Coherent States and Transition Probabilities in a Time-Dependent Electromagnetic Field

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New time-dependent invariants for the N-dimensional nonstationary harmonic oscillator and for a charged particle in a varying axially symmetric classical electromagnetic field are found. For these quantum systems, coherent states are introduced, and the Green's functions are obtained in closed form. For a special type of electromagnetic field which is constant in the remote past and future, the transition amplitudes between both arbitrary coherent states and energy eigenstates are calculated and expressed in terms of classical polynomials. The adiabatic approximation and adiabatic invariants are discussed. In the special case of a particle with time-dependent mass, the solution of the Schrödinger equation is found. The symmetry of nonstationary Hamiltonians is discussed, and the noncompact group $U(N,1)$ is shown to be the group of dynamical symmetry for the time-dependent N-dimensional oscillator.

I. INTRODUCTION

THE Feynman path integrals method provides a simple possibility for considering any quantum system both with constant and time-dependent quadratic Hamiltonians because the Green's functions in these cases are equal to $\exp(iS_{cl}/\hbar)$.[1] Nevertheless, the details of any nonstationary problem are interesting, and there are many attempts to treat different aspects of these problems. The Glauber coherent-states representation[2] proves to be very convenient for many problems of quantum theory. The main aim of this article is to introduce coherent states for some time-dependent Hamiltonians and to use this representation for calculations.

Recently a class of explicitly time-dependent invariants for classical and quantum time-dependent one-dimensional harmonic oscillators was found by Lewis.[3] Lewis and Riesenfeld[4] have developed the theory of time-dependent invariants for nonstationary quantum systems and applied this method to the one-dimensional time-dependent harmonic oscillator and to charged particle motion in a time-dependent electromagnetic field. This paper gives a method for the calculation of the transition probability between energy eigenstates of the quantum systems under consideration. Earlier, the Green's function and the transition amplitudes connecting any arbitrary initial energy eigenstate to a final one for the case of a forced one-dimensional harmonic oscillator with a time-dependent frequency have been calculated in a quite different way by Husimi.[5] Husimi has obtained and used the generating function for the calculation of these amplitudes. Using quadratic invariants found in Refs. 3 and 4, Crosignani, Di Porto, and Solimeno have calculated the evolution of an initial coherent state of the quantum oscillator and considered adiabatic invariants for the oscillator with time-dependent frequency.[6] Dyhne[7] has calculated transition probabilities for the harmonic oscillator in the adiabatic approximation.

The coherent state of the oscillator with a constant frequency is the classical packet, which was considered by Schrödinger.[8] Coherent states for both nonrelativistic and relativistic charged particles moving in a constant classical electromagnetic field ($\mathcal{E}\perp\mathcal{H}$, $\mathcal{H}^2-\mathcal{E}^2>0$) were introduced in Ref. 9. These states are connected with the classical packets constructed by Darwin[10] and Kennard.[11] Coherent states for a time-dependent magnetic field have been briefly discussed in Ref. 12.

Our purpose in the present article is to find all linear independent invariants for the N-dimensional quantum oscillator with time-dependent frequencies and for a charged particle in an axially symmetric and uniform time-dependent electromagnetic field. Using these invariants, we construct coherent states for these quantum systems and the Green's functions for the corresponding Schrödinger equations. Employing the explicit form of coherent states, we get all transition amplitudes for the quantum systems under consideration. We start with a brief discussion of the problem of invariants in quantum mechanics (Sec. II) and give a survey of properties of coherent states, taking the one-dimensional harmonic oscillator with a constant frequency as an example (Sec. III). The N-dimensional nonstationary quantum oscillator is treated in Sec. IV. In Sec. V we consider a charged particle moving in a

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[4] H. R. Lewis and W. B. Risenfeld, J. Math. Phys. **10**, 1458 (1969).
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[8] E. Schrödinger, Naturwiss. **14**, 664 (1926).
[9] I. A. Malkin and V. I. Man'ko, Zh. Eksperim. i Teor. Fiz. **55**, 1014 (1968) [Soviet Phys. JETP **28**, 527 (1969)]; coherent states in this problem for the nonrelativistic states have been also considered by R. Bonifacio (unpublished).
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[11] E. H. Kennard, Z. Physik **44**, 326 (1927).
[12] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, Phys. Letters **30A**, 414 (1969; Zh. Eksperim. i Teor. Fiz. **58**, 721 (1970) [Soviet Phys. JETP (to be translated)].

varying electromagnetic field and we obtain an exact solution in terms of a function which satisfies a linear differential equation describing classical motion. The evolution of adiabatic invariants is calculated in Sec. VI. The results of the previous sections are generalized to the case of a particle with time-dependent mass in Sec. VII.

