evolution. White dwarfs have long since passed the stage of nuclear slow-down.[9]

On the photon-neutrino coupling theory, the brightest white dwarf in the Hyades would be expected to appear at a luminosity of about $\log_{10}(L/L_\odot)=-3.6$ (for $\sim 1M_\odot$). This is 2 magnitudes *fainter* than the luminosity of the faintest observed white dwarf on the blue sequence. The red white dwarfs would be expected to be even fainter.

In order to obtain agreement with the observational data, the neutrino luminosities should be drastically reduced. This allows one to place an upper limit on the square of the photon-neutrino coupling constant, which then becomes approximately $10^{-4}g_{\gamma\nu}^2$ (based on the blue sequence) and $10^{-6}g_{\gamma\nu}^2$ (based on the red sequence).

If the coupling constant were actually this small (or smaller), then the astrophysical consequences of photon-neutrino coupling would be negligible in virtually all situations except possibly supernova implosions. Thus, one would be obliged to discard the current explanation of the scarcity of very massive red supergiants as being due to the neutrino-induced speed-up of the nuclear evolution of a nondegenerate star[10] (to say nothing of the other astrophysical evidence for acceleration of evolution in stars). In this connection, Bandyopadhyay[5,6] seems curiously to have sought to *prolong* the evolution of red supergiants, primarily because he has used the earlier stellar models and arguments of Hayashi *et al.*,[11] which have since been reversed and, in their corrected state, now provide the best evidence in *favor* of neutrino emission.

We conclude, on the basis of the astrophysical data for white dwarfs and red supergiants, that the photon-neutrino coupling theory as proposed by Bandyopadhyay is definitely excluded.

It is a pleasure to thank Dr. C.-W. Chin for deriving the neutrino luminosities of the models.

[9] C.-W. Chin, H.-Y. Chiu, and R. Stothers, Ann. Phys. (N. Y.) **39**, 280 (1966). See also M. Schwarzschild, *Structure and Evolution of the Stars* (Princeton U. P., Princeton, N. J., 1958), Chap. 7 and references.

[10] R. Stothers, Astrophys. J. **155**, 935 (1969); R. Stothers and C.-W. Chin, *ibid.* **158**, 1039 (1969).

[11] C. Hayashi, R. Hōshi, and D. Sugimoto, Progr. Theoret. Phys. (Kyoto) Suppl. **22**, 1 (1962); D. Sugimoto, Y. Yamamoto, R. Hōshi, and C. Hayashi, Progr. Theoret. Phys. (Kyoto) **39**, 1432 (1968).

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Hidden-Variable Example Based upon Data Rejection

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A deterministic local hidden-variable model is presented which describes the simultaneous measurement of the spins of two spin-$\frac{1}{2}$ particles which emerged from the decay of a spin-zero particle. In this model the measurement of the spin of a particle has one of three possible outcomes: spin parallel to the apparatus axis, spin antiparallel to the apparatus axis, or the particle goes undetected. It is shown that agreement with the predictions of quantum theory is obtained provided the experimenter rejects the "anomalous" data in which only one particle is detected. A reasonably model-independent lower bound to the fraction of undetected particles is also computed: It is found that in 14% of the decays or more, one or both of the particles will go undetected.

I. INTRODUCTION

A HIDDEN-VARIABLE description of the measurement of the spins of two widely separated spin-$\frac{1}{2}$ particles which were products of the decay of a single spin-zero particle has been considered by Bell.[1] (This example was first invoked by Bohm[2] to illustrate the Einstein-Podolsky-Rosen[3] argument that quantum theory is not a complete description of nature.) Upon leaving the site of the decay, each particle is presumed to have "made up its mind" as to the spin direction that will be measured by an apparatus (e.g., a Stern-Gerlach apparatus) placed in its path, for any possible orientation of that apparatus. In particular, the response of one particle to the apparatus in its path is unaffected by the orientation of the apparatus encountered by the other particle. (Because of the absence of this long-range interaction, Bell has called a hidden-variable theory of this type "local.") The predetermined responses of one pair of particles arising from a single decay to their two apparatus does not have to be identical to the predetermined responses of another pair. Using the constraints of ordinary probability theory, Bell showed that a model containing the above features cannot produce predictions of the outcome of the spin measurements which are in agreement with the predictions of quantum theory.

We would like to discuss a hidden-variable description of this experiment which is "local" and which ap-

[1] J. S. Bell, Physics **1**, 195 (1965).

[2] D. Bohm, *Quantum Theory* (Prentice-Hall, New York, 1951), p. 614.

[3] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

parently, although not actually, produces predictions in agreement with the predictions of quantum theory. This somewhat enigmatic last statement is explained as follows: Suppose that each particle has three responses to a spin-measuring apparatus instead of two; it can as usual show itself to have spin parallel or antiparallel to the apparatus orientation, or it cannot show itself at all, i.e., cannot be detected.[4] Then instead of four possible experimental outcomes of the measurement of the spins of two particles, there are nine possible outcomes. In one of these outcomes, neither particle is detected, and so the experimenter is unaware that a decay has taken place. In four of these outcomes one of the particles is not detected. If the experimenter rejects these data (in the belief that the apparatus is not functioning properly and that if it had been functioning properly, the data recorded would have been representative of the accepted data), he is left with the usual four possible outcomes. We now suppose that an analysis of these remaining data produces predictions in agreement with the predictions of quantum theory.

The question arises as to whether it is possible to produce a local hidden-variable theory with the properties described in the preceding paragraph. We show that it is indeed possible by explicitly displaying a model[5] with these properties. Then we examine a general restriction on such models.

