Conservation of energy is well satisfied in our model by having the mean multiplicity of pions as the exponent of the Poisson distribution. Compared with the Fermi model,[5] the calculation presented here is much simpler because no tedious phase-space integration is involved. Our model is expected to work well at cosmic-ray energies too. Besides, one can use the model to express all inelastic partial cross sections at various multiplicities in terms of a single one. As the new high-energy accelerators begin to accumulate data, our proposal would provide a convenient way to systematize high-energy data.

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Current Commutators, Dispersion Relations, and K_{l3} Form Factors

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We present a treatment of K_{l3} decays based on current commutators and using a covariant sum-rule formulation. Working in a general kinematical configuration and dispersing on a line (in the plane of the squared pion mass and of the momentum transfer) which belongs to a family of curves all parallel to the "steepest" (kaon rest frame) parabola, we write down definite expressions for the form factors f_+ and f_-. The relation of our procedure to other recently reported approaches is discussed.

I. INTRODUCTION

RECENTLY, Ademollo, Denardo, and Furlan,[1] using the techniques of Ref. 2, gave a generalization of the Callan-Treiman soft-pion relation[3] between the K_{l3} form factors and those of the axial-vector decays of the K and π mesons, in which the pion mass was extrapolated to the physical value by performing a saturation of certain commutators in the kaon rest frame. In order to obtain expressions for the K_{l3} form factors as functions of the momentum transfer, it is of course necessary to work in a more general kinematical configuration. One possibility, actually used by Denardo and Komen,[4] is to saturate the relevant commutators by sandwiching them between the vacuum and a K-meson state of arbitrary energy E. The intergration path is then a curve in the plane of the squared pion four-momentum (q^2) and of the momentum transfer (k^2), which belongs to a family of parabolas (labeled by E) always passing through the Callan-Treiman point [$q^2=0$, $k^2=(\text{kaon mass})^2$]. In this paper, applying the generalized sum-rule formalism given in Ref. 2, we shall consider another possible way of getting the momentum-transfer dependence of the K_{l3} form factors. In order to work covariantly, we shall disperse on a line belonging to a set of parabolas all parallel to the steepest one (that corresponding to the kaon rest frame).

In Sec. II, we start from current- and field-algebra commutators and we derive a set of sum rules, the evaluation of which is treated in Sec. III, where we find, in the one-particle saturation scheme, definite expressions for the K_{l3} form factors $f_\pm$. An alternative procedure, based on the use of dispersion relations for $f_\pm$ in k^2 at $q^2=(\text{pion mass})^2$, together with a current-algebra sum rule and a field-algebra (superconvergent-type) relation, is developed in Sec. IV. The results of this work, as well as its relation to Refs. 1 and 4, are discussed in Sec. V.

II. GENERALIZED SUM RULES

In this section we use the procedure proposed by Fubini and Furlan[2] to derive sum rules for the invariant weak amplitudes related to the K_{l3} decays in the same way as it was done for axial-vector–nucleon-scattering amplitudes in Ref. 5. We start by defining the quantities

$$T_{\mu\nu}=\int d^4z\, e^{iqz}\theta(z_0)\times\langle 0|[A_\mu{}^{(3)}(z),V_\nu{}^{(K^+)}(0)]|K^-(p)\rangle\,, \quad (2.1)$$

$$U_\mu=i\int d^4z\, e^{iqz}\theta(z_0)\times\langle 0|[D_A{}^{(3)}(z),V_\mu{}^{(K^+)}(0)]|K^-(p)\rangle\,, \quad (2.2)$$

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with the corresponding absorptive parts

$$t_{\mu\nu}=\frac{1}{2i}\int d^4z\, e^{iqz}\langle 0|[A_\mu{}^{(3)}(z),V_\nu{}^{(K^+)}(0)]|K^-(p)\rangle\,, \quad (2.1')$$

$$u_\mu=\frac{1}{2}\int d^4z\, e^{iqz}\langle 0|[D_A{}^{(3)}(z),V_\mu{}^{(K^+)}(0)]|K^-(p)\rangle\,, \quad (2.2')$$

where V_μ (A_μ) are the vector (axial-vector) weak currents labeled by the SU_3 superscripts. Equations (2.1) and (2.2) reflect the transitions $K^-(p)\rightarrow A_\mu{}^{(3)}(q)+V_\mu{}^{(K^-)}(k)$ and $K^-(p)\rightarrow D_A{}^3(q)+V_\mu{}^{(K^-)}(k)$, respectively, with $k=p-q$, $p^2=m^2$, and $D_A{}^{(3)}\equiv\partial_\mu A_\mu{}^{(3)}$.

We introduce, as in Ref. 2, the variable x and the four-vector q' through the parametrization

$$q=xp+q',\quad (q'\cdot p)=0 \quad (2.3)$$

$$k=(1-x)p-q'. \quad (2.4)$$

Using the equal-time commutation relation

$$\delta(z_0)[A_0{}^{(3)}(z),V_\mu{}^{(K^+)}(0)]=\tfrac{1}{2}A_\mu{}^{(K^+)}(0)\delta^4(z)\,, \quad (2.5)$$

we have the following Ward identity:

$$q_\mu T_{\mu\nu}=U_\nu-\tfrac{1}{2}f_Kp_\nu,\quad q_\mu t_{\mu\nu}=u_\nu\,, \quad (2.6)$$

where f_K is defined by

$$\langle 0|A_\mu{}^{(K^+)}|K^-(p)\rangle=if_Kp_\mu. \quad (2.7)$$

The Ward identity can be written as

$$xR_\nu+S_\nu=U_\nu-\tfrac{1}{2}f_Kp_\nu,\quad xr_\nu+s_\nu=u_\nu\,, \quad (2.8)$$

where R_ν, S_ν and their corresponding absorptive parts r_ν, s_ν have been defined as

$$R_\nu\equiv p_\mu T_{\mu\nu},\quad S_\nu\equiv q_\mu' T_{\mu\nu}\,, \quad (2.9a)$$

$$r_\nu\equiv p_\mu t_{\mu\nu},\quad s_\nu\equiv q_\mu' t_{\mu\nu}. \quad (2.9b)$$

Projecting on the basis (p_ν,q_ν'), we have the following decompositions into invariant amplitudes:

$$U_\nu=U_1p_\nu+U_2q_\nu',\quad u_\nu=u_1p_\nu+u_2q_\nu'\,, \quad (2.10)$$

$$R_\nu=R_1p_\nu+R_2q_\nu',\quad r_\nu=r_1p_\nu+r_2q_\nu'\,, \quad (2.11)$$

$$S_\nu=S_1p_\nu+S_2q_\nu',\quad s_\nu=s_1p_\nu+s_2q_\nu'. \quad (2.12)$$

The arguments of all the invariant functions above are taken to be x and q'^2. [$U_1=U_1(x,q'^2)$, etc.] In terms of invariant functions, the Ward identity, Eq. (2.8), can be split into the following relations:

$$xR_1+S_1=U_1-\tfrac{1}{2}f_K,\quad xr_1+s_1=u_1\,, \quad (2.13)$$

$$xR_2+S_2=U_2,\quad xr_2+s_2=u_2. \quad (2.14)$$

Supposing now that R_1, S_1, U_1, R_2, S_2, and U_2 satisfy unsubtracted dispersion relations in the variable x at fixed q'^2, the following sum rules are obtained from Eqs. (2.13) and (2.14) by the usual procedure:

