When p^2 is small compared to Λ^2, the integrals can be computed using the approximate form for ρ [Eq. (A2)] except in a constant term (the second integral with p replaced by 0). The result is Eq. (3.21) with

$$c=(8\pi^2\Lambda^2)^{-1}\int_0^\infty q\rho(-q^2,\Lambda^2)dq \qquad \text{(A11)}$$

and $c_1=3(1024\pi^4)^{-1}$. The constant c is independent of Λ because ρ depends only on the ratio (q^2/Λ^2).

In Fourier-transforming $W_c(p_1,p_2)$, the only integral which is not already known is an integral of the form

$$u(x)=\int_p e^{-ip\cdot x}\ln[(-p^2-i\epsilon)/\Lambda^2]. \qquad \text{(A12)}$$

For $x=0$ this is highly divergent, but for $x\neq 0$ the exponent serves as a convergence factor. If one wishes to be careful one can insert an explicit convergence factor, for example, $\exp(-|p_0|\eta-|p_1|\eta-|p_2|\eta-|p_3|\eta)$, with $\eta>0$, $p_0,\ldots,p_3$ being the components of p. Then one writes

$$\ln\left[\frac{-p^2-i\epsilon}{\Lambda^2}\right]=\int_0^\infty \omega^{-1}(e^{-i\omega\Lambda^2}-e^{i\omega(p^2+i\epsilon)})d\omega. \qquad \text{(A13)}$$

After substituting this formula in Eq. (A12), the p integration can be done explicitly, leaving

$$u(x)=(i/16\pi^2)\int_0^\infty \omega^{-3}\exp(-ix^2/4\omega)d\omega. \qquad \text{(A14)}$$

[If the convergence factor is inserted in Eq. (A12), the result is to cutoff the integral (A14) for $\omega<\eta^2$.] One can change variables to $\nu=\omega^{-1}$ and then compute the integral, obtaining

$$u(x)=(1/i\pi^2)(x^2-i\epsilon)^{-2}. \qquad \text{(A15)}$$

The $i\epsilon$ is present because x^2 needs an imaginary part $-i\epsilon$ to ensure that the integral (A14) converges.

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High-Energy Behavior of Total Cross Sections

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A method is presented for obtaining an asymptotic series, for large values of the energy, of a four-dimensional Fourier transform, using only one analyticity assumption. It is shown that this method implies (1) asymptotic constancy of hadron total cross sections, as an "upper bound," and (2) the Pomeranchuk theorem. A consistency check, which lends some plausibility to our assumption, is made. The calculations are done within the context of frame-dependent cutoff quantum field theory.

I. INTRODUCTION

BY using the Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism, one can express a great many physically interesting quantities in terms of a Fourier transform,

$$I=\int d^4x\, e^{\pm iq\cdot x}F(x), \qquad (1)$$

where F is typically a matrix element of a (possibly retarded) commutator or anticommutator, and the four-momentum q is on some mass shell. We shall describe herein a very simple method for obtaining an asymptotic expansion of such a quantity, for large values (this will be made more precise below) of the energy q^4, and shall apply this method to the problem of hadron total cross sections.

The method requires only one assumption, which is, however, rather strong[1]: It is that certain "light-plane integrals" $f_\pm(\xi)$ admit power-series representations about $\xi=0$ which are valid in the interval $\xi=[0,\infty)$. At present, we cannot either prove or disprove this assumption on theoretical grounds, although some indications of its plausibility are available (see below). Its implications are, however, in good agreement with experiment, at least for the processes that we have treated thus far.

Assuming that the leading term in our asymptotic expansion is nonzero, we obtain, in a model-independent fashion, asymptotic constancy of total cross sections.

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[1] The same asymptotic expansion can be obtained also from the considerably weaker assumption that $f_\pm(\xi)$ admit power series in some interval $\xi=[0,a)$, for some $a>0$, and independent of how small a may be, by the use of Watson's lemma [E. T. Copson, *Theory of Functions of a Complex Variable* (Oxford U. P., Oxford, 1935), p. 218]. However, if one uses this method, the physical amplitudes must be defined by a different limit than the one used in the present paper [see Eq. (5)]. The limit defined by Eq. (5) reduces to the conventional one for local field theory.