II. DESCRIPTION OF MODEL

Each pair of decaying particles is represented by a point in a "phase space" consisting of a sphere of unit radius. The probability density that a pair is represented by a point $\mathbf{r}$, whose polar coordinates are r, θ, and φ, is a function $\rho(r)$ which is to be determined; of course,

$$\rho(r)\geq 0\,, \quad 4\pi\int_0^1 r^2 dr\,\rho(r)=1\,. \tag{1}$$

We must give a prescription describing how a particle, in the pair represented by the point $\mathbf{r}$, responds to an apparatus whose axis is oriented along the unit vector $\hat{a}$. Consider the spherical surface of radius r upon which the representative point lies. We divide this surface into three regions determined by its intersection with a double cone whose axis (passing through the origin) lies along $\hat{a}$ and whose opening angle is β ($\beta\leq\pi$). If the point $\mathbf{r}$ lies outside the two circular regions cut on the spherical surface by the cone, the particle—call it A—will not be detected. If the point $\mathbf{r}$ lies inside one of the circular regions cut by the cone—say, the region whose center is pierced by the vector $\hat{a}$—then the particle will be measured as possessing spin parallel to $\hat{a}$. If the point $\mathbf{r}$ lies inside the antipolar circular region, the particle will be measured as possessing spin antiparallel to $\hat{a}$.

Likewise, the response of the other particle—call it B—to an apparatus whose axis is oriented along the unit vector $\hat{b}$ is determined by a similar construction utilizing a double cone, also of opening angle β, whose axis lies along $-\hat{b}$. Thus when both apparatuses are identically oriented ($\hat{a}=\hat{b}$), the surface of the sphere is divided into only three regions (in which spin A is measured parallel and spin B is measured antiparallel to $\hat{a}$, or in which spin B is measured parallel and spin A is measured antiparallel to $\hat{a}$, or in which neither A nor B is detected). However, for arbitrary orientations of $\hat{a}$ and $\hat{b}$, the surface is generally divided into nine regions corresponding to the nine possible outcomes of this experiment. It is important to realize that this construction ensures that the response of particle A (particle B) to its apparatus is independent of $\hat{b}$ ($\hat{a}$), thereby guaranteeing the locality of this model.

Suppose now that the angle between $\hat{a}$ and $\hat{b}$ is α. The area (common to the two circular regions on the spherical surface of radius r) which corresponds to spin A being measured parallel to $\hat{a}$ and spin B being measured antiparallel to $\hat{b}$ is shown in Eq. (A5) of Appendix A to be given by the expression

$$\begin{aligned} I(\beta,\alpha) &= 4r^2\int_{\alpha/2}^{\beta/2} d\lambda\left[1-\left(\frac{\cos\frac{1}{2}\beta}{\cos\lambda}\right)^2\right]^{1/2}, && \alpha\leq\beta \\ &=0\,, && \alpha\geq\beta\,. \end{aligned} \tag{2}$$

The probability $P_{1,-1}(\alpha)$ that spin A is measured parallel to $\hat{a}$ and spin B is measured antiparallel to $\hat{b}$ is therefore

$$P_{1,-1}(\alpha)=\int_0^1 dr\,\rho(r)I(\beta(r),\alpha)\,, \tag{3}$$

where we have permitted the angle β, which determines the size of the circular regions on a surface of radius r, to vary with r. In fact, there is no loss of generality in choosing

$$\beta(r)=\pi r\,, \quad 0\leq r\leq 1 \tag{4}$$

which allows a circular region subtending each permissible angle to exist on some surface of radius r; any $\beta(r)$ can be brought to the form of Eq. (4) without changing the form of Eq. (3) by a suitable (possibly multivalued) transformation on r, together with a redefinition of $\rho(r)$ which does not change the essential requirements on a

[4] E. P. Wigner (private communication) has independently considered this possibility, and obtained a model-independent lower bound for the fraction of undetected particles. The author is grateful to Professor Wigner for informing him that it was possible to obtain such a bound.

[5] The first—and best—hidden-variable model, applicable to quantum systems with a finite number of states, based upon the conceptual structure of statistical mechanics, is due to N. Wiener and A. Siegal, Phys. Rev. **91**, 1551 (1953); Nuovo Cimento Suppl. **11**, 982 (1955); Phys. Rev. **101**, 429 (1956). Bell (Ref. 1) presented a model for measurement of the spin of a single spin-$\frac{1}{2}$ particle. A similar model was found by the author [Harvard University Report, 1965 (unpublished)], and independently by S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967). These spin-$\frac{1}{2}$ models were based on a hidden-variable space consisting of points on the surface of a sphere; the model presented here is a natural extension of this.

probability density expressed in Eq. (1). Thus the points of the sphere which correspond to spin A being measured parallel to axis $\hat{a}$ lie within a mushroom-shaped region that is cylindrically symmetric about $\hat{a}$, and the points contributing to the integral in Eq. (3) lie within the intersection of two such mushroom-shaped regions.

Quantum theory predicts that the probability that spin A is measured parallel to $\hat{a}$ and spin B is measured antiparallel to $\hat{b}$ is $\frac{1}{2}\cos^2(\frac{1}{2}\alpha)$. If we denote the fraction of events in which both particles are detected by $g(\alpha)$ $[0<g(\alpha)\leq 1]$, the requirement that the unrejected data yield the same predictions as quantum theory becomes

$$g(\alpha)\tfrac{1}{2}\cos^2(\tfrac{1}{2}\alpha)=P_{1,-1}(\alpha)=P_{-1,1}(\alpha)=\int_{\alpha/\pi}^{1} dr\,\rho(r)I(\pi r,\alpha). \qquad (5)$$

In Eq. (5) we have utilized the obvious symmetry of the construction to equate $P_{1,-1}$ to $P_{-1,1}$. We have also utilized Eq. (4) and the vanishing of $I(\beta,\alpha)$ for $\alpha\geq\beta$ to set the lower limit of the integral over r.