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} r_1(x,q'^2)\,dx=\tfrac{1}{2}f_K\,, \quad (2.15)$$

$$\int_{-\infty}^{+\infty} r_2(x,q'^2)\,dx=0. \quad (2.16)$$

Defining now

$$\tilde{R}_\mu\equiv p_\nu T_{\mu\nu}=\tilde{R}_1p_\mu+\tilde{R}_2q_\mu',\quad \tilde{r}_\mu\equiv p_\mu t_{\mu\nu}=\tilde{r}_1p_\mu+\tilde{r}_2q_\mu'\,, \quad (2.17)$$

and using the Ward identity

$$k_\nu T_{\mu\nu}=V_\mu+\tfrac{1}{2}f_Kp_\mu\,, \quad (2.18)$$

where

$$V_\mu\equiv -i\int d^4z\, e^{-ikz}\theta(z_0)\times\langle 0|[A_\mu{}^{(3)}(0),\partial_\nu V_\nu{}^{(K^+)}(-z)]|K^-(p)\rangle=V_1p_\mu+V_2q_\mu'\,, \quad (2.19a)$$

$$v_\mu=-\int d^4z\, e^{-ikz}\langle 0|[A_\mu{}^{(3)}(0),\partial_\nu V_\nu{}^{(K^+)}(-z)]|K^-(p)\rangle=v_1p_\mu+v_2q_\mu'\,, \quad (2.19b)$$

under the hypothesis of unsubtracted dispersion relations in x at fixed q'^2 for the relevant amplitudes, we obtain the following sum rules:

$$\frac{1}{\pi}\int \tilde{r}_1(x,q^{12})\,dx=\tfrac{1}{2}f_K\,, \quad (2.20)$$

$$\int \tilde{r}_2(x,q^{12})\,dx=0. \quad (2.21)$$

We shall now consider the Ward identity obtained with a double contraction

$$q_\mu k_\nu T_{\mu\nu}=-W-\tfrac{1}{2}f_K(k\cdot p)+S. \quad (2.22)$$

Here we have denoted

$$W=\int d^4z\, e^{iqz}\theta(z_0)\langle 0|[D_A{}^{(3)}(z),D_V{}^{(K^+)}(0)]|K^-(p)\rangle\quad (D_V{}^{(K^+)}\equiv\partial_\mu V_\mu{}^{(K^+)})\,, \quad (2.23)$$

and S is the "σ" term

$$S=\int d^4z\, e^{iqz}\delta(z_0)\times\langle 0|[D_A{}^{(3)}(z),V_0{}^{(K^+)}(0)]|K^-(p)\rangle. \quad (2.24)$$

Equation (2.22) can be written as

$$q_\mu k_\nu T_{\mu\nu}=x(1-x)C-xD+E\,, \quad (2.25)$$

where

$$C \equiv p_\mu p_\nu T_{\mu\nu}, \quad D \equiv (p_\mu q_\nu' + p_\nu q_\mu') T_{\mu\nu}, \quad E \equiv (q_\mu' p_\nu - q_\mu' q_\nu') T_{\mu\nu}, \tag{2.26}$$

and we shall denote everywhere (as before) by small letters the corresponding absorptive parts of the quantities W, C, D, and E (i.e., by w, c, d, and e). With the same procedure, we get from Eq. (2.22) the following sum rules:

$$\frac{1}{\pi} \int c(x,q'^2)\, dx = \tfrac{1}{2} f_K m^2, \tag{2.27}$$

$$\frac{1}{\pi} \int [d(x,q'^2) + x c(x,q'^2)]\, dx = S. \tag{2.28}$$

Equations (2.15), (2.16), (2.20), (2.21), (2.27), and (2.28) are consequences of the algebra of current densities and the hypothesis of unsubtracted dispersion relations in the variable x at fixed q'^2, made for certain weak amplitudes. Of course, not all these sum rules are independent. For instance, it is immediately seen that, in fact, Eqs. (2.15), (2.20), and (2.24) are all equivalent.

Further, we shall make use of the following set of sum rules of superconvergent type:

$$\int_{-\infty}^{+\infty} u_2(x,q'^2)\, dx = 0, \tag{2.29}$$

$$\int_{-\infty}^{+\infty} v_2(x,q'^2)\, dx = 0, \tag{2.30}$$

$$\int_{-\infty}^{+\infty} w(x,q'^2)\, dx = 0. \tag{2.31}$$

We are postulating these sum rules, taking as a guide the Bjorken procedure to find the asymptotic behavior of U_2, V_2, and W for $x \to \infty$, controlled, respectively, by the equal-time commutators $[\partial_\mu A_\mu{}^{(3)}, V_i{}^{(K^+)}]$, $[A_i{}^{(3)}, \partial_\nu V_\nu{}^{(K^+)}] (i=1,2,3)$ and $[\partial_\mu A_\mu{}^{(3)}, \partial_\nu V_\nu{}^{(K+)}]$. (This can easily be seen by specializing the considerations to the rest frame of the kaon, when $\mathbf{p}=0$, $q_0'=0$, $\mathbf{q}=\mathbf{q}'$, and $q_0 = xm$.) The Bjorken procedure, if these commutators are taken to be 0 or c numbers (as will indeed be the case, for example, in field-algebra models with partial conservation of A_μ or V_μ, when the divergences can be viewed as pseudoscalar or scalar fields), would predict that U_2, V_2, and W vanish, when $x \to \infty$, faster than $1/x$, implying then the superconvergent relations (2.29)–(2.31).

The sum rules derived or postulated in this section represent a generalization to the general kinematical situation of the results found in the collinear configuration ($q'=0$) in Ref. 1. To visualize this better, we rewrite, for instance, Eqs. (2.27), (2.28), and (2.15) in the equivalent form

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{w+e-xd}{x(x-1)}\, dx = \tfrac{1}{2} f_K m^2, \tag{2.27'}$$

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{w+e}{x}\, dx = -\tfrac{1}{2} f_K m^2 + S, \tag{2.28'}$$

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{u_1 - s_1}{x}\, dx = \tfrac{1}{2} f_K, \tag{2.15'}$$

the arguments of the integrands being (x,q'^2). For $q'=0$ we have automatically $s_1=0$, $e=0$, and $d=0$, and our sum rules (2.15), (2.27′), (2.28′), and (2.29) become in this ("collinear") case, respectively, Eqs. (31), (34), (32), and (38) of Ref. 1.

The sum rules (2.15), (2.16), (2.21), and (2.29)–(2.31) represent the basis of our further discussion. By picking up the relevant one-particle contributions, we shall be able to write down definite expressions for the physical K_{l3} form factors.

III. EVALUATION OF SUM RULES

To obtain an insight into the q'^2 dependence of the sum rules derived in Sec. II, we shall now start to study the kinematics of the x integration at fixed q'^2. The dispersion path is a parabola in the q^2, k^2 plane, dependent on the parameter q'^2 and given by the equation

$$(k^2 - q^2 - m^2)^2 = 4m^2(q^2 - q'^2), \tag{3.1}$$

which can be obtained immediately by eliminating x between the expressions of q^2 and $k^2=(p-q)^2$, with q and k parametrized as in Eqs. (2.3) and (2.4). In the plane v, u, where