In addition, our method implies the Pomeranchuk theorem. These results are obtained in Sec. VI. Experimental predictions for elastic scattering and photoproduction differential cross sections, based on a generalization of the present work, are also very encouraging.[2]

These calculations are carried out in the context of "stochastic" field theory,[3] a frame-dependent cutoff theory which is briefly discussed in Sec. II. Our method can also be applied to local field theory, but the results in that case contradict experiment. However, our analyticity assumption is considerably less believable for local field theory than it is for stochastic field theory, on the basis of the theoretical work done thus far on its validity.

Part of this work is described in Sec. V. It consists of applying the method to an I [of the form of Eq. (1)] which can actually be evaluated in terms of an equal-time commutator. For stochastic field theory, no contradiction ensures; for local field theory there is a contradiction. Also, direct checks of our analyticity assumption in terms of calculations with models indicate that the assumption is plausible in stochastic field theory, but not in local field theory.[2]

II. STOCHASTIC FIELD THEORY

Stochastic field theory—a cutoff quantum field theory which is free of ultraviolet divergences and, indeed, generally less "singular" than local field theory—has been extensively discussed in the literature.[3] We provide here only a very brief sketch.

The basic observation is that the conventional view of Lorentz invariance for quantum fields can be generalized without violating the relativity principle, as follows.[4] A local field $\varphi^\Omega(x)$ (Ω representing some set of indices) and its Lorentz transformation law

$$U(L)\varphi^\Omega(x)U(L)^{-1}=S^\Omega{}_{\Omega'}\varphi^{\Omega'}(L^{-1}x)\,,\quad L\in\mathcal{P}\,,$$

where $\mathcal{P}\equiv$ Poincaré group, is replaced by an *ensemble* of fields $\varphi^\Omega(x;\mathfrak{L})$, one field for each equivalence class $\mathfrak{L}$ of Lorentz observers (to be defined shortly) and the transformation law

$$U(L)\varphi^\Omega(x;\mathfrak{L})U(L)^{-1}=S^\Omega{}_{\Omega'}\varphi^{\Omega'}(L^{-1}x;L^{-1}\mathfrak{L}).\quad (2)$$

[2] R. L. Ingraham (unpublished).

[3] R. L. Ingraham, Nuovo Cimento **24**, 1117 (1962); **27**, 303 (1963); *Renormalization Theory of Quantum Field Theory with a Cut-off* (Gordon and Breach, London, 1967), and references therein.

[4] For a detailed discussion of this point, see R. L. Ingraham, Int. J. Theoret. Phys. **2**, 83 (1969); or Proceedings of the Symposium on Elementary Particles, Boulder, Colo., 1968 (unpublished). In this reference, the relativity principle is stated as follows: "If two equivalent observers do the same ($\equiv$ subjectively the same) experiment, they must get identical numbers." Essentially the same statement of relativistic invariance can be found in S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), p. 9.

The equivalence classes $\mathfrak{L}$ are taken to be the following ones. For any Lorentz observer l, let $n(l)$ be the unit four-vector oriented along l's positive time axis. [In l's own coordinates, $n^\mu(l)=(\mathbf{0},1)$.] Then for each fixed timelike unit four-vector $n(\mathfrak{L})$ with positive fourth component, let $\mathfrak{L}\equiv\{l|n(l)=n(\mathfrak{L})\}$. Thus, two observers are in the same equivalence class if and only if they are at rest with respect to one another. In order to define the class $L^{-1}\mathfrak{L}$, for any Poincaré transformation L (consisting of a translation a and a homogeneous transformation Λ), we need only define $n(L^{-1}\Lambda)$. This is done just as one might expect:

$$n^\mu(L^{-1}\mathfrak{L})\equiv(\Lambda^{-1})^\mu{}_\nu n^\nu(\mathfrak{L})\,,$$

where (NB) μ and ν are components referred to frame $\mathfrak{L}$.