II. INTEGRALS OF MOTION IN QUANTUM MECHANICS

First we shall consider the question of how many integrals of motion there are in quantum mechanics. The answer to this question is known. However, sometimes there is certain misunderstanding, especially when one deals with a complex quantum system. It is known that for a classical system which has n degrees of freedom there are exactly $2n$ independent integrals of motion. One may choose n initial coordinates and n initial momenta as these integrals. All the other conserved numbers may be expressed in terms of these integrals of motion. The correspondence between quantum and classical mechanics demands that for any quantum system with n degrees of freedom there should be $2n$ integrals of motion too. They commute with the operator $i\partial/\partial t-H$. It is an obvious fact, but if one looks in any textbook for the two integrals of motion for the one-dimensional quantum oscillator with constant frequency, $H=\omega(a^\dagger a+\frac{1}{2})$, the answer will be difficult to find. These two integrals of motion are $\frac{1}{2}(ae^{i\omega t}+a^\dagger e^{-i\omega t})$, $\frac{1}{2}(ae^{i\omega t}-a^\dagger e^{-i\omega t})$. It is more convenient to consider one non-Hermitian invariant $ae^{i\omega t}$. The real and imaginary parts of it are the two integrals of motion. These integrals are linear forms with respect to coordinate and momentum operators and are physically interpreted as initial coordinates in the phase space. Thus for any Schrödinger equation with any potential (both constant and time-dependent), there must be $2n$ linear integrals of motion. There is an interesting problem of the connection between adiabatic invariants and exact invariants.[13] If one has the time-dependent potential, the quantum numbers are known to be adiabatic invariants.[13] It follows from the above that the quantum numbers are exact invariants. If one had the initial quantum numbers they would determine the state of the system for any time. This becomes very apparent if one proceeds to construct the coherent states for the quantum system. We shall illustrate this statement by the examples of an N-dimensional oscillator and a charge moving in a magnetic field. The adiabatic invariants of all orders may be obtained by expanding the exact invariant in powers of $\dot{\omega}/\omega^2$, where ω is the characteristic frequency. Thus for any quantum system there must be adiabatic invariants corresponding to the exact invariants. If the exact invariants are non-Hermitian and linear with respect to the coordinate and momentum, this should be taken into account when one considers the adiabatic invariant, which in this case contains the phase factor. These ideas will be considered in greater detail in another paper.

[13] M. Born and V. Fock, Z. Physik 51, 165 (1928).

III. COHERENT STATES FOR ONE-DIMENSIONAL QUANTUM OSCILLATOR WITH CONSTANT FREQUENCY

Since in what follows it is necessary to use the problem of the quantum oscillator, we shall briefly review the results referring to the coherent states of a constant-frequency oscillator. A similar discussion of these results is given by Glauber.[2]

The wave equation for the one-dimensional harmonic oscillator has the form

$$\Omega(a^\dagger a+\tfrac{1}{2})\Psi=E\Psi\,,\quad \hbar=c=1 \tag{1}$$

where

$$a=2^{-1/2}[q(M\Omega)^{1/2}+ip(M\Omega)^{-1/2}]\,,\quad a^\dagger=2^{-1/2}[q(M\Omega)^{1/2}-ip(M\Omega)^{-1/2}] \tag{2}$$

are lowering and raising operators:

$$[a,a^\dagger]=1\,. \tag{3}$$

Then the eigenvalues of energy are $E_n=\Omega(n+\frac{1}{2})$ and the energy eigenstates are constructed by means of the operators (2):

$$|n\rangle=\frac{(a^\dagger)^n}{(n!)^{1/2}}|0\rangle,\quad \langle m|n\rangle=\delta_{m,n}\,, \tag{4}$$

where $|0\rangle$ is a vacuum: $a|0\rangle=0$, $\langle 0|0\rangle=1$.

$$a|n\rangle=n^{1/2}|n-1\rangle,\quad a^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle. \tag{5}$$

