Specifically, for the present case,

$$j_{(\phi)}{}^{*}=\frac{\Omega B}{2\pi c}\Omega R\frac{3}{5}b\left(\frac{R_s}{R}\right)\left(\frac{R}{r}\right)^{7}\sin\theta\, P_2(\cos\theta)\,. \tag{14}$$

The virtual charge and current densities illustrate new, previously unnoticed consequences of the dragging of the inertial frame. The interpretations thereof given above are made possible by the fact that these effects are related to the first derivatives of the metric tensor —which can be proved in all generality.[21] Furthermore, these concepts are entirely general and can be applied to any rotating metric with a superposed magnetic field; in fact an application of ρ_* to the Kerr metric[22] is possible. In relation to direct measurements of ρ_* and j_*, the best chances seem offered by pulsars, within the current model, and, because of the sharp radial dependences of (9) and (14), consequences will possibly be felt by the surface-emission theories.[23]

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Derivation of the Equations of Motion of a Gyroscope from the Quantum Theory of Gravitation

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Previous work on the gravitational two-body problem is surveyed. Next, we present a new approach, which we consider to be simpler and more transparent than the usual methods because it is based on a gravitational potential energy. This enables us to carry out our calculations using only the familiar tools of Newtonian mechanics and the Euler-Lagrange equations. Starting from a gravitational potential energy derived from Gupta's quantum theory of gravitation, the classical motion of a spherical gyroscope in the gravitational field of a much larger mass with a quadrupole moment is found. The results of the precession of the spin are compared with those of Schiff, and a detailed derivation of the results of O'Connell for the effect of a quadrupole moment (and higher moments) on the precession of the spin is presented. In addition, we present some new results. First, we show that the quadrupole moment manifests its presence in another way, which also contributes to the precession of the gyroscope a term that is about ten times larger than what could be detected. Second, with regard to the precession of the orbit, in addition to the usual contributions, our results include the effects of the spin of *both* particles (which enables us to calculate the effect of the rotation of Mercury on the precession of its perihelion).

I. INTRODUCTION

THE gravitational two-body equations of motion without spin were first derived by Einstein, Infeld, and Hoffmann[1] using a very lengthy and difficult procedure. A somewhat simplified procedure was used by Fock[2] and further developed by Papapetrou and Corinaldesi[3,4] who derived equations of motion of bodies with spin. Later Corinaldesi,[5] using the quantum theory of gravitation first developed by Gupta,[6] derived the Einstein-Infeld-Hoffmann equations of motion from the one-graviton-exchange interaction.

In this paper we shall be interested in deriving the equations of motion of particles *with* spin using the quantum theory of gravitation.[6] For mathematical

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[3] A. Papapetrou, Proc. Roy. Soc. (London) **A209**, 248 (1951).

[4] E. Corinaldesi and A. Papapetrau, Proc. Roy. Soc. (London) **A209**, 259 (1951).

[5] E. Corinaldesi, Proc. Phys. Soc. (London) **A69**, 189 (1956).

[6] S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 161 (1952); **A65**, 608 (1952); Phys. Rev. **96**, 1683 (1954); Rev. Mod. Phys. **29**, 334 (1957); *Recent Development in General Relativity* (Pergamon, New York, 1962), p. 251; Phys. Rev. **172**, 1303 (1968).

simplicity we shall confine ourselves to the case where one mass is much greater than the other and only the heavy mass has a quadrupole moment.

Our procedure is different from that used by Corinaldesi[5] in his calculation of the equations of motion of nonspinning particles. The essence of our method is the use of a potential[7] derived from Gupta's quantum theory of gravitation. The Lagrangian follows almost immediately, and then using the Euler-Lagrange equations, we obtain—in a manner which we consider to be simpler and more transparent than the usual methods—the equations of motion. In particular, we treat two important consequences of general relativity. First, we derive the precession of the spin of the lighter mass (a gyroscope) and compare our results with those of Schiff.[8] Pustovoĭt and Bautin[9] have also considered the problem of the precession of the gyroscope. However, their starting point is the Lagrangian for a *nonspinning* particle, which they generalize to include spin by an integration over the volume of the moving gyroscope. By contrast, in our method the effects of spin are included *ab initio*. We also derive in detail the results of O'Connell[10] for the effect on the spin of the gyroscope due to an arbitrary multipole potential (and, in particular, a quadrupole potential) of the heavy mass. Second, we derive the precession of the orbit and, in addition to the well-known Einstein and Lense-Thirring[11] contributions, we obtain the contribution of the spin of the light mass, which we apply to a calculation of the effect of the rotation of Mercury on the precession of its perihelion.

II. LAGRANGIAN FOR TWO SPINNING BODIES

Consider two particles of spin $\frac{1}{2}$ and masses m_1 and m_2 with a center-of-mass momentum $\mathbf{P}$ for the first particle. An expression for the Fourier transform of the gravitational potential energy $V(\mathbf{k})$, correct to the first order in the gravitational constant G, has already been obtained.[12] In the nonrelativistic approximation (where $m_1^2c^2 \gg \mathbf{P}^2$, $m_2^2c^2 \gg \mathbf{P}^2$, and the Newtonian and first relativistic terms are kept), the gravitational potential energy itself, which we denote by $V_1(\mathbf{r})$, has also been obtained.[12] The subscript in $V_1(\mathbf{r})$ indicates that it is correct to first order in G. We thus have

$$V_1(\mathbf{r}) = -\frac{Gm_1m_2}{r^2}\left[1+\left(4+\frac{3m_1}{2m_2}+\frac{3m_2}{2m_1}\right)\frac{\mathbf{P}^2}{m_1m_2c^2}\right] +G\left(1+\frac{3m_2}{4m_1}\right)\frac{\hbar\boldsymbol{\sigma}^{(1)}\cdot(\mathbf{r}\times\mathbf{P})}{c^2r^3} +G\left(1+\frac{3m_1}{4m_2}\right)\frac{\hbar\boldsymbol{\sigma}^{(2)}\cdot(\mathbf{r}\times\mathbf{P})}{c^2r^3} +\frac{G\hbar^2}{4c^2r^3}\left(\frac{3(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{r})(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{r})}{r^2}-\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}\right) +\frac{4\pi G\hbar^2}{c^2}\left(1+\frac{3m_2}{8m_1}+\frac{3m_1}{8m_2}\right)\delta(\mathbf{r}) +\frac{2\pi G\hbar^2}{3c^2}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\delta(\mathbf{r}). \quad (1)$$