In a like manner, one can compute the probability $P_{1,1}$ that both spin A and spin B are measured parallel to their respective apparatus axes. It is readily seen that the appropriate integral is that of Eq. (3) or (5) with α replaced by $\pi-\alpha$. Since quantum theory predicts the probability of this outcome of the measurement to be $\frac{1}{2}\sin^2(\frac{1}{2}\alpha)$, we require

$$g(\alpha)\tfrac{1}{2}\sin^2(\tfrac{1}{2}\alpha)=P_{1,1}(\alpha)=P_{-1,-1}(\alpha)=\int_{1-\alpha/\pi}^{1} dr\,\rho(r)I(\pi r,\pi-\alpha). \qquad (6)$$

A comparison of Eqs. (5) and (6) will show that if one of these equations is satisfied, the other equation will be satisfied provided

$$g(\alpha)=g(\pi-\alpha). \qquad (7)$$

All the remaining probabilities predicted by this model can be determined in terms of the probability functions already introduced. Indeed, the probability that the spin of particle A will be measured parallel to $\hat{a}$ (regardless of whether particle B is detected or not) is given by the integral of the probability density over the mushroom-shaped region, and is equal to

$$P_{1,-1}(0)=\tfrac{1}{2}g(0).$$

If we denote by $P_{1,0}(\alpha)$ the probability that spin A will be measured parallel to $\hat{a}$ while particle B goes undetected, we have

$$P_{1,0}(\alpha)=\tfrac{1}{2}g(0)-P_{1,1}(\alpha)-P_{1,-1}(\alpha)=\tfrac{1}{2}[g(0)-g(\alpha)].$$

By similar reasoning we find that

$$P_{1,0}(\alpha)=P_{-1,0}(\alpha)=P_{0,1}(\alpha)=P_{0,-1}(\alpha)=\tfrac{1}{2}[g(0)-g(\alpha)], \qquad (8)$$

$$P_{0,0}(\alpha)=1+g(\alpha)-2g(0). \qquad (9)$$

It is shown in Sec. VII that the relationships between $P_{i,j}(\alpha)$ and $g(\alpha)$ displayed above are relevant to a wider class of models than those explicitly constructed here.

III. SOLUTION OF EQUATIONS

Any function $\rho(r)$ satisfying the probability-density requirements of Eq. (1) will, upon insertion into the integral on the right-hand side of Eq. (5), yield a monotonically decreasing function of α, because the overlap area represented by $I(\pi r,\alpha)$ is a monotonically decreasing function of α. When this integral is divided by $\frac{1}{2}\cos^2(\frac{1}{2}\alpha)$, we obtain a positive function of α which, however, will not ordinarily satisfy the symmetry requirement of Eq. (7). We now proceed to determine the general form for $\rho(r)$ such that Eqs. (1), (5), and (7) are satisfied.

Equation (5) is a difficult integral equation to solve directly because $I(\pi r,\alpha)$ is a complicated function. However, if we take Eq. (5),

$$g(\alpha)\tfrac{1}{2}\cos^2(\tfrac{1}{2}\alpha)=\int_{\alpha/\pi}^{1} 4r^2dr\,\rho(r)\times\int_{\alpha/2}^{\pi r/2} d\lambda\left[1-\left(\frac{\cos\frac{1}{2}\pi r}{\cos\lambda}\right)^2\right]^{1/2}. \qquad (10)$$

and differentiate it with respect to α, we obtain an equivalent integral equation with a much simpler kernel in the integrand:

$$-\frac{1}{\cos\frac{1}{2}\alpha}\frac{d}{d\alpha}g(\alpha)\tfrac{1}{2}\cos^2(\tfrac{1}{2}\alpha)=\frac{4}{\pi}\int_0^1 h(z\cos\tfrac{1}{2}\alpha)(1-z^2)^{1/2}dz. \qquad (11)$$

In Eq. (11) we have introduced a new probability-density function h, which is related to ρ by

$$h(\cos\tfrac{1}{2}\pi r)=\rho(r)r^2/\sin\tfrac{1}{2}\pi r, \qquad (12)$$

and we have introduced a new variable of integration,

$$z=\cos\tfrac{1}{2}\pi r/\cos\tfrac{1}{2}\alpha. \qquad (13)$$

Equation (1), expressed in terms of h, becomes

$$h(x)\geq 0 \quad (0\leq x\leq 1):\quad \int_0^1 h(z)dz=\tfrac{1}{8}. \qquad (14)$$

It is shown in Appendix B, Eq. (B6), that the solution to Eq. (11) is

$$h(x)=\frac{1}{2}\frac{d^2}{dx^2}x^2\int_0^1\frac{wdw}{(1-w^2)^{1/2}}\frac{[1-(xw)^2]^{1/2}}{4xw}\frac{d}{dxw}\times g(xw)x^2w^2. \qquad (15)$$

We have changed the argument of g in Eq. (11) from α to $\cos\frac{1}{2}\alpha\equiv x$ in order to obtain Eq. (15). On account of Eq. (7), $g(x)$ must satisfy the symmetry condition

$$g(x)=g((1-x^2)^{1/2}). \tag{16}$$

We have found it convenient to introduce a function $\mu(x)$,

$$\mu(x)\equiv-\frac{1}{(1-x^2)^{1/2}}\frac{d}{dx}\left[\frac{(1-x^2)^2}{4x}\frac{d}{dx}g(x)x^2\right], \tag{17}$$

satisfying the same symmetry relation as $g(x)$:

$$\mu(x)=\mu((1-x^2)^{1/2}), \tag{18}$$

which follows from Eqs. (16) and (17). Upon multiplying Eq. (17) by $(1-x^2)^{1/2}$ and integrating once, and adding the resultant equation to its symmetric counterpart, we obtain

$$\tfrac{1}{2}g(x)=\left(\int_x^1\mu(z)(1-z^2)^{1/2}dz\right)\Big/(1-x^2)+\left(\int_{(1-x^2)^{1/2}}^1\mu(z)(1-z^2)^{1/2}dz\right)\Big/x^2, \tag{19}$$