$$v = k^2 + q^2, \quad u = k^2 - q^2, \tag{3.2}$$

Eq. (3.1) takes on the form

$$v = u^2/2m^2 + \tfrac{1}{2} m^2 + 2q'^2. \tag{3.3}$$

When q'^2 varies, we have a family of parallel parabolas whose common symmetry axis is the line $u=0$. For $q'^2=0$ we get the parabola corresponding to the (kaon) rest-frame saturation. This was the dispersive path used in Ref. 1, and it passes through the Callan-Treiman point [$q^2=0$, $k^2=m^2$, or, equivalently, $u=v=m^2$ in the (u,v) plane]. As shown in Ref. 1, where a full discussion concerning the frame dependence of the saturation of an equal-time commutator in the case of three-point functions has been given, the most general dispersion path is a parabola in the (q^2,k^2) plane depending on the external kinematical variables. The parabola has a maximum curvature in the case of the (kaon) rest frame and becomes a straight line ($q^2=0$) in the infinite-momentum system. Dispersing on the steepest (rest-frame) parabola in Ref. 1, relations have been obtained for the K_{l3} form factors at the momentum-transfer value $k^2=(m-\mu)^2$. To find the momentum-transfer dependence of the form factors, it is necessary to work in a more general kinematical situation. Our procedure will consist in dispersing on a parabola belonging to the family (3.3), depending parametrically on q'^2. This will allow us to obtain expressions for the K_{l3} form factors working covariantly. Another way to generalize the collinear configuration ($q=xp$) considered in Ref. 1 has been used in Ref. 4, by particularizing the considerations to the case $\mathbf{q}=0$, when one has a set of parabolas tangent to the infinite-momentum line ($u=v$) in the Callan-Treiman point $k^2=m^2$, $q^2=0$ ($u=v=m^2$) and dependent on the parameter E, the energy of the kaon.

The equation of this family of curves is

$$v=u^2/2E^2+m^4/2E^2+u(E^2-m^2)/E^2. \qquad (3.4)$$

Our set of parabolas [Eq. (3.3)] and that given by the above formula have in common only the steepest parabola [obtained for $q'^2=0$ from Eq. (3.3) and for $E=m$ from Eq. (3.4)]:

$$v=u^2/2m^2+\tfrac{1}{2}m^2. \qquad (3.5)$$

From this point of view, our approach and that followed in Ref. 4 are two different possible generalizations of the results found in Ref. 1.

We shall now discuss the contributions to the integrands in our sum rules. Following the results of Ref. 1, the general structure of $t_{\mu\nu}$ can be shown to be

$$t_{\mu\nu}=t_{\mu\nu}{}^{\mathrm{I}}+t_{\mu\nu}{}^{\mathrm{II}},$$

$$\begin{aligned}t_{\mu\nu}{}^{\mathrm{I}}=\frac{(2\pi)^4}{2i}&\Big[\sum_n \delta(q-p_n)\langle 0|A_\mu{}^{(3)}(0)|n\rangle\\ &\times\langle n|V_\nu{}^{(K^+)}(0)|K^-(p)\rangle-\sum_n \delta(q+p_n)\\ &\times\langle 0|V_\nu{}^{(K^+)}(0)|n,K^-(p)\rangle\langle n|A_\mu{}^{(3)}(0)|0\rangle\Big], \qquad (3.6)\end{aligned}$$

$$\begin{aligned}t_{\mu\nu}{}^{\mathrm{II}}=\frac{(2\pi)^4}{2i}&\Big[-\sum_m \delta(k-p_m)\langle 0|V_\nu{}^{(K^+)}|m\rangle\\ &\times\langle m|A_\mu{}^{(3)}|K^-(p)\rangle+\sum_m \delta(k+p_m)\\ &\times\langle 0|A_\mu{}^{(3)}|m,K^-(p)\rangle\langle m|V_\nu{}^{(K^+)}|0\rangle\Big].\end{aligned}$$

The contributions to the sum rules can be computed using the above decomposition as well as the relations between the respective absorptive parts and $t_{\mu\nu}$.

We shall begin by taking into account the pion state contribution (in class I). With Eq. (3.6), we have

$$\begin{aligned}t_{\mu\nu}{}^{\mathrm{I}(\pi)}=\tfrac{1}{2}\pi\delta(q^2-\mu^2)\epsilon(x)f_\pi(xp_\mu+q_\mu')&\\ \times\{f_+(k^2)[(1+x)p_\nu+q_\nu']&\\ +f_-(k^2)[(1-x)p_\nu-q_\nu']\}, &\qquad (3.7)\end{aligned}$$

where we used the notations

$$\langle 0|A_\mu{}^{(3)}|\pi^0(q)\rangle=i(f_\pi/\sqrt{2})q_\mu, \qquad (3.8)$$

$$\begin{aligned}\langle\pi^0(q)|V_\mu{}^{(K^+)}|K^-(p)\rangle=(1/\sqrt{2})\{&f_+[(p-q)^2](p+q)_\mu\\ &+f_-[(p-q)^2](p-q)_\mu\}. \qquad (3.9)\end{aligned}$$

In connection with this pion contribution, we introduce the following kinematical relations:

$$\begin{aligned}\delta(q^2-\mu^2)=(1/2x_\pi m^2)[\delta(x-x_\pi)+\delta(x-\bar{x}_\pi)]&\\ (\mu=\text{pion mass}),&\\ x_\pi\equiv+[(\mu^2-q'^2)/m^2]^{1/2}=-\bar{x}_\pi,&\\ k_\pi{}^2=(1-x_\pi)^2m^2+q'^2,\quad \bar{k}_\pi{}^2=(1-\bar{x}_\pi)^2m^2+q'^2.&\end{aligned} \qquad (3.10)$$

It will be particularly convenient to keep for further discussion the sum rules (2.29) and (2.31), and the following combinations of the sum rules (2.16), (2.21) and (2.27), (2.29):

$$\int[r_2(x,q'^2)-\bar{r}_2(x,q'^2)]\,dx=0, \qquad (3.11)$$

$$\frac{1}{\pi}\int\left[c(x,q'^2)-\frac{x_\pi{}^2m^4}{\mu^2}u_2(x,q'^2)\right]dx=\tfrac{1}{2}f_Km^2. \qquad (3.12)$$

(The other independent sum rules could provide some helpful constraints, e.g., on the continuum contributions to our relations.)

Picking up the pion contributions to Eqs. (2.29), (2.31), (3.11), and (3.12), we obtain the following expressions involving $f_\pm(k_\pi{}^2)$ and $f_\pm(\bar{k}_\pi{}^2)$, respectively:

$$\begin{aligned}\tfrac{1}{2}[f_+(k_\pi{}^2)-f_-(k_\pi{}^2)-f_+(\bar{k}_\pi{}^2)+f_-(\bar{k}_\pi{}^2)]&\\ =-\frac{2x_\pi m^2}{\mu^2f_\pi\pi}\int u_2(x,q'^2)\,dx,&\qquad (3.13)\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}[(m^2-\mu^2)f_+(k_\pi{}^2)+k_\pi{}^2f_-(k_\pi{}^2)&\\ -(m^2-\mu^2)f_+(\bar{k}_\pi{}^2)-\bar{k}_\pi{}^2f_-(\bar{k}_\pi{}^2)]&\\ =\frac{2x_\pi m^2}{\mu^2f_\pi\pi}\int' w(x,q'^2)\,dx,&\qquad (3.14)\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}[f_+(k_\pi{}^2)+f_-(k_\pi{}^2)-f_+(\bar{k}_\pi{}^2)-f_-(\bar{k}_\pi{}^2)]&\\ =\frac{2x_\pi}{\pi f_\pi}\int'[r_2(x,q'^2)-\bar{r}_2(x,q'^2)]\,dx,&\qquad (3.15)\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}[f_+(k_\pi{}^2)+f_-(k_\pi{}^2)+f_+(\bar{k}_\pi{}^2)+f_-(\bar{k}_\pi{}^2)]&\\ =\frac{f_K}{f_\pi}-\frac{2}{m^2f_\pi\pi}\int' c(x,q'^2)dx&\\ +\frac{2x_\pi{}^2m^2}{\mu^2f_\pi\pi}\int' u_2(x,q'^2)\,dx.&\qquad (3.16)\end{aligned}$$

The prime on the integration signs in the right-hand side of the above equations indicates that the pion contributions have already been taken off. The occurrence of the form factors evaluated at two different points results from the contributions of connected and semi-disconnected diagrams to the absorptive parts (Ref. 1).