We can obtain the classical result from the above by letting $\frac{1}{2}\hbar\boldsymbol{\sigma}^{(1)}\to\mathbf{S}^{(1)}$, $\frac{1}{2}\hbar\boldsymbol{\sigma}^{(2)}\to\mathbf{S}^{(2)}$, and dropping the contact terms; $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are the classical spin angular momenta of m_1 and m_2, respectively. Let us also make the large mass approximation, $m_2 \gg m_1$. With this approximation we get, correct to zeroth order in v^2/c^2, $\mathbf{P}=m_1\mathbf{v}$, where $\mathbf{v}$ is the velocity of the first particle. We then have

$$V_1(\mathbf{r}) = -\frac{Gm_1m_2}{r}\left(1+\frac{3}{2}\frac{v^2}{c^2}\right)+\frac{3Gm_2}{2c^2r^3}\mathbf{S}^{(1)}\cdot(\mathbf{r}\times\mathbf{v}) +\frac{2Gm_1}{c^2r^3}\mathbf{S}^{(2)}\cdot(\mathbf{r}\times\mathbf{v})+\frac{G}{c^2r^3} \times\left(\frac{3(\mathbf{S}^{(1)}\cdot\mathbf{r})(\mathbf{S}^{(2)}\cdot\mathbf{r})}{r^2}-\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}\right). \quad (2)$$

The Lagrangian[13] can then be written as

$$\mathcal{L}=\mathcal{L}_{\text{free}}-V_1(\mathbf{r}). \quad (3)$$

In order to find $\mathcal{L}_{\text{free}}$ it is convenient to find $\mathcal{H}_{\text{free}}$ first. The fact that we are using a large-mass approximation in the center-of-mass system ensures that the heavy mass m_2 is at rest. The term m_2c^2 may thus be considered as a constant (as the rotational angular velocity $\boldsymbol{\omega}^{(2)}$ for the heavy mass does not change) and may be dropped from $\mathcal{H}_{\text{free}}$. We then have

$$\mathcal{H}_{\text{free}}=m_1c^2(1-v^2/c^2)^{-1/2}, \quad (4)$$

where m_1 is the relativistic mass of the first particle in

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[8] L. I. Schiff, Proc. Natl. Acad. Sci. U. S. **46**, 871 (1960); in *Proceedings on the Theory of Gravitation*, edited by L. Infeld (Gauthier-Villars, Paris and Warsaw, 1964), p. 71.

[9] V. I. Pustovoĭt and A. V. Bautin, Zh. Eksperim. i Teor. Fiz. **46**, 1386 (1964) [Soviet Phys. JETP **19**, 937 (1964)].

[10] R. F. O'Connell, Astrophys. Space Sci. **4**, 119 (1969); Nuovo Cimento **1**, 933 (1969); for the effect of the Brans-Dicke theory, see R. F. O'Connell, Phys. Rev. Letters **20**, 69 (1968).

[11] J. Lense and H. Thirring, Z. Physik **19**, 156 (1918).

[12] See Eqs. (27)–(34) of Ref. 7 and also the notation of Ref. 7.

[13] What we are actually starting with is the Hamiltonian as a function of r and $\mathbf{P}$ such that $\mathcal{H}(\mathbf{r},\mathbf{P})=\mathcal{H}_{\text{free}}(\mathbf{r},\mathbf{P})+V(\mathbf{r},\mathbf{P})$ and then going to the Lagrangian as a function of r and $\mathbf{v}$ such that $\mathcal{L}(\mathbf{r},\mathbf{v})=\mathcal{L}_{\text{free}}(\mathbf{r},\mathbf{v})-V(\mathbf{r},m_1\mathbf{v})$ + higher-order terms that can easily be shown not to contribute to the results of this paper.

its rest frame. Let m_{01} be the mass of the first particle in its rest frame when it is not spinning. If the first mass is spinning with an angular velocity $\boldsymbol{\omega}_{\rm rest}{}^{(1)}$ about an axis through its center-of-mass which is at rest, the energy of the body is given (to order needed) by

$$E^{(1)}=m_{01}c^2+\tfrac{1}{2}\sum_i m_{0i}r_i''^2\omega_{\rm rest}{}^{(1)2}+\tfrac{3}{8}\sum_i m_{0i}r_i''^4\omega_{\rm rest}{}^{(1)4}/c^2, \quad (5)$$

where m_{0i} is the rest mass of the ith particle in the first body and r_i'' is the distance of the ith particle from the axis of rotation. If we now define

$$I^{(1)}=\sum_i m_{0i}r_i''^2 \quad\text{and}\quad J^{(1)}=\sum_i m_{0i}r_i''^4, \quad (6)$$

we obtain

$$m_1=E^{(1)}/c^2=m_{01}+\tfrac{1}{2}I^{(1)}\omega_{\rm rest}{}^{(1)2}/c^2+\tfrac{3}{8}J^{(1)}\omega_{\rm rest}{}^{(1)4}/c^4. \quad (7)$$

The relation between $\omega^{(1)}$ as measured in the system where the first particle is moving and $\omega_{\rm rest}{}^{(1)}$ as measured in the system where the first particle is at rest is given by

$$\omega_{\rm rest}{}^{(1)2}=(1-v^2/c^2)^{-1}\omega^{(1)2}. \quad (8)$$

Then using (7) and (8) in (4) we obtain, to the order required,

$$\mathcal{H}_{\rm free}=m_{01}c^2+\tfrac{1}{2}I^{(1)}\omega^{(1)2}+\tfrac{1}{2}m_{01}v^2+\tfrac{3}{4}I^{(1)}\omega^{(1)2}v^2/c^2+\tfrac{3}{8}J^{(1)}\omega^{(1)4}/c^2+\tfrac{3}{8}m_{01}v^4/c^4. \quad (9)$$

The corresponding Lagrangian is given by

$$\mathcal{L}_{\rm free}=-m_{01}c^2+\tfrac{1}{2}I^{(1)}\omega^{(1)2}+\tfrac{1}{2}m_{01}v^2+\tfrac{1}{4}I^{(1)}\omega^{(1)2}v^2/c^2+\tfrac{1}{8}J^{(1)}\omega^{(1)4}/c^2+\tfrac{1}{8}m_{01}v^4/c^2. \quad (10)$$

Using (7) and (8) in (2) we obtain

$$V_1(r)=-\frac{Gm_{01}m_2}{r}\left(1+\frac{I^{(1)}\omega^{(1)2}}{2m_{01}c^2}+\frac{3}{2}\frac{v^2}{c^2}\right)+\frac{3Gm_2}{2c^2r^3}\mathbf{S}^{(1)}\cdot(\mathbf{r}\times\mathbf{v})+\frac{2Gm_{01}}{c^2r^3}\mathbf{S}^{(2)}\cdot(\mathbf{r}\times\mathbf{v})+\frac{G}{c^2r^3}\left(\frac{3(\mathbf{S}^{(1)}\cdot\mathbf{r})(\mathbf{S}^{(2)}\cdot\mathbf{r})}{r^2}-\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}\right). \quad (11)$$

We may use $\mathbf{S}^{(1)}=I^{(1)}\boldsymbol{\omega}^{(1)}$ and $\mathbf{S}^{(2)}=I^{(2)}\boldsymbol{\omega}^{(2)}$ in Eq. (11) and be correct to the order that we need. Equations (10) and (11) may now be combined as in (3) to give the total Lagrangian $\mathcal{L}$.