Coherent states are introduced by Glauber[2] as eigenstates of the lowering operator a:

$$a|\alpha\rangle=\alpha|\alpha\rangle\,, \tag{6}$$

where α is a complex number. One can easily check that the normalized eigenstate $|\alpha\rangle$ is given by

$$|\alpha\rangle=\exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n\rangle. \tag{7}$$

This form shows that the average occupation number of the state $|n\rangle$ is given by a Poisson distribution with mean value

$$|\langle\alpha|n\rangle|^2=(|\alpha|^{2n}/n!)e^{-|\alpha|^2}. \tag{8}$$

The coherent state $|\alpha\rangle$ may be obtained from the ground state $|0\rangle$ by acting with the unitary operator

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a)\,,\quad |\alpha\rangle=D(\alpha)|0\rangle. \tag{9}$$

This operator acts as a displacement operator upon a and $a^\dagger$:

$$D^{-1}(\alpha)aD(\alpha)=a+\alpha. \tag{10a}$$

The following formula holds:

$$D(\alpha_1)D(\alpha_2)=D(\alpha_1+\alpha_2)\exp[i\,\mathrm{Im}(\alpha_1\alpha_2^*)]. \tag{10b}$$

Using the operator $D(\alpha)$, one can easily obtain an explicit formula for $|\alpha\rangle$:

$$|\alpha\rangle=(\Omega/\pi)^{1/4}\exp\{-[(\tfrac{1}{2}\Omega)^{1/2}q-\alpha]^2+\tfrac{1}{2}(\alpha^2-|\alpha|^2)\}. \quad (11)$$

Employing the representation (7), the scalar product may be easily calculated:

$$\langle\alpha|\beta\rangle=\exp[\alpha^*\beta-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)],$$
$$|\langle\alpha|\beta\rangle|^2=\exp(-|\alpha-\beta|^2). \quad (12)$$

The coherent state tends to become approximately orthogonal for α and β which are sufficiently different. But they do form a complete set:

$$(1/\pi)\int|\alpha\rangle\langle\alpha|d^2\alpha=1, \quad (13)$$

and for an arbitrary state $|f\rangle$, we get

$$|f\rangle=(1/\pi)\int|\alpha\rangle f(\alpha^*)\exp(-\tfrac{1}{2}|\alpha|^2)d^2\alpha,$$
$$f(\alpha^*)=\langle\alpha|f\rangle\exp(\tfrac{1}{2}|\alpha|^2). \quad (14)$$

Moreover, coherent states form an overcomplete set of functions in the sense that if we have any convergent sequence of complex numbers $\alpha_n\to\alpha_0$, the coherent states $|\alpha_n\rangle$ themselves form a complete set.[14] Thus, in general, expansion (14) is not unique. Equation (14) gives a unique expansion if the expansion amplitudes $f(\alpha^*)$ depend analytically upon the variable α^*. This holds, for example, if $|f\rangle$ is a coherent state, and Eq. (14) gives in this case the expression for any given coherent state in terms of all of the others:

$$|\beta\rangle=(1/\pi)\int|\alpha\rangle\exp[\alpha^*\beta-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)]d^2\alpha. \quad (15)$$

The time-independent states $|\alpha\rangle$ are those characteristic of the Heisenberg picture of quantum mechanics. The corresponding Schrödinger states take the following form:

$$|\alpha\rangle_{\mathrm{Sr}}=|\alpha\exp(-i\Omega t)\rangle\exp(-\tfrac{1}{2}i\Omega t). \quad (16)$$

Furthermore, we shall work only with $|\alpha\rangle_{\mathrm{Sr}}$ and denote it as $|\alpha\rangle$.

The packets $|\alpha\rangle$ describe the most classical states of the quantum oscillator. They represent as close an approach to classical localization of the particle as is possible; in the coherent states, the uncertainty relation reaches its minimum

$$(\Delta p)^2(\Delta q)^2=\tfrac{1}{4}, \quad (17)$$

where

$$(\Delta p)^2=\langle p^2\rangle-\langle p\rangle^2,\quad (\Delta q)^2=\langle p^2\rangle-\langle p\rangle^2.$$

The expectation values of the coordinate and momentum carry out a harmonic motion with amplitudes $|\alpha|(2/M\Omega)^{1/2}$ and $|\alpha|(2M\Omega)^{1/2}$, respectively [see (2)]. In the complex plane α, this is a motion along a circle with radius $|\alpha|$ and frequency Ω. We recall that the motion of a classical oscillator in the complex plane of $\alpha=2^{1/2}[q(M\Omega)^{1/2}+ip(M\Omega)^{-1/2}]$ is a motion along a circle with radius $|\alpha|$ and frequency Ω. Thus the quantity $|\alpha|$ determines the classical amplitudes of the oscillations of the quantum oscillator, and the phase $\varphi(\alpha)$ is the classical phase of the oscillations of the same oscillator. However, the quantum oscillator in the state $|\alpha\rangle$ in the language of classical mechanics corresponds to the motion along a circle in the phase plane of a whole set of classical oscillators each of which oscillates with its own amplitude, which determines its energy.