Solving the system (3.13)–(3.16), we get the following expressions for the K_{l3} form factors as functions of the momentum transfer k_π^2:

$$f_+(k_\pi^2)=\frac{f_K}{f_\pi}+\frac{2}{\pi f_\pi}\int' dx\left\{\frac{1}{2\mu^2}w(x,q'^2)-\frac{1-x_\pi}{2}\times[r_2(x,q'^2)-\tilde{r}_2(x,q'^2)]-(1/m^2)c(x,q'^2)+\frac{1}{\mu^2}u_2(x,q'^2)(m^2x_\pi^2-\tfrac{1}{2}\mu^2-\tfrac{1}{2}m^2x_\pi)\right\}, \quad (3.17)$$

$$f_-(k_\pi^2)=\frac{2}{\pi f_\pi}\int' dx\left\{-\frac{1}{2\mu^2}w(x,q'^2)+\frac{1+x_\pi}{2}[r_2(x,q'^2)-\tilde{r}_2(x,q'^2)]+\frac{m^2x_\pi+\mu^2}{2\mu^2}u_2(x,q'^2)\right\}. \quad (3.18)$$

Up to this point, our equations are exact. Equations (3.17) and (3.18) allow, in principle, the determination of the K_{l3} form factors in terms of the contributions of different intermediate states to the integrals. From now on, we shall adopt the procedure of one-particle saturation. Besides the pion, the other one-particle states contributing to our sum rules are the κ and K^* mesons [in class II, according to the deomposition given in Eq. (3.6)] and the A_1 meson (in class I). The contributions of these states to $t_{\mu\nu}$ are, respectively,

$$t_{\mu\nu}{}^{\mathrm{II}(\kappa)}=-\pi\delta(k^2-m_\kappa^2)\epsilon(1-x)f_\kappa k_\nu\times[g_+(q^2)(2p_\mu-q_\mu)+q_\mu g_-(q^2)], \quad (3.19)$$

$$t_{\mu\nu}{}^{\mathrm{II}(K^*)}=-\pi\delta(k^2-m_*^2)\epsilon(1-x)f_*\{H_1(q^2)\times(-g_{\mu\nu}+k_\mu k_\nu/m_*^2)+[-p_\nu+(k\cdot p/m_*^2)k_\nu]\times[H_2(q^2)(2p_\mu-q_\mu)+H_3(q^2)q_\mu]\}, \quad (3.20)$$

$$t_{\mu\nu}{}^{\mathrm{I}(A_1)}=\pi\delta(q^2-m_A^2)\epsilon(x)f_A\{G_1(k^2)\times(-g_{\mu\nu}+q_\mu q_\nu/m_A^2)+[-p_\mu+(q\cdot p/m_A^2)q_\mu]\times[G_2(k^2)(p_\nu+q_\nu)+G_3(k^2)(p_\nu-q_\nu)]\}, \quad (3.21)$$

where the following definitions have been used:

$$\langle 0|V_\mu{}^{(K^+)}|\kappa(k)\rangle=f_\kappa k_\mu,$$

$$\langle\kappa(k)|A_\mu{}^{(3)}|K^-(p)\rangle=i[g_+(q^2)(p_\mu+k_\mu)+g_-(q^2)(p_\mu-k_\mu)],$$

$$\langle 0|V_\mu{}^{(K^+)}|K^{*-}(k)\rangle=f_*\epsilon_\mu{}^*(k),$$

$$\langle K^*(k)|A_\mu{}^{(3)}|k^-(p)\rangle=i\{H_1(q^2)\epsilon_\mu{}^*+(\epsilon^*\cdot p)[H_2(q^2)(p_\mu+k_\mu)+H_3(q^2)(p_\mu-k_\mu)]\}, \quad (3.22)$$

$$\langle 0|A_\mu{}^{(3)}|A_1{}^0(q)\rangle=f_{A_1}\epsilon_\mu{}^A(q),$$

$$\langle A_1{}^0(q)|V_\mu{}^{(K^+)}|K^-(p)\rangle=i\{\epsilon_\mu{}^A G_1(k^2)+(\epsilon^A\cdot p)[G_2(k^2)(p_\mu+q_\mu)+G_3(k^2)(p_\mu-q_\mu)]\}.$$

In connection with the one-particle contributions written above, we shall, further, need the following kinematical relations:

$$\delta(k^2-m_\kappa^2)=\frac{1}{2m^2(1-x_\kappa)}[\delta(x-x_\kappa)+\delta(x-\bar{x}_\kappa)] \quad (m_\kappa=\kappa\text{ mass}), \quad (3.23)$$

$$1-x_\kappa\equiv[(m_\kappa^2-q'^2)/m^2]^{1/2}\equiv\bar{x}_\kappa-1,$$

$$q_\kappa^2=x_\kappa^2m^2+q'^2,\quad \bar{q}_\kappa^2=\bar{x}_\kappa^2m^2+q'^2.$$

Concerning the A_1 and K^* poles, we introduce the variables x_A, k_A^2, $\bar{k}_A^2$ and x_*, q_*^2, $\bar{q}_*^2$, in analogy with Eqs. (3.10) and (3.23), respectively, with the obvious replacements $\mu\rightarrow m_A$ (m_A is the A_1 meson mass) and $m_\kappa\rightarrow m_*$ (m_* is the mass of the K^* meson). With the aid of Eqs. (3.19)–(3.21), we have immediately the one-particle contributions to all our sum rules, by contracting $t_{\mu\nu}$ with the relevant momenta and projecting the results on the basis p_μ, q_μ'. So we can now write down, using the sum rules (3.17) and (3.18) (in the one-particle saturation scheme), the following expressions for the K_{l3} form factors f_+ and f_-:

$$f_+(k_\pi^2)=f_K/f_\pi+\Delta_{(+)}{}^{(\kappa)}+\Delta_{(+)}{}^{(K^*)}+\Delta_{(+)}{}^{(A_1)}, \quad (3.24)$$

$$f_-(k_\pi^2)=\Delta_{(-)}{}^{(\kappa)}+\Delta_{(-)}{}^{(K^*)}+\Delta_{(-)}{}^{(A_1)}, \quad (3.25)$$

where

$$\Delta_{(+)}{}^{(\kappa)}\equiv\frac{f_\kappa}{m^2\sqrt{2}(1-x_\kappa)}\left(\frac{m_\kappa^2-\mu^2}{2}+m^2x_\pi^2-\frac{m^2x_\pi}{2}\right)\left(\frac{g_\kappa(q_\kappa^2)}{\mu^2-q_\kappa^2}-\frac{g_\kappa(\bar{q}_\kappa^2)}{\mu^2-\bar{q}_\kappa^2}\right)-\frac{f_\kappa}{f_\pi}\frac{1-x_\pi}{1-x_\kappa}\times[g_+(q_\kappa^2)+g_-(q_\kappa^2)-g^+(\bar{q}_\kappa^2)-g_-(\bar{q}_\kappa^2)]+(f_\kappa/f_\pi)[g_+(q_\kappa^2)(2-x_\kappa)+g_-(q_\kappa^2)x_\kappa+g_+(\bar{q}_\kappa^2)(2-\bar{x}_\kappa)+g_-(\bar{q}_\kappa^2)\bar{x}_\kappa], \quad (3.26)$$