III. PRECESSION OF SPIN

For the precession of the spin of the lighter mass, we need only those terms in the Lagrangian of Eq. (3) which depend on $\boldsymbol{\omega}^{(1)}$. These terms may be written as

$$\mathcal{L}(\boldsymbol{\omega}^{(1)})=\tfrac{1}{2}I^{(1)}\omega^{(1)2}\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{Gm_2}{c^2r}\right)+\tfrac{1}{8}J^{(1)}\omega^{(1)4}/c^2-I^{(1)}\boldsymbol{\omega}^{(1)}\cdot\boldsymbol{\Omega}, \quad (12)$$

where

$$\boldsymbol{\Omega}=\boldsymbol{\Omega}_{\rm DS}+\boldsymbol{\Omega}_{\rm LT}, \quad (13)$$

$$\boldsymbol{\Omega}_{\rm DS}\equiv\frac{3Gm_2}{2c^2r^3}(\mathbf{r}\times\mathbf{v}), \quad (14)$$

$$\boldsymbol{\Omega}_{\rm LT}\equiv\frac{G}{c^2r^3}\left(\frac{3\mathbf{r}(\mathbf{S}^{(2)}\cdot\mathbf{r})}{r^2}-\mathbf{S}^{(2)}\right), \quad (15)$$

and $\boldsymbol{\Omega}_{\rm DS}$ and $\boldsymbol{\Omega}_{\rm LT}$ are called the de Sitter[14] and Lense-Thirring[11] terms, respectively. Using the space axes, we can write[15]

$$\begin{aligned}\omega_x{}^{(1)}&=\dot\theta^{(1)}\cos\phi^{(1)}+\dot\psi^{(1)}\sin\theta^{(1)}\sin\phi^{(1)},\\ \omega_y{}^{(1)}&=\dot\theta^{(1)}\sin\phi^{(1)}-\dot\psi^{(1)}\sin\theta^{(1)}\cos\phi^{(1)},\\ \omega_z{}^{(1)}&=\dot\psi^{(1)}\cos\theta^{(1)}+\dot\phi^{(1)},\end{aligned} \quad (16)$$

where $\phi^{(1)}$, $\theta^{(1)}$, and $\psi^{(1)}$ are the Euler angles representing the orientation of the light mass m_{01}. We shall always use a dot to denote differentiation with respect to time. Lagrange's equations for these angles can be written as[16]

$$\frac{d}{dt}\left[I^{(1)}\boldsymbol{\omega}^{(1)}\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{Gm_2}{c^2r}\right)+\frac{1}{2}\left(\frac{J^{(1)}\omega^{(1)2}\boldsymbol{\omega}^{(1)}}{c}\right)-I^{(1)}\boldsymbol{\Omega}\right]=\boldsymbol{\Omega}\times(I^{(1)}\boldsymbol{\omega}^{(1)}). \quad (17)$$

If τ is the proper time as measured by a clock moving in the satellite which contains the lighter mass, we have the relation between t and τ given by[17]

$$\frac{dt}{d\tau}=1+\frac{1}{2}\frac{v^2}{c^2}+\frac{Gm_2}{c^2r}, \quad (18)$$

which is just the first round bracket of Eq. (17). Let us now define $\boldsymbol{\omega}_0{}^{(1)}$ by the relation

$$\boldsymbol{\omega}_0{}^{(1)}=\boldsymbol{\omega}^{(1)}\frac{dt}{d\tau}, \quad (19)$$

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[14] W. de Sitter, Monthly Notices Roy. Astron. Soc. **77**, 155 (1916); **77**, 481 (1916).

[15] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1965), p. 141.

[16] Lagrange's equation for $\phi^{(1)}$ gives the z component of Eq. (17).

[17] This is obtained from the relation $\mathcal{L}_0=-m_{01}c^2d\tau/dt$, where $\mathcal{L}_0$ is the Lagrangian of a nonspinning particle of mass m_{01}. Thus the right-hand side of Eq. (18) follows essentially from Eq. (3) by setting $\boldsymbol{\omega}^{(1)}$ equal to zero.

which means that $\boldsymbol{\omega}_0^{(1)}$ is the angular velocity as measured by a clock moving with the lighter mass. Note that $\boldsymbol{\omega}_0^{(1)}$ refers to the angular velocity in the presence of the gravitational field whereas $\boldsymbol{\omega}_{\text{rest}}^{(1)}$ [see Eq. (8)] is defined in the absence of the gravitational field. As the canonical momentum $P_{\phi^{(1)}}$ is the z component of the expression in the square bracket in Eq. (17), a natural definition for $\mathbf{S}_0^{(1)}$ is

$$\mathbf{S}_0^{(1)}=I^{(1)}\boldsymbol{\omega}_0^{(1)}+\tfrac{1}{2}(J^{(1)}\omega_0^{(1)2}\boldsymbol{\omega}_0^{(1)}/c)-I^{(1)}\boldsymbol{\Omega}, \quad (20)$$

where we have made use of Eqs. (18) and (19). We thus obtain

$$\dot{\mathbf{S}}_0^{(1)}=\boldsymbol{\Omega}\times\mathbf{S}_0^{(1)}, \quad (21)$$

which agrees with the result obtained by Schiff,[8] or, explicitly,

$$I^{(1)}\dot{\boldsymbol{\omega}}_0^{(1)}+J^{(1)}\boldsymbol{\omega}_0^{(1)}\cdot\dot{\boldsymbol{\omega}}_0^{(1)}\boldsymbol{\omega}_0^{(1)}/c^2 +\tfrac{1}{2}J^{(1)}\omega_0^{(1)2}\dot{\boldsymbol{\omega}}_0^{(1)}/c^2-I^{(1)}\dot{\boldsymbol{\Omega}} =\boldsymbol{\Omega}\times(I^{(1)}\boldsymbol{\omega}_0^{(1)}). \quad (22)$$

Using Eq. (22) in itself, the terms involving $J^{(1)}$ can be dropped because they are of higher order. This gives us

$$I^{(1)}\dot{\boldsymbol{\omega}}_0^{(1)}=\boldsymbol{\Omega}\times(I^{(1)}\boldsymbol{\omega}_0^{(1)})+I^{(1)}\dot{\boldsymbol{\Omega}}. \quad (23)$$

