathbf{q};\mathbf{k})]_s$ to $\phi_{(2)}(\mathbf{q};t)$ vanishes as $t\to-\infty$ due to the operation of the Riemann-Lebesgue lemma; $[T_{(2)}{}^{(\epsilon)}(\mathbf{q};\mathbf{k})]_r$, however, contributes a part

$$[\phi_{(2)}(\mathbf{q};t)]_r=-f(\mathbf{q},\mathbf{p}_0)(2\pi)^{-3}\int d\boldsymbol{\kappa}\,|V_\kappa|^2[\omega_\kappa(\omega_\kappa-i\epsilon)]^{-1}\times\exp(-i\omega_q t),$$

which persists as $t\to-\infty$ since the $\delta_{\mathbf{q},\mathbf{k}}$ in Eq. (A5) eliminates the integration in which the Riemann-Lebesgue lemma would force the integral to vanish. $[\phi_{(2)}(\mathbf{q};t)]_r$ therefore augments, or "renormalizes", the assumed incident wave at $t\to-\infty$ to form an incident wave for which the $\lim_{t\to-\infty}\phi(\mathbf{q};t)$, up to this order, is given by

$$\lim_{t\to-\infty}\phi_{(0+2)}(\mathbf{q};t)=\int d\mathbf{p}\,f(\mathbf{p},\mathbf{p}_0)\times\left[1-(2\pi)^{-3}\int d\boldsymbol{\kappa}\,|V_\kappa|^2(\omega_\kappa)^{-2}\right]\exp(-i\omega_p t). \tag{A6}$$

In the case of $\phi(\mathbf{q}_1,\mathbf{q}_2;t)$, we have that $T^{(\epsilon)}(\mathbf{q}_1,\mathbf{q}_2;\mathbf{k})$ to lowest (first) order is $\langle N;\mathbf{q}_1,\mathbf{q}_2|H_1|\mathbf{k};N\rangle=(\delta_{\mathbf{q}(1),\mathbf{k}}V_{\mathbf{q}(2)}+\delta_{\mathbf{q}(2),\mathbf{k}}V_{\mathbf{q}(1)})\sqrt{2}^{-1}$, and this leads to

$$\phi_{(1)}(\mathbf{q}_1,\mathbf{q}_2;t)=-\sqrt{2}^{-1}[f(\mathbf{q}_1,\mathbf{p}_0)V_{\mathbf{q}(2)}(\omega_{\mathbf{q}(2)}-i\epsilon)^{-1}\times\exp(-i\omega_{q(1)}t)+f(\mathbf{q}_2,\mathbf{p}_0)V_{\mathbf{q}(1)}(\omega_{q(1)}-i\epsilon)^{-1}\times\exp(-i\omega_{q(2)}t)], \tag{A7}$$

which also persists at all times and augments the incident wave by contributing to the one-boson component of the "virtual boson cloud" of the physical (dressed) source.

It is easy to reconstruct the entire renormalized incident state by systematically collecting all the parts of the wave function $\psi(t)$ which, in the limit as $t\to-\infty$, are not forced to vanish by the Riemann-Lebesgue lemma. These parts all originate from components of the amplitude $\langle i|T^{(\epsilon)}(M+\omega_k)|\mathbf{k};N\rangle$ in which $a_{\mathbf{k}}{}^\dagger$ commutes past $T^{(\epsilon)}(M+\omega_k)$ and acts on $\langle i|$ to give $\delta_{\mathbf{q},\mathbf{k}}\langle j|$, where $\langle i|=\langle j|a_{\mathbf{q}}$. Since we have that

$$[H_1+H_1(M+\omega_k-H+i\epsilon)^{-1}H_1]a^\dagger_{\mathbf{k}} =a_{\mathbf{k}}{}^\dagger[H_1+H_1(M-H+i\epsilon)^{-1}H_1] +(\text{terms in which } a^\dagger_{\mathbf{k}} \text{ no longer appears}),$$

we can write that the limiting value of the one-boson amplitude $\phi(\mathbf{q};t)$ is given by

$$\lim_{t\to-\infty}\phi(\mathbf{q};t)=f(\mathbf{q},\mathbf{p}_0)(i\epsilon)^{-1}\langle N|T^{(\epsilon)}(M)|N\rangle\times\exp(-i\omega_q t). \tag{A8}$$