$$\Delta_{(+)}{}^{(K^*)}=-\frac{1-x_\pi}{2}\frac{f_*}{f_\pi}\left[-H_2(q_*^2)\left(-\frac{1}{1-x_*}-\frac{m^2}{m_*^2}\right)+H_3(q^{*2})\left(-\frac{1}{1-x_*}+\frac{m^2}{m_*^2}\right)+H_2(\bar{q}_*^2)\left(-\frac{1}{1-x_*}+\frac{m^2}{m_*^2}\right)-H_3(\bar{q}_*^2)\left(-\frac{1}{1-x_*}-\frac{m^2}{m_*^2}\right)\right]+\frac{f_*}{f_\pi}\frac{1}{m^2(1-x_*)}\left[-1+(1-x_*)^2\frac{m^2}{m_*^2}\right]\times\{H_1(q_*^2)+m^2[(2-x_*)H_2(q_*^2)+x_*H_3(q_*^2)]-H_1(\bar{q}_*^2)-m^2[(2-\bar{x}_*)H_2(\bar{q}_*^2)+\bar{x}_*H_3(\bar{q}_*^2)]\}+(m^2x_\pi^2-\tfrac{1}{2}\mu^2-\tfrac{1}{2}m^2x_\pi)\frac{f_*}{m_*^2\sqrt{2}}\left[\frac{g_*(q_*^2)}{\mu^2-q_*^2}+\frac{g_*(\bar{q}_*^2)}{\mu^2-\bar{q}_*^2}\right], \quad (3.27)$$

$$\Delta_{(+)}{}^{(A_1)}=\frac{1-x_\pi}{2}\frac{f_A}{f_\pi}\left[G_2(k_A{}^2)\left(\frac{m^2}{m_A{}^2}+\frac{1}{x_A}\right)+\left(\frac{m^2}{m_A{}^2}-\frac{1}{x_A}\right)G_3(k_A{}^2)+G_2(\bar{k}_A{}^2)\left(\frac{m^2}{m_A{}^2}-\frac{1}{x_A}\right)+G_3(\bar{k}_A{}^2)\left(\frac{m^2}{m_A{}^2}+\frac{1}{x_A}\right)\right]$$
$$-\frac{f_A}{f_\pi}\frac{1}{x_Am^2}\left(-1+\frac{x_A{}^2m^2}{m_A{}^2}\right)[G_1(k_A{}^2)+m^2(1+x_A)G_2(k_A{}^2)+m^2(1-x_A)G_3(k_A{}^2)-G_1(\bar{k}_A{}^2)-m^2$$
$$\times(1-x_A)G_2(\bar{k}_A{}^2)-m^2(1+x_A)G_3(\bar{k}_A{}^2)], \quad (3.28)$$

$$\Delta_{(-)}{}^{(\kappa)}=\frac{f_\kappa}{\sqrt{2}m^2(1-x_\kappa)}\left[\frac{g_\kappa(q_\kappa{}^2)}{\mu^2-q_\kappa{}^2}-\frac{g(\bar{q}_\kappa{}^2)}{\mu^2-\bar{q}_\kappa{}^2}\right]\left(\frac{-m_\kappa{}^2+\mu^2+m^2x_\pi}{2}\right)$$
$$+\frac{1+x_\pi}{2}\frac{f_\kappa}{f_\pi(1-x_\kappa)}[g_+(q_\kappa{}^2)+g_-(q_\kappa{}^2)-g_+(\bar{q}_\kappa{}^2)-g_-(\bar{q}_\kappa{}^2)], \quad (3.29)$$

$$\Delta_{(-)}{}^{(K^*)}=\frac{1+x_\pi}{2}\frac{f_*}{f_\pi}\left[-H_2(q_*{}^2)\left(-\frac{1}{1-x_*}-\frac{m^2}{m_*{}^2}\right)+H_3(q_*{}^2)\left(-\frac{1}{1-x_*}+\frac{m^2}{m_*{}^2}\right)+H_2(\bar{q}_*{}^2)\left(-\frac{1}{1-x_*}+\frac{m^2}{m_*{}^2}\right)\right.$$
$$\left.-H_3(\bar{q}_*{}^2)\left(-\frac{1}{1-x_*}-\frac{m^2}{m_*{}^2}\right)\right]+\frac{f_*}{m_*{}^2\sqrt{2}}\frac{m^2x_\pi+\mu^2}{2}\left[\frac{g^*(q_*{}^2)}{\mu^2-q_*{}^2}+\frac{g_*(\bar{q}_*{}^2)}{\mu^2-\bar{q}_*{}^2}\right], \quad (3.30)$$

$$\Delta_{(-)}{}^{(A_1)}=-\frac{1+x_\pi}{2}\frac{f_A}{f_\pi}\left[G_2(k_A{}^2)\left(\frac{m^2}{m_A{}^2}+\frac{1}{x_A}\right)+G_3(k_A{}^2)\left(\frac{m^2}{m_A{}^2}-\frac{1}{x_A}\right)\right.$$
$$\left.+G_2(\bar{k}_A{}^2)\left(\frac{m^2}{m_A{}^2}-\frac{1}{x_A}\right)+G_3(\bar{k}_A{}^2)\left(\frac{m^2}{m_A{}^2}+\frac{1}{x_A}\right)\right]. \quad (3.31)$$

The form factors $g_\kappa(q^2)$ and $g_*(q^2)$, which appear in the above relations, are defined as follows:

$$\langle\kappa(k)|\partial_\mu A_\mu{}^{(3)}|K^-(p)\rangle=[g_+(q^2)(m^2-m_\kappa{}^2)+q^2g_-(q^2)]$$
$$\equiv\mu^2f_\pi g_\kappa(q^2)/\sqrt{2}(\mu^2-q^2), \quad (3.32)$$

$$\langle K^*(k)|\partial_\mu A_\mu{}^{(3)}|K^-(p)\rangle$$
$$=(\epsilon^*\cdot p)[H_1(q^2)+(m^2-m_*{}^2)H_2(q^2)+q^2H_3(q^2)]$$
$$\equiv(\epsilon^*\cdot p)\mu^2f_\pi g_*(q^2)/\sqrt{2}(\mu^2-q^2). \quad (3.33)$$

For $q^2=\mu^2$, $g_\kappa(q^2)$ and $g_*(q^2)$ become, respectively, the physical $g_{\kappa K\pi}$ and $g_{K^{*(-)}K^{(-)}\pi^0}$ coupling constants

$$g_{\kappa K\pi}=g_\kappa(\mu^2) \quad\text{and}\quad g_{K^*K\pi}=g_*(\mu^2). \quad (3.34)$$

Noting the kinematical relations

$$(k_\pi{}^2-m_*{}^2)(\bar{k}_\pi{}^2-m_*{}^2)=(q_*{}^2-\mu^2)(\bar{q}_*{}^2-\mu^2), \quad (3.35)$$
$$(k_\pi{}^2-m_\kappa{}^2)(\bar{k}_\pi{}^2-m_\kappa{}^2)=(q_\kappa{}^2-\mu^2)(\bar{q}_\kappa{}^2-\mu^2), \quad (3.36)$$
$$(k_A{}^2-m_\kappa{}^2)(\bar{k}_A{}^2-m_\kappa{}^2)=(q_\kappa{}^2-m_A{}^2)(\bar{q}_\kappa{}^2-m_A{}^2), \quad (3.37)$$
$$(k_A{}^2-m_*{}^2)(\bar{k}_A{}^2-m_*{}^2)=(q_*{}^2-m_A{}^2)(\bar{q}_*{}^2-m_A{}^2), \quad (3.38)$$

we see that the poles of f_+ and f_- at $k_\pi{}^2=m_*{}^2$ are given by the explicit factors $1/(q_*{}^2-\mu^2)$ and by the pole of $H_3(q_*{}^2)$ at $q_*{}^2=\mu^2$ in Eqs. (3.27) and (3.30). Also, the pole of f_- at $k_\pi{}^2=m_\kappa{}^2$ is given by the factor $1/(\mu^2-q_\kappa{}^2)$ and by the pole of $g_-(q_\kappa{}^2)$ at $q_\kappa{}^2=\mu^2$. The A_1 poles appearing at $q_\kappa{}^2=m_A{}^2$ in g_+ and g_- and at $q_*{}^2=m_A{}^2$ in $H_{1,2,3}$, of course, are exactly canceled by the poles of $G_{1,2,3}$ at $k_A{}^2=m_\kappa{}^2$ and $k_A{}^2=m_*{}^2$.