Eq. (23) can also be put in the alternative form in terms of $\boldsymbol{\omega}^{(1)}$ rather than $\boldsymbol{\omega}_0^{(1)}$ as

$$I^{(1)}\dot{\boldsymbol{\omega}}^{(1)}=\boldsymbol{\Omega}\times(I^{(1)}\boldsymbol{\omega}^{(1)})+I^{(1)}\dot{\boldsymbol{\Omega}} +\frac{d}{dt}\left[I^{(1)}\boldsymbol{\omega}^{(1)}\left(1-\frac{dt}{d\tau}\right)\right]. \quad (24)$$

In order to obtain the secular precession of the spin, we must average Eq. (23) or (24) over a complete Newtonian orbit. The averaging process is quite straightforward and we have included a useful table of average values in the Appendix. Any term that is a time derivative of some quantity will have a zero average value. As a consequence of this we obtain immediately from Eqs. (23) and (24)

$$\dot{\boldsymbol{\omega}}_0{}^{(1)}{}_{\text{av}}=\dot{\boldsymbol{\omega}}^{(1)}{}_{\text{av}}. \quad (25)$$

We further obtain from Eqs. (13) and (23)

$$\dot{\boldsymbol{\omega}}_0{}^{(1)}{}_{\text{av}}=\boldsymbol{\Omega}_{\text{av}}\times\boldsymbol{\omega}_0^{(1)}, \quad (26)$$

where

$$\boldsymbol{\Omega}_{\text{av}}=\boldsymbol{\Omega}_{\text{DS av}}+\boldsymbol{\Omega}_{\text{LT av}}, \quad (27)$$

$$\boldsymbol{\Omega}_{\text{DS av}}=\frac{3GLm_2/m_1}{2c^2a^3(1-e)^{3/2}}\mathbf{n}, \quad (28)$$

$$\boldsymbol{\Omega}_{\text{LT av}}=\frac{GS^{(2)}}{2c^2a^3(1-e^2)^{3/2}}[\mathbf{n}^{(2)}-3(\mathbf{n}\cdot\mathbf{n}^{(2)})\mathbf{n}], \quad (29)$$

and e is the eccentricity; a is the semimajor axis; $\mathbf{n}^{(1)}$, $\mathbf{n}^{(2)}$, and $\mathbf{n}$ are unit vectors in the $\mathbf{S}^{(1)}$, $\mathbf{S}^{(2)}$, and $\mathbf{L}$ directions, respectively. Also, the orbital angular momentum L is given by

$$\frac{L/m_{01}}{a^2(1-e^2)^{1/2}}=\left(\frac{Gm_2}{a^3}\right)^{1/2}=\frac{2\pi}{T}, \quad (30)$$

where T is the period.

IV. MULTIPOLE EXPANSION OF POTENTIAL

The effects of the nonspherical heavy mass on the precession of the spin of the gyroscope has already been investigated by one of us.[10] We wish to present here a detailed derivation of these results,[10] and in addition to present some new results. We will show that the quadrupole moment actually affects the precession of the gyroscope in two ways. First of all, there is a direct effect,[10] and second, there is an indirect effect which manifests itself only when the principal term (i.e., the de Sitter term) is *averaged* over a period of the motion. We are interested in a generalization of the de Sitter term only as this is the only case of practical importance. Let us divide the heavy mass m_2 into a number of smaller masses m_{2i} such that

$$\mathbf{r}_i=\mathbf{r}-\mathbf{r}_i', \quad (31)$$

where $\mathbf{r}_i$ is the distance from m_{2i} to the gyroscope, $\mathbf{r}$ is the distance from the center of mass of m_2 to the gyroscope, and $\mathbf{r}_i'$ is the distance from the center of mass of m_2 to m_{2i}.

In the potential $V_1(\mathbf{r})$, with $\mathbf{S}^{(2)}=0$ as we are not interested in the Lense-Thirring term here, we have only the two terms Gm_2/r and $Gm_2\mathbf{r}/r^3$ which must be generalized. We thus obtain

$$\frac{Gm_2}{r}\to\int\frac{G\rho_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'=\phi(\mathbf{r}), \quad (32)$$

and

$$-\boldsymbol{\nabla}\frac{Gm_2}{r}=\frac{Gm_2\mathbf{r}}{r^3}\to \int\frac{G\rho_2(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV'=-\boldsymbol{\nabla}\phi(\mathbf{r}), \quad (33)$$

where ρ_2 is the mass density for the heavy mass. Using Eqs. (32) and (33), we obtain the generalizations of $V_1(\mathbf{r})$ and $\mathcal{L}(\boldsymbol{\omega}^{(1)})$ as

$$V_1(\mathbf{r})=-m_{01}\phi(\mathbf{r})\left(1+\frac{I^{(1)}\omega^{(1)2}}{2m_{01}c^2}+\frac{3}{2}\frac{v^2}{c^2}\right) +\mathbf{S}^{(1)}\cdot\boldsymbol{\Omega}, \quad (34)$$

$$\mathcal{L}(\boldsymbol{\omega}^{(1)})=\tfrac{1}{2}I^{(1)}\omega^{(1)2}\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{\phi(\mathbf{r})}{c^2}\right) +\frac{1}{8}\frac{J^{(1)}\omega^{(1)4}}{c^2}-I^{(1)}\boldsymbol{\omega}^{(1)}\cdot\boldsymbol{\Omega}, \quad (35)$$

where

$$\boldsymbol{\Omega}=-\frac{3}{2c^2}\nabla\phi(\mathbf{r})\times\mathbf{v}=-\frac{3}{2c^2}\mathbf{f}\times\mathbf{v}, \qquad (36)$$

and $\mathbf{f}=\nabla\phi(\mathbf{r})$ is the Newtonian force per unit mass. The relation between t and τ is now given by

$$\frac{dt}{d\tau}=1+\frac{1}{2}\frac{v^2}{c^2}+\frac{\phi(\mathbf{r})}{c^2}. \qquad (37)$$

Thus the equations for the precession of the spin are still of the same form as Eqs. (23) and (24).