It follows from Eqs. (15) and (17) that the expression for h in terms of μ is

$$h(x)=\frac{1}{2}\frac{d^2}{dx^2}\frac{x^2}{1-x^2}\left[\int_0^1dz\,\mu(z)(1-z^2)^{1/2}-\int_0^x dz\,\mu(z)\left(1-\frac{z^2}{x^2}\right)^{1/2}\right]. \tag{20}$$

If a function $\mu(z)$ satisfying Eq. (18) but otherwise arbitrary is chosen, and Eq. (20) is solved for $h(x)$, this solution may or may not be positive over the whole range $0\leq x\leq 1$. If it is positive, our task is completed, since the normalization condition (14) can be achieved by a scalar multiplication, and $g(\alpha)$ is given by Eq. (19) in manifestly symmetrical form.

IV. DETERMINATION OF PROBABILITIES

Our purpose is only to demonstrate that a model with the desired properties can be constructed, and so we shall choose the simplest function which satisfies Eq. (18): $\mu=$ constant. When $\mu=C$, Eq. (20) yields

$$h(x)=\tfrac{1}{4}C\pi(1+x)^3, \tag{21}$$

which is clearly positive for $C>0$ and $0\leq x\leq 1$. The normalization condition (14) requires that $C=4/3\pi$, and so by Eq. (12), the probability density $\rho(r)$ is

$$\rho(r)r^2=\frac{4}{3\pi}\frac{\sin\frac{1}{2}\pi r}{(1+\cos\frac{1}{2}\pi r)^3}. \tag{22}$$

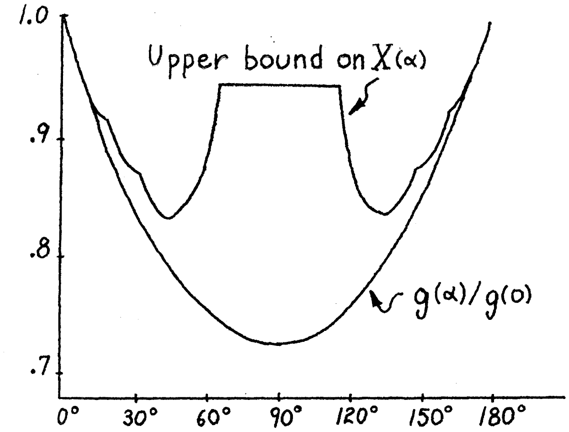

FIG. 1. Graph of the upper bound on $X(\alpha)$ (Sec. VIII), and graph of $X(\alpha)=g(\alpha)/g(0)$ for a particular model [Eq. (27)].

Lastly we determine $g(\alpha)$ from Eq. (19):

$$g(\alpha)=\frac{2}{3\pi}\left[\frac{\alpha-\sin\alpha}{\sin^2(\frac{1}{2}\alpha)}+\frac{(\pi-\alpha)-\sin\alpha}{\cos^2(\frac{1}{2}\alpha)}\right]. \tag{23}$$

V. DISCUSSION OF MODEL

We see from Eq. (23) that the fraction of events in which both particles are detected goes from a maximum of $g(0)=\frac{2}{3}$ to a minimum of $g(\frac{1}{2}\pi)=\frac{4}{3}(1-2/\pi)$ [also see Fig. 1, where a graph of $g(\alpha)/g(0)$ is displayed]. According to Eq. (8), the fraction of events in which only one particle is detected goes from zero (when $\alpha=0$) to a maximum of $\frac{4}{3}(4/\pi-1)$ (when $\alpha=\frac{1}{2}\pi$). Both particles go undetected in a maximum of $\frac{1}{3}$ of the events (when $\alpha=0$), to a minimum of $1-8/3\pi$ events (when $\alpha=\frac{1}{2}\pi$).

One might argue that since the experimenter would be totally unaware of those events in which both particles go undetected, the experimentally important quantity would be the number of events for which both apparatuses detect particles, divided by the total number of events of which he is aware. This ratio is

$$\frac{P_{1,1}(\alpha)+P_{1,-1}(\alpha)+P_{-1,1}(\alpha)+P_{-1,-1}(\alpha)}{1-P_{0,0}(\alpha)}=\frac{g(\alpha)}{2g(0)-g(\alpha)},$$

which goes from a maximum of 1 (when $\alpha=0$) to a minimum of $\frac{1}{2}\pi-1$ (when $\alpha=\frac{1}{2}\pi$).

The fraction of undetected events can be reduced somewhat by a different choice of $\mu(x)$; the extent of this reduction is an open question. Instead of working within the framework of the class of models constructed here, we shall consider a wider class of models for which we can obtain an upper bound on $g(\alpha)$.[4]

VI. INEQUALITY

In order to obtain an upper bound on $g(\alpha)$, we shall utilize inequalities which are a slight generalization of an inequality due to Clauser, Horne, Shimony, and

Holt,[6] which is in turn a generalization of an inequality first introduced by Bell.[1]

Consider a sample space which has a unity-normalized probability-density function $\rho(\lambda)$ defined on it (λ symbolizes the coordinate of an arbitrary point in the space and $d\lambda$ is the volume element). We introduce a function $A(\lambda,a)$ defined on the space, which can only take on the values ± 1, and which depends upon the parameter(s) a; similarly, we introduce another function $B(\lambda,b)$. Because $A^2=B^2=1$, it follows that (suppressing the variable λ)

$$\begin{aligned} A(a_1)B(b_1)-A(a_1)B(b_n) &= A(a_1)B(b_1)[1-A(a_2)B(b_1)] \\ &\quad +A(a_1)A(a_2)[1-A(a_2)B(b_2)] \\ &\quad +A(a_1)B(b_2)[1-A(a_3)B(b_2)]+\cdots \\ &\quad +A(a_1)A(a_n)[1-A(a_n)B(b_n)]. \end{aligned} \tag{24}$$