We shall now consider the poles at $k_\pi{}^2=m_\kappa{}^2$ and $k_\pi{}^2=m_*{}^2$ of the different terms appearing in the expressions of the form factors f_+ and f_-. We shall write, for instance,

$$g_-(q^2)\equiv\frac{f\pi}{\sqrt{2}}\frac{g_\kappa(q^2)}{\mu^2-q^2}+\tilde{g}_-(q^2), \quad (3.39)$$

$$H_3(q^2)\equiv\frac{f\pi}{\sqrt{2}}\frac{g_*(q^2)}{\mu^2-q^2}+\tilde{H}_3(q^2), \quad (3.40)$$

where the new defined quantities $\tilde{g}_-$ and $\tilde{H}_3$ have no poles at $q^2=\mu^2$. We shall also make use of the following convenient expressions:

$$g_+(q_\kappa{}^2)(2-x_\kappa)+g_-(q_\kappa{}^2)x_\kappa=\frac{f\pi}{\sqrt{2}}\frac{g_\kappa(q_\kappa{}^2)}{\mu^2-q_\kappa{}^2}\frac{1}{m(m-m_\kappa)}$$
$$\times\left\{\mu^2+\frac{m+m_\kappa}{2m}[(m-m_\kappa)^2-q_\kappa{}^2]\right\}+q'^2\mathcal{R}, \quad (3.41)$$

$$q'^2\mathcal{R}\equiv\tilde{g}_-(q_\kappa{}^2)\frac{m+m_\kappa}{m-m_\kappa}\frac{(m-m_\kappa)^2-q_\kappa{}^2}{2m^2}$$
$$+g_+(q_\kappa{}^2)\left(2-x_\kappa-\frac{m+m_\kappa}{m}\right), \quad (3.42)$$

$$g_+(\bar{q}_\kappa{}^2)(2-\bar{x}_\kappa)+g_-(\bar{q}_\kappa{}^2)\bar{x}_\kappa=\frac{f\pi}{\sqrt{2}}\frac{g_\kappa(\bar{q}_\kappa{}^2)}{\mu^2-\bar{q}_\kappa{}^2}\frac{1}{m(m+m_\kappa)}$$
$$\times\left\{\mu^2+\frac{m-m_\kappa}{2m}[(m+m_\kappa)^2-\bar{q}_\kappa{}^2]\right\}+q'^2\mathcal{R}^{(Z)}, \quad (3.41')$$

$$q'^2\mathcal{R}^{(Z)}\equiv\tilde{g}_-(\bar{q}_\kappa^2)\frac{m-m_\kappa}{m+m_\kappa}\frac{(m+m_\kappa)^2-\bar{q}_\kappa^2}{2m^2}+g_+(\bar{q}_\kappa^2)[2-\bar{x}_\kappa-(m-m_\kappa)/m]. \quad (3.42')$$

$q'^2\mathcal{R}$ ($q'^2\mathcal{R}^{(Z)}$), defined above, have no pole at $k_\pi^2=m_\kappa^2$ ($\bar{k}_\pi^2=m_\kappa^2$) and vanish at $q'^2=0$.

In the absence of any information on the momentum-transfer dependence of $g_\kappa(q^2)$ and $g_*(q^2)$, we shall make the approximation of taking them as constants:

$$\begin{aligned}g_\kappa(q_\kappa^2)=g_\kappa(\bar{q}_\kappa^2)=g_\kappa(\mu^2)=g_{\kappa K\pi}\equiv g_\kappa,\\ g_*(q_*^2)=g_*(\bar{q}_*^2)=g_*(\mu^2)=g_{K*K\pi}\equiv g_*.\end{aligned} \quad (3.43)$$

Using Eqs. (3.39)–(3.42′) and the approximations (3.43), we finally obtain after some work the following expressions for f_+ and f_-:

$$f_+(k_\pi^2)=\frac{f_K}{f_\pi}-\frac{\sqrt{2}g_\kappa f_\kappa}{m_\kappa^2-m^2}+\frac{g_*f_*k_\pi^2}{\sqrt{2}m_*^2(k_\pi^2-m_*^2)}+\Delta f_+(k_\pi^2), \quad (3.44)$$

$$f_-(k_\pi^2)=-\frac{\sqrt{2}g_\kappa f_\kappa}{k_\pi^2-m_\kappa^2}-\frac{f_*g_*(m^2-\mu^2)}{\sqrt{2}m_*^2(k_\pi^2-m_*^2)}+\Delta f_-(k_\pi^2), \quad (3.45)$$

where

$$\Delta f_+(k_\pi^2)\equiv\Delta_{(+)}{}^{(A_1)}+\tilde{\Delta}_{(+)}{}^{(K^*)}-\frac{f_\kappa}{f_\pi}\frac{1-x_\pi}{1-x_\kappa}\times[g_+(q_\kappa^2)+\tilde{g}_-(q_\kappa^2)-g_+(\bar{q}_\kappa^2)-\tilde{g}_-(\bar{q}_\kappa^2)]+(f_\kappa/f_\pi)q'^2(\mathcal{R}+\mathcal{R}^{(Z)}), \quad (3.46)$$

$$\Delta f_-(k_\pi^2)=\Delta_{(-)}{}^{(A_1)}+\tilde{\Delta}_{(-)}{}^{(K^*)}+\frac{f_\kappa}{2f_\pi}\frac{1+x_\pi}{1-x_\kappa}\times[g_+(q_\kappa^2)+\tilde{g}_-(q_\kappa^2)-g_+(\bar{q}_\kappa^2)-\tilde{g}_-(\bar{q}_\kappa^2)], \quad (3.47)$$

and by $\tilde{\Delta}_{(+),(-)}{}^{(K^*)}$ we have denoted exactly $\Delta_{(+),(-)}{}^{(K^*)}$ given in Eqs. (3.27) and (3.30), with H_3 replaced by $\tilde{H}_3$ and with the last terms containing $g_*(q_*^2)/(\mu^2-q_*^2)$, $g_*(\bar{q}_*^2)/(\mu^2-\bar{q}_*^2)$ omitted.

Equations (3.44) and (3.45) represent the result of the generalized sum-rule formulation in the one-particle saturation procedure and, using the approximation of taking some couplings (which, anyway, are expected to have a smooth variation with the momentum transfer) as purely constants [Eqs. (3.43)], $\Delta f_+(k_\pi^2)$ and $\Delta f_-(k_\pi^2)$ (which, in general, are non-vanishing even for $q'=0$) have no poles and are expected to be smooth functions of k_π^2.