If the heavy mass has an axis of symmetry in the $\mathbf{n}^{(2)}$ direction we can expand $\phi(\mathbf{r})$ as[18]

$$\phi(\mathbf{r})=\frac{Gm_2}{r}\left(1-\frac{1}{r^2}J_2P_2-\frac{1}{r^3}J_3P_3-\cdots\right), \qquad (38)$$

where the J's are constants and the P's are Legendre polynomials. Restricting ourselves to the quadrupole moment contribution (which turns out to be the only one of significance as far as the precession of the gyroscope spin is concerned), we can write $\phi(\mathbf{r})$ as

$$\phi(\mathbf{r})=\frac{Gm_2}{r}+\frac{GJ_2m_2}{2r^3}\left[1-\frac{3(\mathbf{n}^{(2)}\cdot\mathbf{r})^2}{r^2}\right], \qquad (39)$$

where

$$J_2=\frac{1}{2m_2}\int\rho_2(\mathbf{r}')[\mathbf{r}'^2-3(\mathbf{n}^{(2)}\cdot\mathbf{r}')^2]dV'. \qquad (40)$$

Using Eq. (39) in Eq. (36), we obtain

$$\boldsymbol{\Omega}=\boldsymbol{\Omega}_{\mathrm{DS}}+\boldsymbol{\Omega}_Q, \qquad (41)$$

where

$$\boldsymbol{\Omega}_{\mathrm{DS}}=\frac{3Gm_2}{2c^2r^3}(\mathbf{r}\times\mathbf{v}), \qquad (42)$$

$$\boldsymbol{\Omega}_Q=\frac{3Gm_2}{2c^2r^3}(\mathbf{R}\times\mathbf{v}), \qquad (43)$$

and

$$\mathbf{R}=\frac{3J_2}{2r^2}\left[1-5\frac{(\mathbf{n}^{(2)}\cdot\mathbf{r})^2}{r^2}\right]\mathbf{r}+\frac{3J_2}{r^2}(\mathbf{n}^{(2)}\cdot\mathbf{r})\mathbf{n}^{(2)}. \qquad (44)$$

Equations (43) and (44) agree with the results of O'Connell.[10]

Averaging over an elliptic orbit, we find

$$\boldsymbol{\Omega}_{Q\,\mathrm{av}}=\frac{3GLm_2/m_{01}}{2c^2a^3(1-e^2)^{3/2}}\left[\frac{3J_2}{16a^2(1-e^2)^2}\right]\{[-(4+e^2)+(20+15e^2)(\mathbf{n}^{(2)}\cdot\mathbf{n})^2-10e^2(\mathbf{n}^{(2)}\cdot\mathbf{n}^{(A)})^2]\times\mathbf{n}-(8+2e^2)(\mathbf{n}^{(2)}\cdot\mathbf{n})\mathbf{n}^{(2)}-20e^2(\mathbf{n}^{(2)}\cdot\mathbf{n}^{(A)})\times(\mathbf{n}^{(2)}\cdot\mathbf{n})\mathbf{n}^{(A)}\}, \qquad (45)$$

where $\mathbf{n}^{(A)}$ is a unit vector in the direction of the perihelion. Setting $e=0$, we obtain the result for a circular orbit as

$$\Omega_{Q\,\mathrm{av}}=\frac{3GLm_2/m_{01}}{2c^2a^3}\left(\frac{3J_2}{2a^2}\right)\{[\tfrac{5}{2}(\mathbf{n}^{(2)}\cdot\mathbf{n})^2-\tfrac{1}{2}]\times\mathbf{n}-(\mathbf{n}^{(2)}\cdot\mathbf{n})\mathbf{n}^{(2)}\}. \qquad (46)$$

As we are interested in results correct to the first power of J_2, it will *not* suffice to average $\boldsymbol{\Omega}_{\mathrm{DS}}$ over an elliptic orbit. We must average $\boldsymbol{\Omega}_{\mathrm{DS}}$ over a distorted elliptic orbit which is the result of using the complete Newtonian potential energy $-m_{01}\phi(\mathbf{r})$. We will consider two types of orbits for which the plane of the orbit does not change its orientation in space and so a "period" is a readily definable quantity: (a) a distorted circular polar orbit (the *polar* orbit is of particular importance because this is the orbit selected for the Stanford experimental test,[19,20] primarily because it enables one to measure $\boldsymbol{\Omega}_{\mathrm{DS}}$ and $\boldsymbol{\Omega}_{\mathrm{LT}}$ separately), and (b) a distorted circular equatorial orbit. We have averaged $\boldsymbol{\Omega}_{\mathrm{DS}}$ for these two special cases.

A. Distorted Circular Polar Orbit

From the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}m_{01}(\dot{r}^2+r^2\dot{\phi}^2)+\frac{Gm_{01}m_2}{r}+\frac{GJ_2m_{01}m_2}{2r^3}(1-3\cos^2\phi), \qquad (47)$$

where ϕ is the angle between $\mathbf{n}^{(2)}$ and $\mathbf{r}$ and $0\leq\phi\leq 2\pi$, we find the solution

$$r=a-(J_2/2a)\cos^2(\bar{\omega}t),\quad \phi=\bar{\omega}t-(J_2/8a^2)\sin(2\bar{\omega}t). \qquad (48)$$

In the above, a is a constant and $\bar{\omega}$ is the average angular velocity. We also have

$$\bar{\omega}=2\pi/T=(Gm_2/a^3)^{1/2}. \qquad (49)$$

Notice that our definition of the constant a is such as to ensure that Kepler's law holds in its normal form, as in Eq. (49). From Eq. (48), we have

$$r_{\mathrm{equatorial}}=a,\quad r_{\mathrm{polar}}=a-J_2/2a. \qquad (50)$$

Noting that

$$|\mathbf{r}\times\mathbf{v}|/r^3=\dot{\phi}/r, \qquad (51)$$

[18] By this definition J_2 will be positive for an oblate spheroid.

[19] C. W. F. Everitt and V. M. Fairbank, in *Proceedings of the Tenth International Conference on Low-Temperature Physics, Moscow, 1966*, edited by M. P. Malkov (Proizvodstrenno-Izdatel'-skii Kombinat, VINITI, Moscow, 1967).

[20] W. M. Fairbank, in *Proceedings of the Eleventh International Conference on Low-Temperature Physics*, edited by J. F. Allen, D. M. Finlayson, and D. M. McCall (University of St. Andrews Printing Department, St. Andrews, Scotland, 1969), pp. 14, 15.

and using Eqs. (42), (48), and (49), we obtain

$$\mathbf{\Omega}_{\mathrm{DS\ av}}=\frac{3Gm_2\bar{\omega}}{2c^2a}\left(1+\frac{J_2}{4a^2}\right)\mathbf{n}. \tag{52}$$

For the special case of the polar circular orbit, Eq. (46) reduces to

$$\mathbf{\Omega}_{Q\ \mathrm{av}}=\frac{3Gm_2\bar{\omega}}{2c^2a}\left(-\frac{3J_2}{4a^2}\right)\mathbf{n}. \tag{53}$$

Note that the magnitude of the direct quadrupole-moment effect, as given by Eq. (53), is of the same order as the indirect effect, as given by the second term in Eq. (52). This will be true in general.