The left-hand side of Eq. (24) is dominated by the absolute magnitude of the sum of the individual terms on the right-hand side. Realizing that

$$|A(a_i)A(a_j)|=|A(a_i)B(b_j)|=1$$

and

$$|1-A(a_i)B(b_j)|=1-A(a_i)B(b_j),$$

we find

$$\begin{aligned} A(a_1)B(b_1)+B(b_1)A(a_2)+A(a_2)B(b_2)+\cdots \\ +A(a_n)B(b_n)\leq 2n-2+A(a_1)B(b_n). \end{aligned} \tag{25}$$

We now multiply Eq. (25) by $\rho(\lambda)$ and integrate over λ. By further supposing that a and b are unit vectors $\hat{a}$ and $\hat{b}$, respectively, and that

$$\int d\lambda\rho(\lambda)A(\lambda,\hat{a})B(\lambda,\hat{b})=S(\hat{a}\cdot\hat{b}), \tag{26}$$

we then obtain the inequality

$$\begin{aligned} S(\hat{a}_1\cdot\hat{b}_1)+S(\hat{b}_1\cdot\hat{a}_2)+\cdots+S(\hat{a}_n\cdot\hat{b}_n) \\ \leq 2n-2+S(\hat{a}_1\cdot\hat{b}_n). \end{aligned} \tag{27}$$

If the angles between adjacent vectors in the sequence $\hat{a}_1$, $\hat{b}_1$, $\hat{a}_2$, ..., $\hat{b}_n$ are identical, Eq. (27) becomes the inequality

$$(2n-1)S(\hat{a}\cdot\hat{b})\leq(2n-2)+S(\hat{a}_1\cdot\hat{b}_n) \tag{28}$$

(where $\hat{a}$ and $\hat{b}$ are any two adjacent vectors in the sequence).

VII. DEFINITION OF CLASS OF MODELS

In order to apply Eq. (28) to the present problem, we suppose that a hidden-variable model has been constructed satisfying three conditions:

1. Predictions are made in agreement with quantum theory, based upon the data-rejection hypothesis.
2. Predictions possess the same symmetries as do the predictions of quantum theory.
3. It is a local hidden-variable model.

We now show that the relations [Eqs. (5)–(9)] between $P_{i,j}(\alpha)$ and $g(\alpha)$ are very nearly satisfied for this wider class of models.

The first condition stipulates that if one apparatus axis is along $\hat{a}$ and the other apparatus axis is along $\hat{b}$ ($\hat{a}\cdot\hat{b}=\cos\alpha$), then

$$\begin{aligned} P_{1,-1}(\hat{a},\hat{b})=P_{-1,1}(\hat{a},\hat{b})=g(\hat{a},\hat{b})\tfrac{1}{2}\cos^2(\tfrac{1}{2}\alpha), \\ P_{1,1}(\hat{a},\hat{b})=P_{-1,-1}(\hat{a},\hat{b})=g(\hat{a},\hat{b})\tfrac{1}{2}\sin^2(\tfrac{1}{2}\alpha), \end{aligned} \tag{29}$$

where $g(\hat{a},\hat{b})$ is the fraction of detected particles. The second condition requires the dependence of $P_{i,j}$ and g to be upon α only, rather than upon the absolute orientation of the apparatus axes. It also requires that the predictions be invariant under exchange of the two apparatuses and invariant under inversion of both apparatus axes, from which we conclude that

$$P_{1,0}(\alpha)=P_{-1,0}(\alpha)=P_{0,1}(\alpha)=P_{0,-1}(\alpha). \tag{30}$$

Likewise, if only one apparatus axis is inverted, the predictions for an angle α must be the same as for an uninverted apparatus axis with an angle $\pi-\alpha$, so by Eq. (29),

$$g(\alpha)=g(\pi-\alpha). \tag{31}$$

The third condition states that a detection rate at one apparatus does not depend upon the orientation of the other apparatus axis, e.g.,

$$P_{1,1}(\alpha)+P_{1,0}(\alpha)+P_{1,-1}(\alpha)=C(\hat{a}), \tag{32}$$

$$P_{0,1}(\alpha)+P_{0,0}(\alpha)+P_{0,-1}(\alpha)=C'(\hat{a}), \tag{33}$$

where $C(\hat{a})$ and $C'(\hat{a})$ are functions which can only depend upon the orientation vector $\hat{a}$. However, since the left-hand sides of Eqs. (32) and (33) depend on the scalar product $\hat{a}\cdot\hat{b}$, we see that C and C' are merely constants. Putting Eq. (29) into Eq. (32), we find that

$$P_{1,0}(\alpha)=C-\tfrac{1}{2}g(\alpha). \tag{34}$$

Putting Eqs. (30) and (34) into Eq. (33), we obtain

$$P_{0,0}(\alpha)=C'-2C+g(\alpha). \tag{35}$$

The normalization condition

$$\sum_{i,j}P_{i,j}(\alpha)=g(\alpha)+4P_{1,0}+P_{0,0}=1$$

tells us that

$$2C+C'=1, \tag{36}$$

which, when inserted into Eq. (35), yields

$$P_{0,0}(\alpha)=1-4C+g(\alpha). \tag{37}$$

It is a consequence of the requirement $P_{i,j}(\alpha)\geq 0$ and Eqs. (34) and (37) that

$$\tfrac{1}{4}+\tfrac{1}{4}\min[g(\alpha)]\geq C\geq\tfrac{1}{2}\max[g(\alpha)]. \tag{38}$$

[6] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Letters 23, 880 (1969).