[14] K. Cahill, Phys. Rev. **138**, 1566 (1965).

IV. N-DIMENSIONAL TIME-DEPENDENT QUANTUM OSCILLATOR

A. Invariants

The Hamiltonian of this quantum system is

$$H=\sum_{k=1}^{N}\left[\frac{p_k^2}{2M_k}+\tfrac{1}{2}M_k\Omega_k^2(t)q_k^2\right], \quad (18)$$

where q_k are canonical coordinates, p_k are their conjugate momenta, $\Omega_k(t)$ are arbitrary continuous functions of time, and M_k are mass parameters. For the sake of simplicity, we shall take $M_k=1$, i.e., we shall use new coordinates $M_k^{1/2}q_k$ which we also denote as q_k. The wave equation is

$$\left\{i\frac{\partial}{\partial t}-\frac{1}{2}\sum_{k=1}^{N}[p_k^2+\Omega_k^2(t)q_k^2]\right\}\Psi=0,\quad \hbar=c=1. \quad (19)$$

In accordance with Sec. II, there must be N independent (non-Hermitian) invariants, and one can make sure that the operators

$$A_k(t)=i2^{-1/2}[\epsilon_k(t)p_k-\dot\epsilon_k(t)q_k],\quad k=1,\ldots,N \quad (20)$$

where $\epsilon_k(t)$ are special solutions of the equations[7]

$$\ddot\epsilon_k+\Omega_k^2(t)\epsilon_k=0,\quad \epsilon_k=|\epsilon_k|\exp\left(i\int^t\frac{d\tau}{|\epsilon_k|^2}\right), \quad (21)$$

commute with the operator $i\partial/\partial t-H$ and, thus, are invariants. The following commutation relations hold:

$$[A_k,A_l^\dagger]=\delta_{k,l},\quad [A_k,A_l]=0. \quad (22)$$

Equations (21) are equivalent to the nonlinear ones[4]

$$\frac{d^2}{dt^2}|\epsilon_k|+\Omega_k^2(t)|\epsilon_k|-\frac{1}{|\epsilon_k|^3}=0. \quad (23)$$

B. Coherent States

One can observe now the manner of construction of coherent states for the N-dimensional time-dependent quantum oscillator, i.e., the normalized eigenstates of invariants (20). We define the unitary operators

$$D(\boldsymbol{\alpha})=\prod_{k=1}^{N}\exp(\alpha_k A_k^\dagger-\alpha_k^* A_k),\quad \boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_N) \tag{24}$$

where α_k are arbitrary complex numbers, and construct the normalized vacuum state

$$|\mathbf{0};t\rangle=\prod_{k=1}^{N}\pi^{-1/4}\epsilon_k^{-1/2}\exp\left(\frac{i\dot{\epsilon}_k}{2\epsilon_k}q_k^2\right),\quad A_k|\mathbf{0};t\rangle=0 \tag{25}$$

which obeys the wave equation $(i\partial/\partial t-H)|\mathbf{0};t\rangle=0$.

By acting upon vacuum (25) with unitary operator (24), we get the required coherent states

$$|\boldsymbol{\alpha};t\rangle=\prod_{k=1}^{N}\pi^{-1/4}\epsilon_k^{-1/2}\exp\left[\frac{i\dot{\epsilon}_k}{2\epsilon_k}\left(q_k-i\frac{\sqrt{2}\alpha_k}{\dot{\epsilon}_k}\right)^2-\frac{1}{2}\left(\frac{\dot{\epsilon}_k^*}{\dot{\epsilon}_k}\alpha_k^2+|\alpha_k|^2\right)\right]. \tag{26}$$