Similarly, one can compute the probability amplitude of finding n bosons and the bare source in the scattering wave function $\psi(t)$ as $t\to-\infty$. In that case one has

$$\lim_{t\to-\infty}\phi(\mathbf{q}_1,\ldots,\mathbf{q}_n,t) =\int d\mathbf{k}\,\langle N;\mathbf{q}_1,\ldots,\mathbf{q}_n|T^{(\epsilon)}(M+\omega_k)|\mathbf{k};N\rangle \times(\omega_k-\omega_{q(1)}-\cdots-\omega_{q(n)}+i\epsilon)^{-1}f(\mathbf{k},\mathbf{p}_0) \times\exp(-i\omega_k t) \tag{A9}$$

and

$$\lim_{t\to-\infty}\phi(\mathbf{q}_1,\ldots,\mathbf{q}_{n-1},\mathbf{q};t) =\sum f(\mathbf{q},\mathbf{p}_0)(n)^{-1/2}\langle N;\mathbf{q}_1,\ldots,\mathbf{q}_{n-1}|(M-H_0+i\epsilon)^{-1} \times T^{(\epsilon)}(M)|N\rangle\exp(-i\omega_q t), \tag{A10}$$

where $\sum$ indicates summation over the n terms in which $\mathbf{q}_i$'s are permuted with $\mathbf{q}$.

Summing the contributions to $|\psi(t)\rangle$ as $t\to-\infty$ over the entire spectrum of boson states yields the following expression for $|\psi_s(t)\rangle$, the scattered wave:

$$\lim_{t\to-\infty}|\psi_s(t)\rangle$$

$$=\sum_{n=1}^{\infty}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n} f(\mathbf{q}_n,\mathbf{p}_0)a^\dagger{}_{\mathbf{q}(n)}|\mathbf{q}_1,\ldots,\mathbf{q}_{n-1};N\rangle$$

$$\times\langle N;\mathbf{q}_1,\ldots,\mathbf{q}_{n-1}|(M-H_0+i\epsilon)^{-1}T^{(\epsilon)}(M)|N\rangle$$

$$\times\exp(-i\omega_{q(n)}t) \quad \text{(A11a)}$$

$$=\sum_{\mathbf{q}} f(\mathbf{q},\mathbf{p}_0)a^\dagger{}_{\mathbf{q}}(M-H_0+i\epsilon)^{-1}T^{(\epsilon)}(M)|N\rangle$$

$$\times\exp(-i\omega_q t).$$

The "dressed" source state $|\mathfrak{N}\rangle$, for which $(H-M)\times|\mathfrak{N}\rangle=0$, can be represented by[23]

$$|\mathfrak{N}\rangle=\lim_{\epsilon\to0}|\mathfrak{N};\epsilon\rangle, \quad \text{(A11a')}$$

where

$$|\mathfrak{N};\epsilon\rangle=[\mathfrak{z}(\epsilon)]^{-1/2}[1+(M-H+i\epsilon)^{-1}H_1]|N\rangle \quad \text{(A11b)}$$

and

$$\lim_{\epsilon\to0}\mathfrak{z}(\epsilon)=Z_2. \quad \text{(A11c)}$$

From Eq. (A11b) it follows that

$$(M-H_0+i\epsilon)^{-1}T^{(\epsilon)}(M)|N\rangle$$
$$=[\mathfrak{z}(\epsilon)]^{1/2}|\mathfrak{N}_{;\epsilon}\rangle-|N\rangle. \quad \text{(A12)}$$

This result establishes that the residual "scattered" wave which persists as $t\to-\infty$ is given by

$$\lim_{t\to-\infty}|\psi_s(t)\rangle=\sum_{\mathbf{q}} f(\mathbf{q},\mathbf{p}_0)a^\dagger{}_{\mathbf{q}}[(Z_2)^{1/2}|\mathfrak{N}\rangle-|N\rangle],$$

and combining this with the originally assumed incident wave leads us to the expression for the limiting form for the entire wave function,

$$\lim_{t\to-\infty}|\psi(t)\rangle$$
$$=(Z_2)^{1/2}\sum_{\mathbf{q}} f(\mathbf{q},\mathbf{p}_0)a^\dagger{}_{\mathbf{q}}|\mathfrak{N}\rangle\exp(-i\omega_q t). \quad \text{(A13)}$$

In Sec. III, Eqs. (30) and (31) show the wave function generated from the assumed incident state of electrons and transverse photons by the interaction term $[H_0,D]$. $[H_0,D]$ can only connect a state $\langle\varphi_A{}^\star|a_{\mathbf{k},Q}$ with a state $|\varphi_B\rangle$ since $[H_0,D]$ can create a Q-type photon or annihilate an R-type photon. However, R-type photons can never occur when $[H_0,D]$'s operate on electron–transverse-photon states, so that only the $\frac{1}{2}\sum_{\mathbf{k}} a^\dagger{}_{\mathbf{k},Q}\times\rho(\mathbf{k})k^{-3/2}$ part of D ever operates. The incident transverse photons therefore emerge unscattered in such Feynman graphs, and the resulting δ functions obliterate integrations in which the Riemann-Lebesgue lemma might operate. For example, for $\phi_R(t)=\langle e_{\mathbf{k}'}\gamma_{\mathbf{q}',t}\gamma_{\mathbf{p}',R}{}^\star\times|\psi(t)\rangle=\langle p_{\mathbf{k}'}\gamma_{\mathbf{q}',t}|a_{\mathbf{p}',Q}|\psi(t)\rangle$, where $\psi(t)$ originates from Compton scattering, we have for lowest (second) order