In the model previously constructed, the constant C was equal to $\frac{1}{2}g(0)=\frac{1}{2}\max[g(\alpha)]$. In the more general class of models now under consideration, such an equality is not assumed to hold.

VIII. USE OF INEQUALITY

We shall take the probability density $\rho(\lambda)$ in Eq. (26) to be the probability density of a hidden-variable model satisfying the three conditions of Sec. VII. Upon choosing $A(\lambda,\hat{a})$ $[B(\lambda,\hat{b})]$ to have the value $+1$ at all points λ for which particle A (particle B) is detected with spin parallel (antiparallel) to the apparatus axis, and to have the value -1 at all points λ for which the particle is detected with spin antiparallel (parallel) to the apparatus axis *or* for which the particle is not detected at all, we see that the value of the integral in Eq. (26) is

$$\begin{aligned}S&=(1)(1)P_{1,-1}+(1)(-1)[P_{1,1}+P_{1,0}]\\&\quad+(-1)(1)[P_{0,-1}+P_{-1,-1}]\\&\quad+(-1)(-1)[P_{0,1}+P_{0,0}+P_{-1,0}+P_{-1,1}]\\&=2g(\alpha)\cos^2(\tfrac{1}{2}\alpha)+1-4C \qquad (39)\end{aligned}$$

When Eq. (39) is inserted into Eq. (28), we obtain an inequality which the $g(\alpha)$ of this model must satisfy, viz.,

$$(2n-1)X(\alpha)\cos^2(\tfrac{1}{2}\alpha)\leq 2(n-1)+X(\beta)\cos^2(\tfrac{1}{2}\beta)\,, \qquad (40)$$

where we have introduced the new variable

$$X(\alpha)\equiv g(\alpha)/2C\,, \qquad (41)$$

which satisfies $1\geq X(\alpha)\geq 0$ [by Eq. (38)]. In Eq. (40), α is the angle between any two adjacent vectors in a sequence of $2n$ vectors, and β is the angle between the first and last vectors of the sequence.

Similarly by choosing $A(\lambda,\hat{a})$ $[B(\lambda,\hat{b})]$ to have the value $+1$ at all points λ for which particle A (B) is not detected, and to have the value -1 at all other points, we find that in this case the value of the integral in Eq. (30) is

$$\begin{aligned}S(\alpha)&=(1)(1)P_{0,0}+(1)(-1)(P_{0,1}+P_{0,-1})\\&\quad+(-1)(1)(P_{1,0}+P_{-1,0})\\&\quad+(-1)(-1)(P_{1,1}+P_{1,-1}+P_{-1,1}+P_{-1,-1})\\&=4g(\alpha)+1-8C. \qquad (42)\end{aligned}$$

Upon inserting Eq. (44) into Eq. (31), we obtain a second inequality which $g(\alpha)$ must satisfy:

$$(2n-1)X(\alpha)\leq 2(n-1)+X(\beta)\,. \qquad (43)$$

Two new inequalities can be obtained if we replace $A(\lambda,\hat{a})$ by $-A(\lambda,\hat{a})$ in the previous two paragraphs, but we have not found these inequalities to be useful, so we shall not include them here. All other natural choices of definitions for $A(\lambda,\hat{a})$ and $B(\lambda,\hat{b})$ produce inequalities which are identical to those already mentioned.

The upper bound on $X(\alpha)$ obtained from Eqs. (40) and (43) is drawn in Fig. 1; the procedure by which this graph was obtained is outlined in Appendix C. Here we will merely note that since $X(\alpha)=X(\pi-\alpha)$, if we set $\beta=\pi-\alpha$ we obtain inequalities involving $X(\alpha)$ alone. For $\alpha=\frac{1}{4}\pi$, $\beta=\frac{4}{3}\pi$, and $n=2$, we find from Eq. (40) that

$$X(45°)\leq 0.83\,, \qquad (44)$$

which is the smallest bound we have been able to obtain from any angle.

It is now possible to find an upper bound on $g(\alpha)$ itself by employing the inequality (38) involving C, and the smallest bound (44):

$$\tfrac{1}{4}+\tfrac{1}{4}\times 0.83\times 2C\geq C \quad\text{or}\quad 0.43\geq C. \qquad (45)$$

Since $g(\alpha)=2CX(\alpha)$, we see from the upper bound displayed in Fig. (1) that $g(\alpha)$ must be less than 0.86 everywhere: In particular, this is our upper bound at $\alpha=0°$ (and 180°), while at $\alpha=45°$ (and 135°), g must be less than $0.86\times 0.83\approx 0.72$.

IX. REMARKS

(A) We have shown that it is possible to make a local hidden-variable theory, based upon the data-rejection hypothesis, by constructing an explicit model. We have obtained an upper bound on the fraction of events in which both particles are detected, for any such model in a wide class. Because we found that in 14% or more of the events one or both particles will go undetected, it is difficult to take this hypothesis seriously as a physical principle capable of extension to a large group of phenomena; had such large fractions of undetected events occurred in other already performed correlation experiments, it is hard to see how such behavior would have gone unnoticed.

(B) A correlation experiment of the type considered here (utilizing photons whose polarizations are measured by their being passed or stopped by a polaroid filter) has been recently proposed by Clauser, Horne, Shimony, and Holt.[6] This experiment will test whether nature chooses to satisfy an inequality [of the type of Eq. (27), with $n=2$] which must be satisfied by a local hidden-variable theory (as was first shown by Bell[1]) but which is not satisfied by quantum theory.

In the language of the comparable measurement on spin-$\frac{1}{2}$ particles, the experimentally obtainable quantities are (1) the rate of events in which both particles are measured with spins parallel to their respective apparatus axes, (2) the rate of events in which one particle's spin is measured parallel to its apparatus axis while the other apparatus is removed and the other particle is detected (without having its spin measured), and (3) the rate of events in which both particles are detected (without having their spins measured) while both apparatuses are removed.