Since we have

$$\Delta f_+(k_\pi^2)+\frac{k_\pi^2}{m^2-\mu^2}\Delta f_-(k_\pi^2)=0$$

for

$$q'^2=0 \quad [k_\pi^2=(m-\mu)^2],$$

we obtain from Eqs. (3.44) and (3.45) the relation

$$f_+[(m-\mu)^2]+\frac{m-\mu}{m+\mu}f_-[(m-\mu)^2]=\frac{f_K}{f_\pi}+\frac{\sqrt{2}g_\kappa f_\kappa\mu[2m_\kappa^2+\mu(m-\mu)]}{(m_\kappa^2-m^2)(m+\mu)[(m-\mu)^2-m_\kappa^2]}, \quad (3.48)$$

which is the last equation of Ref. 4 or essentially Eq. (47) of Ref. 1. The point is that, unlike the case of Ref. 1, we obtain it without any use of fixed-q^2 ($q^2=\mu^2$) dispersion relations in the momentum-transfer variable k^2.

IV. DISPERSION RELATIONS IN k^2 AT $q^2=\mu^2$

We present in this section another way of deriving formulas for the K_{l3} form factors, based on the use of once-subtracted dispersion relations for $f_\pm$ in the momentum-transfer variable k^2, along the line $q^2=\mu^2$. This procedure, which makes it possible to connect $f_\pm(k_\pi^2)$ with $f_\pm(\bar{k}_\pi^2)$, provides two supplementary relations which, together with the current-algebra sum rule (2.27) and Eq. (2.31) (of superconvergent type), will again allow us to solve the system for $f_\pm(k_\pi^2)$ and $f_\pm(\bar{k}_\pi^2)$. As we shall see below, this treatment limits somewhat the model dependence of the results, in the sense that the contributions related to the A_1 meson will vanish now, unlike the case of Sec. III for $q'^2=0$.

We begin by displaying the sum rules (2.27) and (2.31) with all one-particle contributions explicitly written down. We have, respectively,

$$\begin{aligned}&\tfrac{1}{2}[f_+(k_\pi^2)+f_-(k_\pi^2)+f_+(\bar{k}_\pi^2)+f_-(\bar{k}_\pi^2)]\\&+\tfrac{1}{2}x_\pi[f_+(k_\pi^2)-f_-(k_\pi^2)-f_+(\bar{k}_\pi^2)+f_-(\bar{k}_\pi^2)]\\&=f_K/f_\pi+(f_\kappa/f_\pi)[g_+(q_\kappa^2)(2-x_\kappa)+g_-(q_\kappa^2)x_\kappa\\&+g_+(\bar{q}_\kappa^2)(2-\bar{x}_\kappa)+\bar{x}_\kappa g_-(\bar{q}_\kappa^2)]\\&+q'^2\mathcal{L}(q'^2)-\frac{2}{m^2\pi f_\pi}\int c^{\text{cont}}(x,q'^2)\,dx, \end{aligned}\quad (4.1)$$

$$\begin{aligned}&\tfrac{1}{2}[(m^2-\mu^2)f_+(k_\pi^2)+k_\pi^2f_-(k_\pi^2)\\&-(m^2-\mu^2)f_+(\bar{k}_\pi^2)-\bar{k}_\pi^2f_-(\bar{k}_\pi^2)]\\&-\frac{m_\kappa^2x_\pi f_\kappa}{\sqrt{2}(1-x_\kappa)}\left[\frac{g_\kappa(q_\kappa^2)}{\mu^2-q_\kappa^2}-\frac{g_\kappa(\bar{q}_\kappa^2)}{\mu^2-\bar{q}_\kappa^2}\right]\\&-\frac{2m^2x_\pi}{\mu^2\pi f_\pi}\int w^{\text{cont}}(x,q'^2)\,dx=0, \end{aligned}\quad (4.2)$$

where $\mathcal{L}(q'^2)$ is defined by

$$\mathcal{L}(q'^2)\equiv \frac{f_A}{f_\pi}\frac{1}{m^2m_A{}^2x_A}[G_1(k_A{}^2)+m^2(1+x_A)G_2(k_A{}^2)+m^2(1-x_A)G_3(k_A{}^2)-G_1(\bar{k}_A{}^2)-m^2(1+\bar{x}_A)G_2(\bar{k}_A{}^2)-m^2(1-\bar{x}_A)G_3(\bar{k}_A{}^2)]-\frac{f_*}{f_\pi}\frac{1}{m^2m_*{}^2(1-x_*)}\times\{H_1(q_*{}^2)+m^2[(2-x_*)H_2(q_*{}^2)+x_*H_3(q_*{}^2)]-H_1(\bar{q}_*{}^2)-m^2[(2-\bar{x}_*)H_2(\bar{q}_*{}^2)+\bar{x}_*H_3(\bar{q}_*{}^2)]\}\,, \quad (4.3)$$

and the last integrals in Eqs. (4.1) and (4.2) indicate the continuum parts remaining after the one-particle pieces have been taken off.

Writing now once-subtracted dispersion relations for $f_\pm$ in k^2, along the line $q^2=\mu^2$,

$$f_\pm(\bar{k}_\pi{}^2)=f_\pm(k_\pi{}^2)+\frac{\bar{k}_\pi{}^2-k_\pi{}^2}{\pi}\int\frac{\mathrm{Im}f_\pm(k^2)\,dk^2}{(k^2-k_\pi{}^2)(k^2-\bar{k}_\pi{}^2)}, \quad (4.4)$$

we have the following two additional relations:

$$f_+(\bar{k}_\pi{}^2)=f_+(k_\pi{}^2)-\frac{(\bar{k}_\pi{}^2-k_\pi{}^2)f_*g_*}{\sqrt{2}(k_\pi{}^2-m_*{}^2)(\bar{k}_\pi{}^2-m_*{}^2)}+C_{(+)}\,, \quad (4.5)$$

$$f_-(\bar{k}_\pi{}^2)=f_-(k_\pi{}^2)+\frac{(\bar{k}_\pi{}^2-k_\pi{}^2)\sqrt{2}f_\kappa g_\kappa}{(k_\pi{}^2-m_\kappa{}^2)(\bar{k}_\pi{}^2-m_\kappa{}^2)}+\frac{(\bar{k}_\pi{}^2-k_\pi{}^2)(m^2-\mu^2)f_*g_*}{m_*{}^2\sqrt{2}(k_\pi{}^2-m_*{}^2)(\bar{k}_\pi{}^2-m_*{}^2)}+C_{(-)}\,. \quad (4.6)$$

Again, by $C_{(\pm)}$ we have conventionally denoted the remaining continuum contributions to the dispersive integrals.

Using now the same approximation as in Sec. III [Eqs. (3.43)] of setting $g_\kappa(q^2)$ and $g_*(q^2)$ as constants and taking also into account Eqs. (3.41), (3.42), (3.41′), and (3.42′), we get from the system of equations (4.1), (4.2), (4.5), and (4.6) (in the one-particle saturation scheme) the following expressions for the form factors $f_\pm$:

$$f_+(k_\pi{}^2)=\frac{f_K}{f_\pi}-\frac{\sqrt{2}g_\kappa f_\kappa}{m_\kappa{}^2-m^2}+\frac{k_\pi{}^2g_*f_*}{m_*{}^2\sqrt{2}(k_\pi{}^2-m_*{}^2)}+\delta f_+(k_\pi{}^2)\,, \quad (4.7)$$

$$f_-(k_\pi{}^2)=-\frac{f_\kappa g_\kappa\sqrt{2}}{k_\pi{}^2-m_\kappa{}^2}+\frac{(m^2-\mu^2)f_*g_*}{m_*{}^2\sqrt{2}(m_*{}^2-k_\pi{}^2)}. \quad (4.8)$$

The quantity $\delta f_+(k_\pi{}^2)$ appearing in Eq. (4.7) is defined as

$$\delta f_+(k_\pi{}^2)\equiv q'^2\rho(k_\pi{}^2)=-(1/4m^2)[k_\pi{}^2-(m-\mu)^2]\times[k_\pi{}^2-(m+\mu)^2]\rho(k_\pi{}^2)\,, \quad (4.9)$$

$$\rho(k_\pi{}^2)\equiv\mathcal{L}'+(f_\kappa/f_\pi)(\mathcal{R}+\mathcal{R}^{(Z)})\,.$$

(By $\mathcal{L}'$ we denote $\mathcal{L}$ [given by Eq. (4.3)] with H_3 replaced by $\tilde{H}_3$ defined in Eq. (3.40).)