$$\phi_{(2),R}(t)=\int d\mathbf{q}d\mathbf{k}\,\langle e_{\mathbf{k}'}\gamma_{\mathbf{q}',t}|a_{\mathbf{p}',Q}[H_0,D]|\gamma_{\mathbf{q},t}e_{\mathbf{k}}\rangle$$

$$\times(\omega_k+q-\omega_{k'}-q'-p'+i\epsilon)^{-1}f(\mathbf{k},\mathbf{k}_0)$$

$$F(\mathbf{q},\mathbf{q}_0)\exp[-i(\omega_k+q)t], \quad \text{(A14)}$$

where $f(\mathbf{k},\mathbf{k}_0)$ and $F(\mathbf{q},\mathbf{q}_0)$ are spectral packets for the incident electron and the incident transverse photon, respectively. It is easy to see that

$$\lim_{\epsilon\to0}\langle e_{\mathbf{k}'}\gamma_{\mathbf{q}',t}|a_{\mathbf{p}',Q}[H_0,D]|\gamma_{\mathbf{q},t}e_{\mathbf{k}}\rangle$$
$$\times(\omega_k+q-\omega_{k'}-q'-p'+i\epsilon)^{-1}$$
$$=\bar{u}_{\mathbf{k}'}\gamma_4 u_{(\mathbf{k}'+\mathbf{p}')}(2p^{3/2})^{-1}\delta_{(\mathbf{k}-\mathbf{k}'),\mathbf{p}}\delta_{\mathbf{q},\mathbf{q}'},$$

so that

$$\phi_{(2),R}(t)=\bar{u}_{\mathbf{k}'}\gamma_4 u_{(\mathbf{k}'+\mathbf{p}')}(2p^{3/2})^{-1}f(\mathbf{k}'+\mathbf{q}',\mathbf{k}_0)$$
$$\times F(\mathbf{q}',\mathbf{q}_0)\exp[-i(\omega_{|\mathbf{k}'+\mathbf{p}'|}+q')t]. \quad \text{(A15)}$$

Here again, the δ functions have absorbed all integrations leaving an oscillatory time dependence which does not vanish at $t\to-\infty$; the corresponding state vector augments or renormalizes the assumed incident wave as in the previous example and as discussed in Sec. III. We note that the identical effect occurs for $[H_0,D]^n$ for arbitrary n.

APPENDIX B

In this appendix we will evaluate the lowest-order contributions to Z_2 in various gauges to illustrate the validity of the results presented in Sec. II C. For this purpose we will here use the usual covariant definition of the self-energy terms and the covariant definition of Z_2, although we will integrate the infinite integrals to noninvariant cutoffs.

The second-order self-energy in the Coulomb gauge includes contributions from transverse photons and from the H_C interaction Hamiltonian. The lowest-order Z_2 stems from transverse photons entirely and the contribution to $\Sigma_T{}^{(2)}(p_\mu)$ from transverse photons is given by

$$\Sigma_T{}^{(2)}(p_\mu)=\frac{ie^2}{(2\pi)^4}\int\frac{d_4k}{k^2}\gamma\cdot\epsilon$$
$$\times\frac{i\gamma\cdot(p-k)-m}{m^2+(p-k)^2}\gamma\cdot\epsilon-\delta m_T{}^{(2)}, \quad \text{(B1a)}$$

where $\Sigma_T{}^{(2)}$ is the second-order self-energy correction to an electron of momentum p_μ due to transverse photons, and where δm_t is the part of the self-energy due to trans-

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[23] B. S. DeWitt, in *Quantum Mechanics* (Benjamin, New York, 1966), Chap. 10.

verse photons. We will express this quantity in the rest frame of the electron and rewrite Eq. (B1a) as

$$\Sigma_T^{(2)}(p_\mu)=\frac{-2ie^2}{(2\pi)^4}\times\int_0^1 dx\left\{\int d^4k\frac{m+i\gamma\cdot p(1-x)}{[k^2+m^2x+p^2x(1-x)]^2}-\tfrac{1}{2}\pi^2\gamma\cdot px\right\}-\delta m_T^{(2)}, \quad \text{(B1b)}$$

where p is the vector $(0,0,0,ip_0)$. We will evaluate the integrals by defining

$$\int d^4k=\int d\Omega\int_\lambda^L k^2dk\int_{-\infty}^{+\infty}dk_0,$$

where $\int dk_0$ is performed over the Feynman contour. This leads to

$$\Sigma_T^{(2)}(p_\mu)=(\alpha/2\pi)(m+i\gamma\cdot p)[\ln(2L/m)-\tfrac{1}{4}]. \quad \text{(B2)}$$