In order to apply the three-outcome local hidden-variable model presented here to this experiment, we must make additional assumptions about the counting rates in (2) and (3), which require experimental setups different from that considered for our model. It appears most natural to assume that when both spin-measuring apparatuses are removed, all particles are detected at a rate in agreement with that predicted by quantum theory, and also that when one apparatus is removed, the rate at which the other apparatus measures the particle's spin component is unaffected. This means that the counting rate in (2) is

$$P_{1,1}(\alpha)+P_{1,0}(\alpha)+P_{1,-1}(\alpha)=C$$

times the counting rate in (3).

With these assumptions, it is readily seen that this experiment cannot distinguish between a two-outcome and a three-outcome local hidden-variable model. Indeed, the experimentally measured quantities do not distinguish between a particle whose spin is measured antiparallel to the apparatus axis and a particle which is not detected at all. Therefore the experiment effectively turns the three-outcome model into a two-outcome model. It is amusing that the three-outcome model appears to yield the predictions of quantum theory in the more difficult experiment where all spin components are measured (because the extra information encourages one to selectively reject data), while this relatively simpler experiment distinguishes the three-outcome model from quantum theory (because no data is rejected); of course the former experiment would perform the same service as the latter experiment if the data were properly handled.

Thus if the outcome of this experiment is in agreement with the predictions of quantum mechanics, both local hidden-variable models will be rejected. But if the inequality is satisfied, further experimentation will be necessary to determine which model is correct. A crucial test of the three-outcome model with the above assumptions would be to compare the counting rate in (2) with that in (3). According to quantum theory, the ratio of these rates should be 0.5; according to the model, this ratio should be 0.43 or less. Indeed, sufficient data to settle this question may have been taken in already performed photon-correlation experiments.[7]

(C) The model presented here is not complete in that its extension to measurements on more complicated physical systems is not readily apparent and such an extension would lead to new unresolved difficulties. For example, if the two spin-$\frac{1}{2}$ particles are oppositely charged, and one of them is detected while the other is not, does this mean that the model predicts an experimentally measurable violation of charge conservation? Or shall we interpret the words "particle is undetected" to mean that the particle will only be undetected by an apparatus which is capable of measuring its spin, but that an apparatus incapable of measuring its spin can detect it? This rescues charge conservation at the expense of introducing an unusual kind of incompatibility between spin measurements and charge measurements. However, the resolution of these difficulties does not presently appear to be an urgent problem.

[7] C. S. Wu and I. Shaknov, Phys. Rev. **77**, 136 (1950); C. A. Kocher and E. D. Commins, Phys. Rev. Letters **18**, 575 (1967).

APPENDIX A

We wish to calculate the area on a spherical surface of radius r, which is common to the interior of two cones each of opening angle $\beta\leq\pi$, whose axes pass through the origin at a relative angle α. When $\alpha\geq\beta$, the area of overlap of the two circular regions cut by the cones on the surface is zero. In order to calculate the overlap area when $\alpha<\beta$, we choose a coordinate system in which the two cone axes $\hat{a}$ and $\hat{b}$ lie in the xy plane, each making an angle $\frac{1}{2}\alpha$ with the y axis. The boundaries of the two circular regions are then given by the expressions $\hat{a}\cdot\mathbf{r}=r\cos\frac{1}{2}\beta$ and $\hat{b}\cdot\mathbf{r}=r\cos\frac{1}{2}\beta$, which become (using $\hat{a}=\hat{\imath}_x\sin\frac{1}{2}\alpha+\hat{\imath}_y\cos\frac{1}{2}\alpha$, $\hat{b}=-\hat{\imath}_x\sin\frac{1}{2}\alpha+\hat{\imath}_y\cos\frac{1}{2}\alpha$, and polar coordinates)

$$\begin{aligned}\cos\tfrac{1}{2}\beta&=\sin\theta\sin(\varphi+\tfrac{1}{2}\alpha)\\&=\sin\theta\sin(\varphi-\tfrac{1}{2}\alpha).\end{aligned}\tag{A1}$$

These boundaries intersect at $\varphi=\frac{1}{2}\pi$ in two points which, according to Eq. (A1), are characterized by

$$\sin\theta=\cos\tfrac{1}{2}\beta/\cos\tfrac{1}{2}\alpha.\tag{A2}$$

The area $I(\beta,\alpha)$ between these boundaries is four times the area lying within the octant $x\geq0$, $y\geq0$, $z\geq0$, so

$$I(\beta,\alpha)=4r^2\int_{\sin^{-1}(\cos\frac{1}{2}\beta/\cos\frac{1}{2}\alpha)}^{\frac{1}{2}\pi}\sin\theta d\theta\times\int_{\frac{1}{2}\alpha+\sin^{-1}(\cos\frac{1}{2}\beta/\sin\theta)}^{\frac{1}{2}\pi}d\varphi.\tag{A3}$$

After integration over φ followed by an integration by parts with respect to θ, we obtain

$$I(\beta,\alpha)=4r^2\int_{\sin^{-1}(\cos\frac{1}{2}\beta/\cos\frac{1}{2}\alpha)}^{\frac{1}{2}\pi}d\theta\cot^2\theta\times\frac{\cos\frac{1}{2}\beta}{[1-(\cos^2(\frac{1}{2}\beta)/\sin^2\theta)]^{1/2}}.\tag{A4}$$

Finally, a change of variable of integration to $\lambda=\cos^{-1}(\cos\frac{1}{2}\beta/\sin\theta)$ yields

$$I(\beta,\alpha)=4r^2\int_{\frac{1}{2}\alpha}^{\frac{1}{2}\beta}d\lambda\left[1-\left(\frac{\cos\frac{1}{2}\beta}{\cos\lambda}\right)^2\right]^{1/2}.\tag{A5}$$

This integral can be evaluated, but we will not find it necessary to do so.