From Eq. (2) it follows that, if Ψ and Φ are any two states,

$$(\Phi,\varphi^\Omega(x;\mathfrak{L})\Psi)=S^\Omega{}_{\Omega'}(L^{-1}\Phi,\varphi^{\Omega'}(L^{-1}x;L^{-1}\mathfrak{L})L^{-1}\Psi)\,,$$

which expresses the requirement of relativistic form invariance in the extended set of variables x, $n(\mathfrak{L})$. By $L^{-1}\Psi$ we mean $U(L)\Psi$: Note that for two observers whose reference frames are connected by the Lorentz transformation L, the states Ψ and $L^{-1}\Psi$ are subjectively identical.[4]

For the (divergence-free, unitary) scattering operator $S(\mathfrak{L})$, which now has a dependence on the equivalence classes $\mathfrak{L}$, one has

$$(\Phi,S(\mathfrak{L})\Psi)=(L^{-1}\Phi,S(L^{-1}\mathfrak{L})L^{-1}\Psi)\,,$$

which again expresses relativistic form invariance. The physical meaning of this equation is that if two Lorentz observers each perform the same (meaning subjectively the same) scattering experiment in their respective frames, they each measure the same numbers.

A theory of this form can be obtained by postulating that the space-time coordinates have an inherent fluctuation, the dispersion of which is a *fundamental length* λ. All calculations done to date indicate that $\lambda\approx10^{-14}$ cm. This corresponds (see below) to an effective cutoff in momentum space above about 2 GeV/c.

The stochastic fields $\varphi^\Omega(x;\mathfrak{L})$ are obtained by a certain averaging procedure from local fields. For the details of this procedure, see Refs. 3 and 5. For the purposes of the present work, it suffices to point out that the averaging gives rise to *kinematical form factors* $g(k;\mathfrak{L})$ in the Fourier decomposition of quantum fields. The possible forms for $g(k;\mathfrak{L})$ are severely limited by theoretical arguments.[3,5] The presently favored form has absolute square[3]

$$|g(k;\mathfrak{L})|^2=|(k_+/2k_\perp)[\exp(-\tfrac{1}{2}k_+{}^2\lambda^2)+iZ(k_+\lambda/\sqrt{2})]+(k_-/2k_\perp)[\exp(-\tfrac{1}{2}k_-{}^2\lambda^2)-iZ(k_-\lambda/\sqrt{2})]|^2\,,$$

[5] R. L. Ingraham, Nuovo Cimento **34**, 182 (1964).

where

$$k_{\pm}\equiv k_{\perp}\pm(k_{\perp}^2+\mu^2)^{1/2}, \quad \mu\equiv\text{mass of the field},$$
$$k_{\perp}^2\equiv k^2+(k\cdot n)^2, \quad k_{\perp}\equiv+(k_{\perp}^2)^{1/2},$$
$$Z(x)\equiv 2\pi^{-1/2}e^{-x^2}\int_0^x dy\, e^{y^2}.$$

It has the asymptotic form, valid for $k_{\perp}^2\gg\lambda^2$,

$$|g(k;\mathfrak{L})|^2\sim 1/(2\pi\lambda^2k_{\perp}^2). \tag{3}$$

Indeed, the results of Sec. VI, which depend critically upon Eq. (3), strongly indicate that this form factor, which was originally obtained from purely theoretical considerations,[3] is the correct one.

The LSZ reduction formalism goes through in stochastic field theory, with the following changes to the contraction formulas of local field theory: (a) For each particle which has been contracted from an out-state, multiply the local formula by a factor $g(k;\mathfrak{L})^{-1}$, where k is the momentum of the particle. (b) For each particle which has been contracted from an in-state, multiply the local formula by a factor $g^*(k;\mathfrak{L})^{-1}$ (* denotes complex conjugate), where k is the momentum of the particle. (c) Replace the Dyson T product of the local formula by the *stochastic T product* $T_{\mathfrak{L}}$, which orders the operators relative to the time of the class $\mathfrak{L}$ of observers. For instance,

$$\begin{aligned}T_{\mathfrak{L}}(A(x;\mathfrak{L})B(y;\mathfrak{L}))&=\theta(-n(\mathfrak{L})\cdot(x-y))A(x;\mathfrak{L})B(y;\mathfrak{L})\\&\quad+\theta(-n(\mathfrak{L})\cdot(y-x))B(y;\mathfrak{L})A(x;\mathfrak{L}),\end{aligned}$$

for boson fields A and B. If one uses coordinates in which $n^{\mu}(\mathfrak{L})=(\mathbf{0},1)$, this looks the same as the Dyson product.