For the case of the Lorentz gauge, $\Sigma_L^{(2)}(p_\mu)$ is given by

$$\Sigma_L^{(2)}(p_\mu)=\frac{ie^2}{(2\pi)^4}\int\frac{d^4k}{k^2}\gamma_\mu\frac{i\gamma\cdot(p-k)-m}{m^2+(p-k)^2}\gamma_\mu-\delta m \quad \text{(B3a)}$$

and this can be rewritten as

$$\Sigma_L^{(2)}(p_\mu)=\frac{-ie^2}{(2\pi)^4}\int dx\times\left\{\int d^4k\frac{2m+i\gamma\cdot p(1-x)}{[k^2+m^2x+p^2x(1-x)]^2}-\tfrac{1}{2}\pi^2\gamma\cdot px\right\}-\delta m. \quad \text{(B3b)}$$

The integrals are evaluated as before, leading to

$$\Sigma_L^{(2)}(p_\mu)=(\alpha/2\pi)(m+i\gamma\cdot p)\times[(\ln(2L/m)+7/4+2\ln(2\lambda/m)]. \quad \text{(B4)}$$

The preceding equations lead to the following expressions for the lowest-order terms in Z_2:

$$(\mathfrak{z}_2)_C=1-(\alpha/2\pi)[\ln(2L/m)-\tfrac{1}{4}], \quad \text{(B5a)}$$

$$(\mathfrak{z}_2)_L=1-(\alpha/2\pi)[\ln(2L/m)+7/4+2\ln(2\lambda/m)]. \quad \text{(B5b)}$$

The difference between $(\mathfrak{z}_2)_C$ and $(\mathfrak{z}_2)_L$ is given by

$$\Delta\mathfrak{z}_2=(\mathfrak{z}_2)_L-(\mathfrak{z}_2)_C=-(\alpha/\pi)[\ln(2\lambda/m)+1]. \quad \text{(B6)}$$

This shows that $\Delta\mathfrak{z}_2\neq 0$, and, moreover, that

$$\lim_{\lambda\to 0}[\Delta\mathfrak{z}_2/(\mathfrak{z}_2)_C]\neq 0.$$

It is the latter equation which is significant, since in an expression for a divergent renormalization constant, the finite parts are dependent on the details of the method of calculation and not significant to the final result.[24]

The quantity $(\bar{\mathfrak{z}}_2)_L$ is given by

$$(\bar{\mathfrak{z}}_2)_L=(\mathfrak{z}_2)_L-\lim_{\epsilon\to 0}\tau_{nt}^{(\epsilon)}(E_p),$$

where $\tau_{nt}^{(\epsilon)}(E_p)$ is given by Eq. (20c). $\tau_{nt}^{(\epsilon)}(E_p)$ can be shown to be given by

$$\tau_{nt}^{(\epsilon)}(E_p)=-e^2\bar{u}_{\mathbf{p}}\gamma_4\frac{1}{(2\pi)^3}\int d\mathbf{k}(2k^3)^{-1}\times\left\{u_{(\mathbf{k}-\mathbf{p})}\bar{u}_{(\mathbf{k}-\mathbf{p})}\frac{\omega_p-\omega_{|\mathbf{k}-\mathbf{p}|}+k}{\omega_p-\omega_{|\mathbf{k}-\mathbf{p}|}-k+i\epsilon}-v_{(\mathbf{p}-\mathbf{k})}\bar{v}_{(\mathbf{p}-\mathbf{k})}\times\frac{\omega_p+\omega_{|\mathbf{k}-\mathbf{p}|}-k}{\omega_p+\omega_{|\mathbf{k}-\mathbf{p}|}+k-i\epsilon}\right\}\gamma_4u_{\mathbf{p}}, \quad \text{(B7)}$$

and, for $\mathbf{p}=0$, this has the value

$$\tau_{nt}^{(\epsilon)}(E_p)=-(\alpha/\pi)[\ln(\lambda/2m)+\tfrac{1}{2}],$$

so that

$$(\bar{\mathfrak{z}}_2)_L=1-(\alpha/2\pi)[\ln(2L/m)+\tfrac{3}{4}]$$

and

$$\lim_{L\to\infty}\frac{(\bar{\mathfrak{z}}_2)_L-(\mathfrak{z}_2)_C}{(\mathfrak{z}_2)_C}=0.$$

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[24] J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1967), p. 283.