APPENDIX B

We wish to solve the integral equation

$$\lambda(x)=\frac{4}{\pi}\int_0^1 h(zx)(1-z^2)^{1/2}dz\tag{B1}$$

for $h(x)$ (here λ is an arbitrary analytic function over the interval $0\leq x\leq 1$). We begin by expanding both sides of this equation in powers of x and equating terms. Upon inserting the series

$$\lambda(x)=\sum_{n=0}^{\infty}\lambda^{(n)}x^n\,,\quad h(zx)=\sum_{n=0}^{\infty}h^{(n)}x^nz^n \qquad \text{(B2)}$$

into Eq. (B1), performing the integrations, and equating terms, we obtain

$$h^{(n)}=\tfrac{1}{2}\pi\lambda^{(n)}\frac{\Gamma(\frac{1}{2}n+2)}{\Gamma(\frac{1}{2}(n+1))\Gamma(\frac{3}{2})}\,. \qquad \text{(B3)}$$

It will be useful for us to recognize that Eq. (B3) can also be written as

$$h^{(n)}=\tfrac{1}{2}\pi\lambda^{(n)}\frac{(\frac{1}{2}n+1)(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}\frac{\Gamma(\frac{1}{2}n+1)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}n+\frac{3}{2})}$$

$$=\tfrac{1}{2}\lambda^{(n)}(n+2)(n+1)\int_0^1\frac{w^{n+1}dw}{(1-w^2)^{1/2}}\,. \qquad \text{(B4)}$$

Upon forming the sum $\sum h^{(n)}x^n$, we obtain

$$h(x)=\frac{1}{2}\int_0^1\frac{wdw}{(1-w^2)^{1/2}}\sum_{n=0}^{\infty}\lambda^{(n)}(xw)^n(n+2)(n+1)\,, \qquad \text{(B5)}$$

which can be written in terms of λ itself by replacing $x^n(n+2)(n+1)$ by d^2x^{n+2}/dx^2:

$$h(x)=\frac{1}{2}\frac{d^2}{dx^2}x^2\int_0^1\frac{\lambda(xw)wdw}{(1-w^2)^{1/2}}\,. \qquad \text{(B6)}$$

Eq. (B6) is the desired solution.

APPENDIX C

Here we outline the procedure by means of which the upper bound on $X(\alpha)$ illustrated in Fig. 1 can be obtained from the inequalities (40) and (43):

$$(2n-1)X(\alpha)\cos^2(\tfrac{1}{2}\alpha)\leq 2(n-1)+X(\beta)\cos^2(\tfrac{1}{2}\beta)\,, \qquad \text{(C1)}$$

$$(2n-1)X(\alpha)\leq 2(n-1)+X(\beta)\,. \qquad \text{(C2)}$$

We first consider the use of Eq. (C1), and remark that the inequality is certainly satisfied, regardless of the value of $X(\alpha)$ (or β), if $(2n-1)\cos^2(\frac{1}{2}\alpha)<2(n-1)$. Thus we will only find the inequality useful for

$$\begin{array}{ll} 0\leq\alpha\lesssim 70°\,, & n=2\\ 0\leq\alpha\lesssim 53°\,, & n=3\\ 0\leq\alpha\lesssim 44°\,, & n=4\\ \vdots & \vdots \end{array}$$

Since $X(\alpha)=X(\pi-\alpha)$, an upper bound for $\alpha\lesssim 70°$ can be turned into an upper bound for $\alpha\gtrsim 110°$; for α between 70° and 110° we will use the inequality (C2).

The inequality (C1) is most powerful when α is as small as possible and β is as large as possible, because we find that $\cos^2(\frac{1}{2}\alpha)$ is a much more rapidly varying function of α than is $X(\alpha)$.

We first utilize Eq. (C1) by setting $\beta=\pi-\alpha$; since $X(\alpha)=X(\pi-\alpha)$ we obtain an inequality involving a single variable, and a consequent upper bound

$$X(\alpha)\leq\frac{2}{1+[n/(n-1)]\cos\alpha}\,, \qquad \text{(C3)}$$

for α such that $\pi/2n\leq\alpha\leq\cos^{-1}(n-1/n)$: these limits on α stem from the restriction that α can be no less than $\beta/(2n-1)$, and from the knowledge that $X(\alpha)\leq 1$. We compute an upper bound on $X(\alpha)$ from Eq. (C3) for each value of n, and take the least of these upper bounds.

Thus we obtain an initial upper bound for all $0\leq\alpha\leq 60°$ (and by the symmetry relation for $120°\geq\alpha\geq 180°$) which, as expected, is especially good when α takes on its minimum possible value $\pi/2n$.

For $60°\leq\alpha\leq 70°$, we may obtain a useful upper bound from Eq. (C1) by setting $\beta=\pi$.

Turning to Eq. (C2), we set $\beta=135°$ and $n=2$; because $X(135°)\leq 0.83$ [Eq. (44)], we obtain

$$X(\alpha)\leq 0.943\,,\quad 45°\leq\alpha\leq 90° \qquad \text{(C4)}$$

which is our lowest bound in the region $70°\lesssim\alpha\leq 90°$ (and by symmetry for $90°\leq\alpha\lesssim 110°$), where Eq. (C1) is not useful. We now have an initial upper bound for all α.

Returning to Eq. (C1), we set $\alpha=\beta/(2n-1)$ and by letting β successively take on values between 180° and 0°, and $X(\beta)$ take on the values of the initial upper bound, we can obtain an upper bound on $X(\alpha)$ which, for certain n and certain ranges of α, is an improvement over the initial upper bound. Eventually we end up with the upper bound illustrated in Fig. 1.