III. CONVENTIONS AND NOTATION

Our space-time metric has signature $+2$, and we set $x^4\equiv$time. We use the summation convention for repeated indices, Greek indices running from 1 to 4, Latin ones from 1 to 3. The symbol

$$\epsilon(\mathbf{q},m)\equiv+(\mathbf{q}^2+m^2)^{1/2}$$

occurs often; when there is no danger of confusion it will be written ϵ_q. By * we denote complex conjugation on c numbers and Hermitian conjugation on q numbers, and by $^-$ we denote Dirac adjoint on spinors. Our conventions on spinors are the same as those of Jauch and Rohrlich.[6] The symbol λ will always mean the fundamental length, and $g(k)$ the kinematical form factor. In the remainder of this paper, we shall not write out the $\mathfrak{L}$ dependences: for example, a stochastic field $\varphi^{\Omega}(x;\mathfrak{L})$ will simply be written $\varphi^{\Omega}(x)$, and the kinematical form factor will be written $g(k)$ instead of $g(k;\mathfrak{L})$.

[6] J. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955).

IV. ASYMPTOTIC EXPANSION

Let I be defined by Eq. (1), with q on some mass shell, and suppose for the moment that the sign in the exponent is negative. Since we shall be interested only in very large values of $\omega\equiv q^4$, we may write

$$q\cdot x=(\hat{q}\cdot\mathbf{x}-x^4)\omega, \quad \hat{q}=\mathbf{q}/|\mathbf{q}|.$$

Let $\xi\equiv x^4-\hat{q}\cdot\mathbf{x}$. Then

$$\begin{aligned}I(\omega)&=\int d^3x d\xi\, e^{i\omega\xi}F(\mathbf{x},\xi+\hat{q}\cdot\mathbf{x})\\&=I_+(\omega)+I_-(\omega),\end{aligned}$$

with

$$I_{\pm}(\omega)\equiv\int_0^{\infty}d\xi\, e^{\pm i\omega\xi}f_{\pm}(\xi), \tag{4}$$

$$\begin{gathered}f_+(\xi)\equiv f(\xi),\ \xi>0;\quad f_-(\xi)\equiv f(-\xi),\ \xi>0;\\ f(\xi)\equiv\int d^3x\, F(\mathbf{x},\xi+\hat{q}\cdot\mathbf{x}).\end{gathered} \tag{5}$$

We shall treat ω as a complex variable, and define the physical amplitude, for real ω, as

$$I(\omega)=\lim_{\epsilon\to 0+}[I_+(\omega+i\epsilon)+I_-(\omega-i\epsilon)]. \tag{6}$$

We shall assume that

$f_{\pm}(\xi)$ admit power series $\sum_{n=0}^{\infty}b_n^{(\pm)}\xi^n$ which converge in the interval $[0,\infty)$.

It then follows that, for ω real, positive, and large,

$$I(\omega)\sim\frac{i}{\omega}\sum_{m=0}^{\infty}\left(\frac{i}{\omega}\right)^m C^{(m)}, \tag{7}$$

$$C^{(m)}\equiv\frac{d^m}{d\xi^m}f(0+)-\frac{d^m}{d\xi^m}f(0-). \tag{8}$$

With the obvious changes in Eqs. (4) and (6) and hypothesis (1), one has also

$$\int d^4x\, e^{+iq\cdot x}F(x)\sim\frac{-i}{\omega}\sum_{m=0}^{\infty}\left(\frac{-i}{\omega}\right)^m C^{(m)}. \tag{9}$$

By "large ω" we mean $\omega\gg m$, where $q^2=-m^2$.

The proof runs as follows. We insert the power series for $f_{\pm}$ into Eq. (4), and integrate term by term,[7] which

[7] We can do this for finite upper limits of integration because the series in the integrand converges uniformly. In a rigorous proof, we would need also some uniform convergence hypothesis for the approach of the upper limit to infinity, and also as $\mathrm{Im}\omega\to 0$ in the two series in Eq. (10).

yields

$$I_{\pm}(\omega)\sim\frac{i}{\omega}\sum_{n=0}^{\infty}\left(\frac{\pm i}{\omega}\right)^n n!b_n^{(\pm)}. \qquad (10)$$

Noting that

$$b_n^{(\pm)}=(n!)^{-1}\frac{d^n}{d\xi^n}f_{\pm}(0),$$

and using Eq. (5), we immediately get Eqs. (7) and (8). Equation (9) is obtained in the same manner.