B. Distorted Circular Equatorial Orbit

From the Lagrangian

$$\mathfrak{L}=\tfrac{1}{2}m_{01}(\dot{r}^2+r^2\dot{\phi}^2)+\frac{Gm_{01}m_2}{r}+\frac{GJ_2m_{01}m_2}{2r^3}, \tag{54}$$

where $0\leq\phi\leq 2\pi$, we find the solution

$$r=a+J_2/2a,\quad \phi=\bar{\omega}t, \tag{55}$$

and Eq. (49) holds here also.

Using Eqs. (42), (55), and (49), we obtain

$$\mathbf{\Omega}_{\mathrm{DS\ av}}=\frac{3Gm_2\bar{\omega}}{2c^2a}\left(1-\frac{J_2}{2a^2}\right)\mathbf{n}. \tag{56}$$

For the special case of the equatorial circular orbit, Eq. (46) reduces to

$$\mathbf{\Omega}_{Q\ \mathrm{av}}=\frac{3Gm_2\bar{\omega}}{2c^2a}\left(\frac{3J_2}{2a^2}\right)\mathbf{n}. \tag{57}$$

For the case of the earth, we have[21]

$$J_2/R^2=(1082.64\pm.08)\times10^{-6}, \tag{58}$$

where R is the earth's equatorial radius. Hence $\Omega_{Q\ \mathrm{av}}\simeq10^{-3}\Omega_{\mathrm{DS\ av}}$ for a gyroscope in orbit close to the earth. For a gyroscope in a circular orbit 300 miles above the earth $\Omega_{\mathrm{DS\ av}}\simeq7''/\mathrm{yr}$ and thus $\Omega_{Q\ \mathrm{av}}\simeq0.01''/\mathrm{yr}$. Since measurements accurate to $0.001''/\mathrm{yr}$ will be feasible[20] by use of the London moment-readout technique, we see that the quadrupole-moment contribution is about 10 times larger than what can be measured.

V. PRECESSION OF ORBIT

For the precession of the orbit, we shall use the Lagrangian

$$\mathfrak{L}=\mathfrak{L}_{\mathrm{free}}-V(\mathbf{r}), \tag{59}$$

with

$$V(\mathbf{r})=V_1(\mathbf{r})+V_2(\mathbf{r})+V_Q(\mathbf{r}), \tag{60}$$

where $V_1(\mathbf{r})$ is given by Eq. (11),

$$V_2(\mathbf{r})=\frac{G^2m_{01}m_2^2}{2c^2r^2}, \tag{61}$$

and

$$V_Q(\mathbf{r})=\frac{GJ_2m_{01}m_2}{2r^2}\left(\frac{3(\mathbf{n}^{(2)}\cdot\mathbf{r})^2}{r^2}-1\right). \tag{62}$$

The $V_2(\mathbf{r})$ term[22] was not necessary for the precession of the spin since it is not a function of $\boldsymbol{\omega}^{(1)}$.

Using the above Lagrangian, Eq. (59), we find by the use of the Euler-Lagrange equations that the equations of motion can be put in the form

$$\dot{\mathbf{v}}+Gm_2\mathbf{r}/r^3=\mathbf{B}, \tag{63}$$

where

$$\mathbf{B}=\mathbf{B}^{(E)}+\mathbf{B}^{(1)}+\mathbf{B}^{(2)}+\mathbf{B}^{(1,2)}+\mathbf{B}^{(Q)}, \tag{64}$$

and

$$\mathbf{B}^{(E)}=\frac{Gm_2}{c^2r^3}\left[\frac{4Gm_2\mathbf{r}}{r}-v^2\mathbf{r}+4(\mathbf{v}\cdot\mathbf{r})\mathbf{v}\right], \tag{65a}$$

$$\mathbf{B}^{(1)}=\frac{3Gm_2}{c^2r^5m_{01}}\{\tfrac{3}{2}[\mathbf{S}^{(1)}\cdot(\mathbf{r}\times\mathbf{v})]\mathbf{r}+r^2\mathbf{S}^{(1)}\times\mathbf{v}-\tfrac{3}{2}(\mathbf{v}\cdot\mathbf{r})\mathbf{S}^{(1)}\times\mathbf{r}\}, \tag{65b}$$

$$\mathbf{B}^{(2)}=\frac{4G}{c^2r^5}\{\tfrac{3}{2}[\mathbf{S}^{(2)}\cdot(\mathbf{r}\times\mathbf{v})]\mathbf{r}+r^2\mathbf{S}^{(2)}\times\mathbf{v}-\tfrac{3}{2}(\mathbf{v}\cdot\mathbf{r})\mathbf{S}^{(2)}\times\mathbf{r}\}, \tag{65c}$$

$$\mathbf{B}^{(1,2)}=-\frac{3G}{c^2r^5m_{01}}[(\mathbf{S}^{(2)}\cdot\mathbf{r})\mathbf{S}^{(1)}+(\mathbf{S}^{(1)}\cdot\mathbf{r})\mathbf{S}^{(2)}-5(\mathbf{S}^{(1)}\cdot\mathbf{r})\times(\mathbf{S}^{(2)}\cdot\mathbf{r})\mathbf{r}/r^2+(\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)})\mathbf{r}], \tag{65d}$$

$$\mathbf{B}^{(Q)}=-\frac{3GJ_2m_2}{2r^5}\times\left\{\left[1-\frac{5(\mathbf{n}^{(2)}\cdot\mathbf{r})^2}{r^2}\right]\mathbf{r}+2(\mathbf{n}^{(2)}\cdot\mathbf{r})\mathbf{n}^{(2)}\right\}. \tag{65e}$$

$\mathbf{B}$ thus represents the correction to the Newtonian force per unit mass demanded by the general theory of relativity.

For a Newtonian elliptic orbit around a spherically symmetric body, the energy E, the orbital angular momentum $\mathbf{L}$, and the Runge-Lenz vector $\mathbf{A}$ are constants of the motion. They can be written as

$$E=m_{01}(v^2/2-Gm_2/r), \tag{66}$$

[21] D. W. Smith, Planet. Space Sci. **13**, 1151 (1965).