Since A_k are invariants, $|\boldsymbol{\alpha};t\rangle$ obeys the Schrödinger equation and this may be checked by direct differentiation of $|\boldsymbol{\alpha};t\rangle$. The following formulas hold:

$$A_k|\boldsymbol{\alpha};t\rangle=\alpha_k|\boldsymbol{\alpha};t\rangle, \tag{27}$$

$$\langle\boldsymbol{\beta};t|\boldsymbol{\alpha};t\rangle=\prod_{k=1}^{N}\exp[\beta_k^*\alpha_k-\tfrac{1}{2}(|\alpha_k|^2+|\beta_k|^2)], \tag{28}$$

$$\frac{1}{\pi^N}\int|\boldsymbol{\alpha};t\rangle\langle\boldsymbol{\alpha};t|d\mu(\boldsymbol{\alpha})=1,\quad d\mu(\boldsymbol{\alpha})=\prod_{k=1}^{N}d^2\alpha_k. \tag{29}$$

The coherent state $|\boldsymbol{\alpha};t\rangle$ is a generating function for the eigenstates of the Hermitian operators $I_k=A_k^\dagger A_k$:

$$|\boldsymbol{\alpha};t\rangle=\exp(-\tfrac{1}{2}|\boldsymbol{\alpha}|^2)\times\sum_{n_k=0}^{\infty}\frac{\alpha_1^{n_1}\cdots\alpha_k^{n_k}\cdots\alpha_N^{n_N}}{(n_1!\cdots n_N!)^{1/2}}|\mathbf{n};t\rangle, \tag{30}$$

$$I_k|\mathbf{n};t\rangle=n_k|\mathbf{n};t\rangle,\quad \langle\mathbf{m};t|\mathbf{n};t\rangle=\delta_{\mathbf{n},\mathbf{m}}, \tag{31}$$

where

$$|\boldsymbol{\alpha}|^2=\sum_{k=1}^{N}|\alpha_k|^2,\quad \mathbf{n}=(n_1,\cdots,n_N),$$

and n_k are positive integers. The eigenstates $|\mathbf{n};t\rangle$ are constructed by means of invariants A_k as follows:

$$|\mathbf{n};t\rangle=\prod_{k=1}^{N}|n_k;t\rangle=\prod_{k=1}^{N}\frac{(A_k^\dagger)^{n_k}}{(n_k!)^{1/2}}|0;t\rangle, \tag{32}$$

and their explicit form may be obtained from the explicit form of $|\boldsymbol{\alpha};t\rangle$, if we recall the generating functions of the Hermite polynomials,[15]

$$|\mathbf{n};t\rangle=\prod_{k=1}^{N}\left(\frac{\epsilon_k^*}{2\epsilon_k}\right)^{n_k/2}(n_k!\epsilon_k\pi^{1/2})^{-1/2}\times\exp\left(\frac{i\dot{\epsilon}_k}{2\epsilon_k}q_k\right)H_{n_k}\left(\frac{q_k}{|\epsilon_k|}\right). \tag{33}$$

These formulas are the trivial generalizations of Husimi's results for the one-dimensional oscillator.[5] We observe that the coherent states $|\boldsymbol{\alpha};t\rangle$ and the eigenstates of I_k, $|\mathbf{n};t\rangle$, for the N-dimensional oscillator are obtained as the product of the corresponding states for the one-dimensional oscillator. It is apparent that the coherent states for the one-dimensional time-dependent oscillator are solutions of Gaussian type, i.e., of the following form: an exponential of a quadratic. Gaussian packets for a time-dependent oscillator were first found by Husimi,[5] who also found the Green's function for this system. In terms of coherent states, this Green's function for our system can be easily obtained as $\pi^{-N}\int d\mu(\boldsymbol{\alpha})|\boldsymbol{\alpha};\mathbf{q}_2,t_2\rangle\langle\boldsymbol{\alpha};\mathbf{q}_1,t_1|$:

$$G(\mathbf{q}_2,t_2;\mathbf{q}_1,t_1)=\prod_{k=1}^{N}(2\pi i|\epsilon_{k;1}\epsilon_{k;2}|\sin\gamma_k)^{-1/2}\exp\left(-\frac{i}{\sin\gamma_k}Q_{k;2}Q_{k;1}\right)\times\exp\left[\tfrac{1}{2}i\cot\gamma_k(Q_{k;2}{}^2+Q_{k;1}{}^2)+\tfrac{1}{4}i\left(Q_{k;2}{}^2\frac{d|\epsilon_k|^2}{dt_2}-Q_{k;1}{}^2\frac{d|\epsilon_k|^2}{dt_1}\right)\right], \tag{34}$$