The equation $\xi=0$ defines a plane in space-time which is tangent to the light cone; as ξ varies in a neighborhood of zero, this plane moves from intersecting to nonintersecting the forward light cone. According to Eq. (8), the high-energy behavior is governed by the discontinuities of f and its derivatives across the plane $\xi=0$. These discontinuities are in turn governed by the behavior of F near the light cone. Indeed, for the case where $F(x)$ has no angular dependence, $F(x)=F(r,t)$, and it is easy to show that $C^{(m)}$ depends only on the jump of $\partial^{m-1}F/\partial t^{m-1}$ across the forward light cone [see Eq. (21)].

That high-energy behavior is related to the behavior of source current commutators near the light cone is, of course, nothing new[8]; what is new is the sharpening of this notion that follows from our analyticity assumption.

V. CONSISTENCY CHECK

Let ψ be an interacting Dirac field, with source current j:

$$D_x\psi(x)\equiv(\gamma\cdot\partial+m)\psi(x)=j(x),$$

where m is the renormalized mass. In this section, we shall apply our method to the quantity

$$I=\int d^4x\, e^{-ip\cdot x}\theta(x)(\Psi_0,\{j(x),\bar{j}(0)\}\Psi_0),$$

where $p^2=-m^2$ and Ψ_0 is the vacuum, and check the resulting prediction. Since the source current j is built of interacting fields, this is a nontrivial check.

We can actually establish the asymptotic behavior of this I as follows. Let $\Psi(p)$ be a one-particle state with momentum p, the one particle being of the type described by ψ. Then the mass renormalization condition is

$$(\Psi(p),\bar{j}(0)\Psi_0)=0. \qquad (11)$$

Contracting out the particle,

$$(\Psi(p),\bar{j}(0)\Psi_0)=i(2\pi)^{-3/2}g(p;\mathfrak{L})^{-1} \times\int d^4x\, e^{-ip\cdot x}\bar{u}(p)D_x(\Psi_0,T_{\mathfrak{L}}(\psi(x)\bar{j}(0))\Psi_0). \qquad (12)$$

Working in coordinates in which $n^\mu=(\mathbf{0},1)$, expanding the $T_{\mathfrak{L}}$ product, and inserting a sum over a complete set of intermediate states into one term to show that it is zero,[9] we finally obtain from (11) and (12)

$$\int d^4x\, e^{-ip\cdot x}\theta(x)(\Psi_0,\{j(x),\bar{j}(0)\}\Psi_0) =\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\gamma^4(\Psi_0,\{\psi(x),\bar{j}(0)\}|_{x^4=0}\Psi_0),$$

where here equality means that both sides give the same result when acting on the spinor $\bar{u}(p)$.

The left-hand side of this equation is just I. The right-hand side can be evaluated for model interactions, and typically is found to behave like ω^{-2} for large ω. This result depends upon the fact that in the canonical equal-time commutation relations for stochastic fields one has, in place of the Dirac δ distribution of local field theory, the stochastic δ function

$$\delta_{\rm sto}(\mathbf{x})\equiv(2\pi)^{-3}\int d^3k|g(k)|^2e^{i\mathbf{k}\cdot\mathbf{x}},$$

where the kinematical form factor g has the asymptotic behavior (3).

Thus, in this case I verifiably has an asymptotic behavior of the form predicted by our method, with $C^{(0)}=0$, and $C^{(1)}\neq0$. A similar verification can be done for an interacting Klein-Gordon field.

This result could, of course, be mere coincidence. Nevertheless, it is probably the only nontrivial matrix element for which one can explicitly check a prediction of our method. It is encouraging that no contradiction ensues.

VI. TOTAL CROSS SECTIONS

In this section we shall apply the lemma of Sec. IV to show that total cross section of hadron A on target hadron B behaves asymptotically like

$$\sigma(AB)\sim a(AB)+\omega^{-1}b(AB)+O(1/\omega^2) \qquad (13)$$

as the lab momentum $\omega\to\infty$. The constants have the symmetry

$$a(AB)=a(\bar{A}B),\quad b(AB)=-b(\bar{A}B), \qquad (14)$$

where $\bar{A}$ means antiparticle of A. The first equation of (14) is just the Pomeranchuk theorem. We treat $AB=\pi p$ and pp here, but the method should be general.

The coefficients a and b in (13) are given by certain integrals which depend only on the integrated behavior of source current commutators infinitesimally near the light cone. These result from going to $\omega=\infty$ through mass-shell values, unlike the more tractable equal-time commutators which would be yielded by the unphysical Bjorken-type limit $\omega\to\infty$, $|\mathbf{q}|$ fixed.