Looking at the above expressions, we see that, in the one-particle approximation plus the assumptions (3.43), the current-algebra sum rule (2.27) and the "field-algebra" relation (2.31), supplemented by the usual subtracted dispersion relations for $f_\pm(k^2)$ (in k^2 at $q^2=\mu^2$), provide us with formulas for K_{l3} form factors less affected by the A_1 contributions than in the approach of Sec. III, which was based entirely on current- and field-algebra sum rules, all written in the dispersion variable x, at fixed q'^2. Indeed, here f_- is completely determined in terms of κ and k^* parameters only, while f_+ is affected by the axial-vector contributions through the nonpolar and presumably smooth quantity δf_+, which vanishes for $q'^2=0$ [that is, for $k_\pi{}^2=(m-\mu)^2$, the upper limit of the physical region].

V. DISCUSSION OF RESULTS AND CONCLUSIONS

We have presented in this paper two possible sum-rule formulations of K_{l3} decays, both of them furnishing definite expressions for the form factors f_+ and f_- as functions of the momentum transfer k^2. Their results are, of course, not identical, the hypotheses and the methods used being different. In the first approach, given in Sec. III and based entirely on current- and field-algebra sum rules, we obtained expressions for $f_\pm(k^2)$ containing some contributions related to the A_1 meson which do not vanish in general, even if we put $q'^2=0$ [$k^2=(m-\mu)^2$]. In the second approach (Sec. IV), we wrote once-subtracted dispersion relations for $f_\pm$ and by combining them with a current-algebra sum rule and a field-algebra relation involving, respectively, the amplitudes c and w, we succeeded in minimizing the appearance of A_1-related contributions. Indeed, this time, while f_- is not affected at all by such contributions, f_+ is affected only by the piece $\delta f_+(k^2)$ which vanishes at $q'^2=0$. Moreover, the derivative of $\delta f_+(k^2)$ with respect to k^2 at $k^2=(m-\mu)^2$ has in front a factor μ/m:

$$\left(\frac{d}{dk^2}[\delta f_+(k^2)]\right)_{k^2=(m-\mu)^2}=\frac{\mu}{m}\rho[(m-\mu)^2]. \quad (5.1)$$

The results of Ref. 4, in which another possible way (based on somewhat different postulates) of generalizing the collinear configuration is explored, differ from our results only by the terms $\Delta f_\pm$ in Eqs. (3.44) and (3.45), or δf_+ in Eq. (4.7). Our Eqs. (4.7) and (4.8) coincide with the results of Ref. 4 for $k^2=(m-\mu)^2$ and are essentially the same as Eqs. (26) and (27) of Ref. 4 for k^2 very near the boundary value $(m-\mu)^2$ of the physical region [in the sense that the derivative of $\delta f_+(k^2)$ at k^2 in $k^2=(m-\mu)^2$ can presumably be considered negligible]. The procedure of Ref. 4, making use of quantities involving only divergences of the axial-vector currents, can avoid the consideration of A_1 contributions to the sum rules, but one must then disperse noncovariantly. The appearance of A_1 contributions (or, in general, the consideration of matrix elements of the axial-vector densities) seems to be a price to be paid if one wishes to disperse on a parabola belonging to a family parametrized in terms of Lorentz invariants.

From Eqs. (4.7) and (4.8) we can eliminate the contributions of the K^* mesons by considering the particular combination $f_+(k^2)+(k^2/m^2-\mu^2)f_-(k^2)$. One can further eliminate the product $g_\kappa f_\kappa$ using the partially conserved vector-current (PCVC) hypothesis

$$f_+(k^2)+\frac{k^2}{m^2-\mu^2}f_-(k^2)=\frac{f_\kappa g_\kappa m_\kappa^2\sqrt{2}}{(m^2-\mu^2)(m_\kappa^2-k^2)}. \quad (5.2)$$

If this relation holds over a large k^2 region, one arrives at the conclusion that $\delta f_+(k^2)$ and $\Delta f_+ + [k^2/(m^2-\mu^2)] \times \Delta f_-$ are zero [because with Eq. (5.2) they must be constants and they vanish at the point $k^2=(m-\mu)^2$]. Thus one gets the formula

$$f_+(k^2)+\frac{k^2}{m^2-\mu^2}f_-(k^2)=\frac{f_K}{f_\pi}\frac{m_\kappa^2(m_\kappa^2-m^2)}{(m_\kappa^2-\mu^2)(m_\kappa^2-k^2)}. \quad (5.3)$$

This relation is a generalization of Eq. (48) of Ref. 1 and coincides with it for $k^2=(m-\mu)^2$.

In our approach, without PCVC or further arguments, $\delta f_+(k^2)$ does not seem to be negligible. Perhaps the best thing to do is to take into consideration Eqs. (4.7) and (4.8) (as well as their derivatives with respect to k^2) only for $k^2=(m-\mu)^2$ when $\delta f_+(k^2=(m-\mu)^2)=0$ and its derivative at that point can, hopefully, be neglected. In the linear fit procedures, such relations could be used in order to make a comparison with the data. Also, one could try to fix some less-known parameters by considering (for arbitrary q'^2) other sum rules which can be written down in connection, for instance, with the additional vertex functions

$$\langle 0|[A_\mu(z),A_\nu(0)]|\kappa\rangle, \quad \langle 0|[A_\mu(z),V_\nu(0)]|\pi\rangle,$$

as has been done, in the collinear configuration, in Ref. 6.

[6] D. W. McKay, J. M. McKisic, and W. W. Wada, Phys. Rev. **184**, 1609 (1969).

The experimental situation is also unclear. In this respect, as pointed out in Ref. 7 [where a consistent explanation is given of different ξ measurements, using a linear fit for $\xi(k^2)\equiv f_-(k^2)/f_+(k^2)$], a precise experimental determination of the λ_+ parameter (introduced by the expansion $f_+(k^2)=f_+(0)[1+(\lambda_+/\mu^2)k^2]$) is highly desirable.

Setting $k^2=0$ in Eq. (5.3) and using the recently determined number $\sigma\equiv f_K/f_\pi f_+(0)=1.23$ (the value quoted in Ref. 8), one gets a value of the κ-meson mass $m_\kappa=[(\sigma m^2-\mu^2)/(\sigma-1)]^{1/2}\simeq 1150$ MeV, in good agreement with the experimental evidence (Ref. 9). Dividing by $f_+(k^2)$ in Eq. (5.3), taking the derivative with respect to k^2 at $k^2=0$, and using for m_κ the expression found above, one obtains the following relation between $\xi(0)$ and λ_+:

$$\xi(0)=\frac{(m^2-\mu^2)(\sigma-1)}{\sigma m^2-\mu^2}-\frac{m^2-\mu^2}{\mu^2}\lambda_+.$$

Taking $\lambda_+=0.06$, as suggested in Ref. 7 (this value of λ_+ corresponds to a linear fit of the combined data with a χ^2 probability of 98%), one finds $\xi(0)=-0.56$, which agrees within 1-standard-deviation error with the results of Ref. 7 for $\xi(0)$.

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