where

$$\gamma_k=\int_{t_1}^{t_2}|\epsilon_k|^{-2}dt,\quad Q_{k;1,2}=q_k|\epsilon_k(t_{1,2})|^{-1},$$

and

$$\mathbf{q}=(q_1,\ldots,q_N).$$

In order to elucidate the physical meaning of the coherent states $|\boldsymbol{\alpha};t\rangle$ and of the invariants $A_k(t)$, let us consider their limits for constant frequencies Ω_k. For simplicity we shall suppose that $\Omega(t)=\Omega_k{}^i$ for $t<0$, and $\Omega_k(t)=\Omega_k{}^f$ for $t\to\infty$, where $\Omega_k{}^{i,f}$ are constants. Then for the limits $t\to\mp\infty$, there exists a complete set of coherent states $|\boldsymbol{\alpha};i\rangle$ and $|\boldsymbol{\beta};f\rangle$, and of orthonormalized energy eigenstates $|\mathbf{n};i\rangle$ and $|\mathbf{m};f\rangle$, and the transition amplitudes between these states may be calculated. The general expression of the transition amplitude connecting an initial state $|i\rangle$ and a final one $|f\rangle$ is given by the matrix element

$$T_i{}^f=\langle f|t\to\infty\rangle, \tag{35}$$

[15] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by Erdélyi (McGraw-Hill, New York, 1953), Vol. II.

where $|t\rightarrow\infty\rangle$ is the $t\rightarrow\infty$ limit of the state $|t\rangle$, which has as its $t\rightarrow-\infty$ limit the state $|i\rangle$. We are free to choose the initial conditions for the solutions of Eqs. (21) or (23) in order to get correct limits as $t\rightarrow-\infty$ of the states $|\boldsymbol{\alpha};t\rangle$ and $|\mathbf{n};t\rangle$:

$$\epsilon_k(-\infty)=(\Omega_k{}^i)^{-1/2}\exp(i\Omega_k{}^i t),\quad \dot{\epsilon}_k(-\infty)=i(\Omega_k{}^i)^{1/2}\exp(i\Omega_k{}^i t). \tag{36}$$

The above choice for $\epsilon_k(-\infty)$ leads to the following limits for the invariants:

$$A_k(-\infty)=A_k{}^i=2^{-1/2}\exp(i\Omega_k{}^i t)\times[q_k(\Omega_k{}^i)^{1/2}+ip_k(\Omega_k{}^i)^{-1/2}]. \tag{37}$$

It is clear, then, that $|\boldsymbol{\alpha};-\infty\rangle$ and $|\mathbf{n};-\infty\rangle$ will coincide with the initial states $|\boldsymbol{\alpha};i\rangle$ and $|\mathbf{n};i\rangle$, which are constructed by the operators $A_k{}^i$ in the same manner as $|\boldsymbol{\alpha};t\rangle$ and $|\mathbf{n};t\rangle$ are constructed by the operators $A_k(t)$. We give the expression for the initial states related to the constant frequencies $\Omega_k{}^i$:

$$|\boldsymbol{\alpha};i\rangle=\prod_{k=1}^{N}\left(\frac{\Omega_k{}^i}{\pi}\right)^{1/2}\times\exp\left\{-\tfrac{1}{2}\Omega_k{}^i\left[q_k-\left(\frac{2}{\Omega_k{}^i}\right)^{1/2}\alpha_k e^{-i\Omega_k{}^i t}\right]^2-\tfrac{1}{2}i\Omega_k{}^i t\right\}\times\exp\{\tfrac{1}{2}(\alpha_k{}^2e^{-2i\Omega_k{}^i t}-|\alpha_k|^2)\},\tag{38}$$

$$|\mathbf{n};i\rangle=\prod_{k=1}^{N}\left[\frac{(\Omega_k{}^i/\pi)^{1/2}}{2^{n_k}n_k!}\right]^{1/2}\times\exp[-\tfrac{1}{2}\Omega_k{}^i q_k{}^2-i(n_k+\tfrac{1}{2})\Omega_k{}^i t]H_{n_k}(q_k(\Omega_k{}^i)^{1/2}),$$