[8] See, e.g., K. Symanzik, Phys. Rev. 105, 743 (1957).

[9] Essentially the same calculation is carried out in G. Källén, Helv. Phys. Acta 25, 417 (1952).

Treat first πp. The standard amplitude,[10] after reducing the two pions, spin-averaging, and specializing to the forward direction, is[11]

$$T(\omega,\alpha)=\frac{-i}{(2\pi)^3}|g(q)|^{-2}\int d^4x\, e^{-iq\cdot x}\theta(x)F_\alpha(x)+\text{ETC}\,,$$
$$F_\alpha(x)\equiv\tfrac{1}{2}\sum_{\text{spin}}(\Psi_p,[j_\pi{}^\alpha(x),j_\pi{}^{\alpha *}(0)]\Psi_p)\,. \tag{15}$$

Here Ψ_p is a one-proton state at rest in the lab frame, j_{π^α} is the source current of the incident π of momentum $q^\mu=(\mathbf{q},\omega)$ and isotopic index α. ETC means a term involving an equal-time commutator which can be shown to be real. In the local quantum-field-theory (QFT) case $|g(q)|^2\rightarrow 1$. The optical theorem reads[12]

$$\sigma(\omega,\alpha)=-[(2\pi)^6/|\mathbf{q}|]\,\text{Im}T(\omega,\alpha)\quad(\text{lab})\,. \tag{16}$$

We get $\text{Im}T(\omega,\alpha)$ by deleting $\theta(x)$ in (15) and dividing by $2i$.

Thus $-2(2\pi)^3|g(q)|^2\,\text{Im}T(\omega,\alpha)$ is given by an integral of precisely the form (1) with a minus sign and

$$F_\alpha(x)\equiv\tfrac{1}{2}\sum_{\text{spin}}(\Psi_p,[j_\pi{}^\alpha(x),j_\pi{}^{\alpha *}(0)]\Psi_p)\,. \tag{17}$$

Thus specifically making our analyticity assumption at this point, we can apply the lemma and get (7). Via the optical theorem (16), this gives[13]

$$\sigma(\omega,\alpha)\sim\frac{-(2\pi)^3}{2\omega^2|g(q)|^2}\bigg[\text{Im}C_\alpha{}^{(0)}+\frac{1}{\omega}\text{Re}C_\alpha{}^{(1)}+O\Big(\frac{1}{\omega^2}\Big)\bigg]\,. \tag{18}$$

But now the kinematical form factor has exactly the right asymptotic behavior to cancel the energy dependence of the leading term in (18), since according to (3) and $q_\perp{}^2\equiv\mathbf{q}^2\sim\omega^2$,

$$|g(q)|^2\sim 1/(2\pi\lambda^2\omega^2)\,,\quad \omega^2\lambda^2\gg 1\,.$$

Therefore, we get Eq. (13) with

$$a(\pi^\alpha p)=-8\pi^4\lambda^2\,\text{Im}C_\alpha{}^{(0)}\,,$$
$$b(\pi^\alpha p)=-8\pi^4\lambda^2\,\text{Re}C_\alpha{}^{(1)}\,. \tag{19}$$

It will be proved below that $\text{Im}C_\alpha{}^{(0)}<0$.

Since we are dealing with elastic scattering and the forward direction, the matrix elements $F_\alpha(x)$, Eq. (17), have a high degree of symmetry, namely, (a) $F_\alpha(x)$ is a function of r and t only, (b) $F_\alpha{}^*(x)=F_\alpha(-x)$, and (c) $F_\alpha{}^*(x)=-F_{\bar\alpha}(x)$, where $\bar\alpha$ means the antiparticle of α.

Using (c), we derive immediately

$$\text{Im}C_\alpha{}^{(m)}=+\text{Im}C_{\bar\alpha}{}^{(m)}\,,\quad \text{Re}C_\alpha{}^{(m)}=-\text{Re}C_{\bar\alpha}{}^{(m)}\,. \tag{20}$$

This proves Eq. (14), in particular, the Pomeranchuk theorem for πp.