[22] This is not one of the higher-order terms mentioned in footnote 13. For a classical derivation of this term (for nonspinning particles), use the Lagrangian $\mathfrak{L}_0=-m_0c(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}$, where $g_{\mu\nu}$ is the Schwarzschild solution expressed in isotropic or harmonic coordinates, and then expand $\mathfrak{L}_0$ as a power series in G and v^2/c^2. Note that the field theory results are derived in the harmonic coordinate system[6,7] and, in addition, this system is identical with the isotropic system up to terms of order Gv^2/c^2 or G^2.

$$\mathbf{L}=m_{01}(\mathbf{r}\times\mathbf{v}), \tag{67}$$

$$\mathbf{A}=m_{01}[\mathbf{v}\times(\mathbf{r}\times\mathbf{v})-Gm_2\mathbf{r}/r]. \tag{68}$$

Taking the time derivative of Eqs. (66)–(68) and using Eq. (63), we obtain

$$\dot{E}=m_{01}(\mathbf{v}\cdot\mathbf{B}), \tag{69}$$

$$\dot{\mathbf{L}}=m_{01}(\mathbf{r}\times\mathbf{B}), \tag{70}$$

$$\dot{\mathbf{A}}=m_{01}[\mathbf{v}\times(\mathbf{r}\times\mathbf{B})+\mathbf{B}\times(\mathbf{r}\times\mathbf{v})]. \tag{71}$$

Explicitly writing Eqs. (69)–(71) with the values of Eq. (65) leads to rather lengthy expressions. The results that are of interest are the secular results, the time average of Eqs. (69)–(71) over a complete ellipse. After a rather lengthy calculation, we find that

$$\dot{E}_{\text{av}}=0, \tag{72}$$

$$\dot{\mathbf{L}}_{\text{av}}=\mathbf{\Omega}^*\times\mathbf{L}, \tag{73}$$

$$\dot{\mathbf{A}}_{\text{av}}=\mathbf{\Omega}^*\times\mathbf{A}, \tag{74}$$

where

$$\mathbf{\Omega}^*=\mathbf{\Omega}^{(E)}+\mathbf{\Omega}^{(1)}+\mathbf{\Omega}^{(2)}+\mathbf{\Omega}^{(1,2)}+\mathbf{\Omega}^{(Q)}, \tag{75}$$

and $\mathbf{\Omega}^{(E)}$, $\mathbf{\Omega}^{(1)}$, $\mathbf{\Omega}^{(2)}$, $\mathbf{\Omega}^{(1,2)}$, and $\mathbf{\Omega}^{(Q)}$ are the results corresponding to $\mathbf{B}^{(E)}$, $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$, $\mathbf{B}^{(1,2)}$, and $\mathbf{B}^{(Q)}$, respectively. The final form is

$$\mathbf{\Omega}^{(E)}=\frac{3GLm_2/m_{01}}{c^2a^3(1-e^2)^{3/2}}\mathbf{n}=2\mathbf{\Omega}_{\text{DS av}}, \tag{76a}$$

$$\mathbf{\Omega}^{(1)}=\frac{3GS^{(1)}m_2/m_{01}}{2c^2a^3(1-e^2)^{3/2}}[\mathbf{n}^{(1)}-3(\mathbf{n}\cdot\mathbf{n}^{(1)})\mathbf{n}], \tag{76b}$$

$$\mathbf{\Omega}^{(2)}=\frac{2GS^{(2)}}{c^2a^3(1-e^2)^{3/2}}[\mathbf{n}^{(2)}-3(\mathbf{n}\cdot\mathbf{n}^{(2)})\mathbf{n}] =4\mathbf{\Omega}_{\text{LT av}}, \tag{76c}$$

$$\mathbf{\Omega}^{(1,2)}=\frac{-3GS^{(1)}S^{(2)}/L}{2c^2a^3(1-e^2)^{3/2}}\{(\mathbf{n}\cdot\mathbf{n}^{(1)})\mathbf{n}^{(2)}+(\mathbf{n}\cdot\mathbf{n}^{(2)})\mathbf{n}^{(1)} +[\mathbf{n}^{(1)}\cdot\mathbf{n}^{(2)}-5(\mathbf{n}\cdot\mathbf{n}^{(1)})(\mathbf{n}\cdot\mathbf{n}^{(2)})]\mathbf{n}\}, \tag{76d}$$

$$\mathbf{\Omega}^{(Q)}=\frac{-3Gm_{01}m_2J_2c^2/L}{4c^2a^3(1-e^2)^{3/2}} \times\{2(\mathbf{n}\cdot\mathbf{n}^{(2)})\mathbf{n}^{(2)}+[1-5(\mathbf{n}\cdot\mathbf{n}^{(2)})^2]\mathbf{n}\}. \tag{76e}$$

The factors in Eq. (76) can be altered in various ways by using the relations of Eq. (30). The term $\mathbf{\Omega}^{(E)}$ gives the familiar result of Einstein, while the term $\mathbf{\Omega}^{(2)}$ was first given by Lense and Thirring.[11] The term due the quadrupole moment, $\mathbf{\Omega}^{(Q)}$, is also a standard result. The terms $\mathbf{\Omega}^{(1)}$ and $\mathbf{\Omega}^{(1,2)}$ are new results. As $\mathbf{\Omega}^*$ is the same for Eqs. (73) and (74) the ellipse precesses as a whole with the angular velocity Ω^*.

We can write Ω^* in the form that the astronomers or experimentalists use as

$$\Omega^*=\frac{d\Omega'}{dt}\mathbf{n}^{(2)}+\frac{d\omega'}{dt}\mathbf{n}+\frac{di'}{dt}\frac{(\mathbf{n}^{(2)}\times\mathbf{n})}{|\mathbf{n}^{(2)}\times\mathbf{n}|}, \tag{77}$$

where Ω', ω', and i' denote the longitude of the ascending node, the argument of the perihelion, and the inclination of the orbit, respectively, in the heavy mass's equatorial system.[23,24] The terms $\mathbf{\Omega}^{(E)}$, $\mathbf{\Omega}^{(2)}$, and $\mathbf{\Omega}^{(Q)}$ depend only on $\mathbf{n}^{(2)}$ and $\mathbf{n}$, but, since the presence of $(\mathbf{n}^{(2)}\times\mathbf{n})$ in $\mathbf{\Omega}^*$ is necessary to change the inclination of the orbit, it follows that these terms do not cause the angle i' to change. The terms $\mathbf{\Omega}^{(1)}$ and $\mathbf{\Omega}^{(1,2)}$ depend also on $\mathbf{n}^{(1)}$, which can be written as

$$\mathbf{n}^{(1)}=\alpha\mathbf{n}^{(2)}+\beta\mathbf{n}+\gamma(\mathbf{n}^{(2)}\times\mathbf{n})/|\mathbf{n}^{(2)}\times\mathbf{n}|, \tag{78}$$

where

$$\begin{aligned}\alpha&=[\mathbf{n}^{(1)}\cdot\mathbf{n}^{(2)}-(\mathbf{n}^{(1)}\cdot\mathbf{n})(\mathbf{n}^{(2)}\cdot\mathbf{n})]/[1-(\mathbf{n}^{(2)}\cdot\mathbf{n})^2],\\ \beta&=[\mathbf{n}^{(1)}\cdot\mathbf{n}-(\mathbf{n}^{(1)}\cdot\mathbf{n}^{(2)})(\mathbf{n}^{(2)}\cdot\mathbf{n})]/[1-(\mathbf{n}^{(2)}\cdot\mathbf{n})^2],\\ \gamma&=\mathbf{n}^{(1)}\cdot(\mathbf{n}^{(2)}\times\mathbf{n})/|\mathbf{n}^{(2)}\times\mathbf{n}|.\end{aligned} \tag{79}$$

Thus the terms $\mathbf{\Omega}^{(1)}$ and $\mathbf{\Omega}^{(1,2)}$ can cause a change in the inclination of the orbit.