Finally, by using the symmetries (a) and (b) of $F_\alpha(x)$, one can cast the $C_\alpha{}^{(m)}$ into a form which shows that they depend only on the value of $F_\alpha(x)$ or of its time derivatives in an infinitesimally thin region around the forward light cone, namely,

$$C_\alpha{}^{(m)}=2\pi\int_0^\infty dr\,r\int_{r-\epsilon}^{r+\epsilon}dt\,F_\alpha(r,t)^{(m)},\quad \epsilon\rightarrow 0+\,. \tag{21}$$

Note that for $m\geqslant 1$, the time integral is just the jump of $F_\alpha(r,t)^{(m-1)}$ across the light cone at $t=r$.

We treat next pp. This is typical of the cases in which the projectile has spin and nonzero mass. $T(\omega,A=p)$ is given by Eq. (15), with

$$F_p(x;q)\equiv\tfrac{1}{4}\sum_{\text{spins}}(\Psi_p,\{\bar u(q)f_p(x),\bar f_p(0)u(q)\}\Psi_p)\,, \tag{22}$$

where f_p is the proton source current. This depends on energy through the spinors; thus[14] $F_p=F_{p0}(x)+\omega F_{p1}(x)$ and let the corresponding constant $C_{p0,1}{}^{(m)}$ be formed from the $F_{p0,1}$. The $F_{p0,1}$ have[15] the symmetries (a) and (b) above, while symmetry (c) reads $F_{p0}{}^*(x)=-F_{p0}(x)$, $F_{p1}{}^*(x)=+F_{p1}(x)$.

Hence, following the same[16] procedure as for πp, we get Eq. (13) with

$$a(pp)=-(2\pi)^4\lambda^2M(\text{Im}C_{p0}{}^{(0)}+\text{Re}C_{p1}{}^{(1)})\,,$$
$$b(pp)=-(2\pi)^4\lambda^2M(\text{Re}C_{p0}{}^{(1)}-\text{Im}C_{p1}{}^{(2)})\,, \tag{23}$$

if we assume

$$\text{Im}C_{p1}{}^{(0)}=0\,. \tag{24}$$

If this method is valid, (24) must be true; otherwise by (7) and (16), for sufficiently large ω, one of $\sigma(pp)$ or $\sigma(\bar pp)$ would be negative, contradicting unitarity. Thus we expect that (24) will follow from some symmetry not yet utilized.

Using symmetry (c), we get an equation like (20) for $C_{p0}{}^{(m)}$, while for $C_{p1}{}^{(m)}$ the signs are reversed. Thus $a(pp)$ and $b(pp)$ satisfy Eq. (14), in particular, the Pomeranchuk theorem for pp. From symmetries (a) and (b) we get Eq. (21) for each of $C_{p0,1}{}^{(m)}$.

To check the reality and sign properties assumed above [namely, Ref. 13 and the positivity of $a(\pi p)$ and $a(pp)$], insert a sum of intermediate states into the $F_A(x)$ ($A=\alpha$, p, or $\bar p$) and evaluate the spatial integral in $f_A(\xi)$, Eq. (5). If it is legitimate to inter-

[10] See Eq. (21.5) of Gasiorowicz (Ref. 4).

[11] See the end of Sec. II.

[12] The optical theorem has the same form as in local QFT, since $S(\mathfrak{L})$ is unitary and we are (provisionally) defining the cross section in terms of $|T|^2$ just as in local QFT.

[13] We are using here $\text{Re}C_\alpha{}^{(m)}=0$ (m even), $\text{Im}C_\alpha{}^{(m)}=0$ (m odd), to be proved below.

[14] $F_{p1}=i(8M)^{-1}\,\text{tr}[\gamma^4\sum_{\text{spin}}(\Psi_p,\{f_p(x),\bar f_p(0)\}\Psi_p)]$ and F_{p0} is given by letting $i\gamma^4/M\rightarrow 1$ in this formula.

[15] Use $\bar f_p u(q)=-\bar v(q)f_{\bar p}$, where $u(p)$ and $v(p)$ are the standard particle and antiparticle 4-spinors.

[16] The optical theorem (16) has an extra factor $2M$ in the right member for pp.

change $\int d^3x$ and $\sum_n$, we get after some algebra

$$C_A{}^{(m)}=(2\pi)^3(-2i)(-i)^m\sum_n \delta(\mathbf{P}_n-(E_n-M)\hat{q}) \times(E_n-M)^m \sin\epsilon(E_n-M) \times\{|M_{pn,A}|^2\pm(-)^m|M_{np,A}|^2\},$$

$$A=\begin{pmatrix}\text{boson}\\ \text{fermion}\end{pmatrix},\quad \epsilon\to 0+$$

where the $|M_{pn,A}|^2$ and $|M_{np,A}|^2$ are spin-averaged absolute squares of matrix elements of source currents $[j_{\pi^\alpha}(0)$ for $A=\alpha$, $\bar{u}(q)f_p(0)$ for $A=p]$ between Ψ_p and intermediate state Ψ_n, Ψ_n and Ψ_p, respectively.

We find immediately that $C_A{}^{(m)}$ is pure imaginary for m even, pure real for m odd, as used in (18) and (23).

Note that $E_n-M\geqslant 0$ for $A=$boson, while $(E_n-M)^m \times\sin\epsilon(E_n-M)$ is even in E_n-M for m odd. Thus every summand of $\sum_n$ is positive for $A=$boson, m even, or $A=$fermion, M odd. Putting $m=0$, $A=$boson, $m=1$, $A=$proton, this proves that $\mathrm{Im}C_\alpha{}^{(0)}<0$ and also $\mathrm{Re}C_{p1}{}^{(1)}<0$.

VII. REMARKS

(a) Equation (21) shows that the asymptotic behavior depends critically on the smoothness of the relevant current commutator across the light cone. For example, if $F(r,t)$ is continuous in t,

$$\int_{r-\epsilon}^{r+\epsilon} F(r,t)=2\epsilon F(r,t(r)),\quad r-\epsilon\leqslant t(r)\leqslant r+\epsilon$$

by the theorem of the mean. Then $C^{(0)}=0$ if $\int dr\, rF[r,t(r)]<\infty$. On the other hand, if $F(r,t)=g(r)\delta(r-t)/r$ with $\int dr\, g(r)<\infty$ and $\neq 0$, then $C^{(0)}\neq 0$.

(b) The necessity of the frame-dependent cutoff for constant nonzero asymptotic σ is striking. For from (18) and its pp analog, if $|g(q)|^2$ were replaced by unity, $\sigma(\pi p)\sim O(1/\omega^2)$ and $\sigma(pp)\sim O(1/\omega)$ at most. However, it may well be that the assumption of the analyticity everywhere of $f_\pm(\xi)$ is not justified in local QFT. Models have been examined in another work,[2] which suggests that it is not, or perhaps better said, that the question has not much meaning in local QFT because of its divergent and ill-defined nature. Tanaka[17] gives examples of light-cone behavior of source current commutators $[\propto \partial^m\delta(-x^2)/(\partial t)^m,\ m=0, 1, 2, \ldots]$ which can yield $\mathrm{Im}T\sim O(\omega)$ or even a higher power and thus constant asymptotic σ.

(c) Recent Serpukhov data[18] on total cross sections for π^-p, K^-p, $\bar{p}p$ up to $\omega=65$ GeV show some waveiness at these very high energies. This behavior can be fitted by power series in $1/\omega$ as given by this theory.

The theoretical values of the constants $C_A{}^{(m)}$, in particular, the values $a(AB)$ of the asymptotic cross sections, will have to await a reliable way to calculate the current comutators on the light cone. But it is seen from (19) and (23) that their scale is given by the square of the cutoff length λ.

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Renormalization Constants, Wave Functions, and Energy Shifts in the Coulomb and Lorentz Gauges

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The scattering wave functions for quantum electrodynamics are examined in the Coulomb gauge, in the conventional Lorentz gauge, and in a reformulated version of the Lorentz gauge. It is shown that when the Lorentz gauge is formulated so that Maxwell's equations hold even when the Green's-function pole is displaced off the real axis, Z_2 is identical in the Coulomb gauge and in the Lorentz gauge. It is also shown that the unrenormalized asymptotic states in the reformulated Lorentz gauge include a partial dressing of the bare electrons by longitudinal and timelike photons sufficient to generate the electron's static electric field. It is proven that for true bound states the radiative energy shifts in the reformulated Lorentz gauge and in the conventional formulation agree.

I. INTRODUCTION

IN earlier work[1–3] we introduced a reformulation of quantum electrodynamics in the Lorentz gauge, in which the physical states $|\bar{\varphi}\rangle$ are defined by

$$\Omega^{(+)}(\mathbf{x})|\bar{\varphi}\rangle=0,\tag{1a}$$

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