We will now apply our result for $\mathbf{\Omega}^{(1)}$ to a calculation of the effect of the rotation of Mercury on the precession of its perihelion, i.e., the effect of $\mathbf{\Omega}^{(1)}$ on the angle ω'. Now it is clear from Eq. (76) that $\Omega^{(1)}/\Omega^{(2)}$ is of the order of magnitude of $(R^{(1)}/R^{(2)})^2$, where $R^{(1)}$ and $R^{(2)}$ are the radii of masses m_{01} and m_2, respectively. In the case of Mercury (m_{01}) and the Sun (m_2), we have

$$R^{(1)}/R^{(2)}=3.6\times10^{-3}, \tag{80}$$

and since the Lense-Thirring term $\mathbf{\Omega}^{(2)}$ only contributes about $-0.003''$/century to the precession of the perihelion, it is clear that the contribution of $\mathbf{\Omega}^{(1)}$ is negligibly small. The situation will be similar in the case of the precession of the gyroscope around the Earth. Due to the presence of the factor $S^{(1)}/L$ in the ratio $\Omega^{(1,2)}/\Omega^{(2)}$, it is clear that the contribution from $\mathbf{\Omega}^{(1,2)}$ is even smaller still. For the same reason, the effects of $\mathbf{\Omega}^{(1)}$ and $\mathbf{\Omega}^{(1,2)}$ on the angles Ω' and i' are negligible.

VI. CONCLUSION

By using a potential[12] derived from the quantum theory of gravitation, we have found the classical motion of a spherical gyroscope in the gravitational field

[23] Whereas observations of Earth satellite orbits are referred to the Earth's equatorial system, it should be noted that the observations of all planetary orbits are described with respect to the equinox and ecliptic of a given epoch. A discussion of this point as well as details of the equations needed to transform from the Sun's equational system to a system based on the equinox and ecliptic of a given epoch may be found in Ref. 24.

[24] I. I. Shapiro, Icarus **4**, 549 (1965); R. F. O'Connell, Astrophys. J. Letters **152**, L11 (1968).

of a much larger mass with a quadrupole moment. Our method of derivation is shorter and more straightforward than that of Papapetrou and Corinaldesi[3,4] as we have made use of familiar Lagrangian concepts.

We made the approximation of one mass much larger than the other only for mathematical simplicity and to be able to compare our results with previous results. In fact, Eq. (1) from which we start has *not* been subject to the large-mass approximation, and thus we could have proceeded in the same manner without this approximation.

For the precession of the spin, besides the usual de Sitter and Lense-Thirring terms, we derived in detail the effect of the quadrupole moment of the earth. This result gives a contribution of about 0.01″/yr, which is about 10 times larger than the expected experimental error.[20]

For the precession of the orbit, besides the usual results, we found the effect of the spin of the lighter mass. Applying this to the case of Mercury we found that the rotation of Mercury had a negligible effect on the precession of its orbit.

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APPENDIX

In calculating the expressions in Eqs. (28), (29), (45), and (76), the time average value of a number of quantities had to be determined. If we introduce a special coordinate system so that the orbit is in the x-y plane with the perihelion in the x direction and the orbital angular momentum in the z direction, we obtain

$$(r^{-2})_{\text{av}}=\frac{1}{a^2(1-e^2)^{1/2}},\quad (r^{-3})_{\text{av}}=\frac{1}{a^3(1-e^2)^{3/2}},\quad (r^{-4})_{\text{av}}=\frac{2+e^2}{2a^4(1-e^2)^{5/2}},$$
$$(r^{-5})_{\text{av}}=\frac{2+3e^2}{2a^5(1-e^2)^{7/2}},\quad (r^{-6})_{\text{av}}=\frac{8+24e^2+3e^4}{8a^6(1-e^2)^{9/2}},\tag{A1}$$

$$(xr^{-3})_{\text{av}}=0,\quad (xr^{-4})_{\text{av}}=\frac{e}{2a^3(1-e^2)^{3/2}},\quad (xr^{-5})_{\text{av}}=\frac{e}{a^4(1-e^2)^{5/2}},\quad (yr^{-3})_{\text{av}}=0,\quad (yr^{-4})_{\text{av}}=0,\quad (yr^{-5})_{\text{av}}=0,\tag{A2}$$

$$(x^2r^{-5})_{\text{av}}=\frac{1}{2a^3(1-e^2)^{3/2}},\quad (y^2r^{-5})_{\text{av}}=\frac{1}{2a^3(1-e^2)^{3/2}},\quad (xyr^{-5})_{\text{av}}=0,$$
$$(x^2r^{-7})_{\text{av}}=\frac{4+9e^2}{8a^5(1-e^2)^{7/2}},\quad (y^2r^{-7})_{\text{av}}=\frac{4+3e^2}{8a^5(1-e^2)^{7/2}},\quad (xyr^{-7})_{\text{av}}=0,\tag{A3}$$

$$(\dot{x}r^{-3})_{\text{av}}=0,\quad (\dot{y}r^{-3})_{\text{av}}=\frac{3eL/m_{01}}{2a^4(1-e^2)^{5/2}},\tag{A4}$$

$$(\dot{x}x^2r^{-5})_{\text{av}}=0,\quad (\dot{x}y^2r^{-5})_{\text{av}}=0,\quad (\dot{x}xyr^{-5})_{\text{av}}=\frac{-eL/m_{01}}{8a^4(1-e^2)^{5/2}},\quad (\dot{y}x^2r^{-5})_{\text{av}}=\frac{7eL/m_{01}}{8a^4(1-e^2)^{5/2}},$$
$$(\dot{y}y^2r^{-5})_{\text{av}}=\frac{5eL/m_{01}}{8a^4(1-e^2)^{5/2}},\quad (\dot{y}xyr^{-5})_{\text{av}}=0,\tag{A5}$$

$$(x\dot{x}r^{-5})_{\text{av}}=0,\quad (y\dot{y}r^{-5})_{\text{av}}=0,\quad (x\dot{y}r^{-5})_{\text{av}}=\frac{(4+11e^2)L/m_{01}}{8a^5(1-e^2)^{7/2}},\quad (y\dot{x}r^{-5})_{\text{av}}=\frac{-(4+e^2)L/m_{01}}{8a^5(1-e^2)^{7/2}}.\tag{A6}$$