which are again the products of the corresponding states for the one-dimensional oscillators. The final states $|\boldsymbol{\gamma};f\rangle$ and $|\mathbf{m};f\rangle$ are given by the same equations (38) with the replacement $\Omega_k{}^i\rightarrow\Omega_k{}^f$. The coherent states for a time-dependent oscillator describe most classical states in a similar manner to the coherent states for the oscillator with constant frequency. We have discussed in Sec. III the physical meaning of the eigenvalue α_k of the operator $A_k{}^i$, k fixed; $|\alpha_k|$ determines the classical amplitude of the oscillations in the phase plane (p_k,q_k), and the phase $\varphi(\alpha_k)$ is the classical phase of the oscillators of the same oscillation. Classical motion in the phase plane $(p_k(2\Omega_k{}^i)^{-1/2},q_k(\tfrac{1}{2}\Omega_k{}^i)^{1/2})$ is along a circle with radius $|\alpha_k|$. Since $A_k(t)$ is an invariant, it follows then that its eigenvalue α_k in the state $|\alpha_k;t\rangle$ is connected with the p_k,q_k coordinates of the initial point where the classical motion has started. Of course, for a varying $\Omega_k(t)$, this motion is more complicated, but if we introduce the time-dependent coordinates

$$q_k'=q_k(|\epsilon_k|2^{1/2})^{-1},\quad p_k'=2^{-1/2}(|\epsilon_k|p_k-q_kd|\epsilon_k|/dt),$$

the classical motion along a circle with the radius $|\alpha_k|$ is defined in these coordinates, which is apparent from Eq. (20). Now for the N-dimensional oscillator the phase space is $2N$ dimensional, and in the time-dependent coordinates p_k', q_k', $k=1,\ldots,N$, we get the classical motion on a $2N$-dimensional sphere with the radius $|\boldsymbol{\alpha}|$. The initial point of this motion is determined by $2N$ real numbers and we have N complex integrals of motion α_k, $k=1,\ldots,N$, the eigenvalues of N non-Hermitian invariants $A_k(t)$.

C. Transition Amplitudes

Let us now turn to the calculation of the transition probabilities (35). We shall need the following expression for the operators $A_k(t)$ in terms of the final operators $A_k{}^f$, related to the constant frequencies $\Omega_k{}^f$ [3,4]:

$$A_k(t)=\xi_k(t)A_k{}^f+\eta_k(t)(A_k{}^f)^\dagger, \tag{39}$$

where

$$\xi_k(t)=\tfrac{1}{2}\exp(-i\Omega_k{}^f t)[\epsilon_k(\Omega_k{}^f)^{1/2}-i\dot{\epsilon}_k(\Omega_k{}^f)^{-1/2}],\quad \eta_k(t)=-\tfrac{1}{2}\exp(i\Omega_k{}^f t)[\epsilon_k(\Omega_k{}^f)^{1/2}+i\dot{\epsilon}_k(\Omega_k{}^f)^{-1/2}]. \tag{40}$$

The commutation relations (22) require

$$|\xi_k|^2-|\eta_k|^2=1, \tag{41}$$

which is clearly satisfied. The general solutions of Eq. (21) for constant $\Omega_k{}^f$ (in the limit $t\rightarrow\infty$) may be written in the form

$$\epsilon_k{}^f=\frac{\xi_k}{(\Omega_k{}^f)^{1/2}}\exp(i\Omega_k{}^f t)-\frac{\eta_k}{(\Omega_k{}^f)^{1/2}}\exp(-i\Omega_k{}^f t). \tag{42}$$

Therefore, the transition amplitudes that we shall calculate are completely determined by these constants ξ_k and η_k [since the solutions of the wave equation are given in terms of $\epsilon_k(t)$].

We will also use the compact (θ_k) and noncompact (δ_k) parameters:

$$\cos\theta_k=1-2|\eta_k/\xi_k|^2,\quad \cosh\delta_k=|\xi_k|^2+|\eta_k|^2. \tag{43}$$

In the case of the one-dimensional time-dependent oscillator, the noncompact parameter $\cosh\delta$ has been used in Refs. 3 and 4. The solutions $|\boldsymbol{\alpha};t\rangle$ and $|\mathbf{n};t\rangle$ for the N-dimensional oscillator are the products of the solutions for the one-dimensional oscillators and, therefore, the transition amplitudes for the N-dimensional oscillator are products of those of the one-dimensional oscillators.

The transition amplitudes for the one-dimensional quantum oscillator between the energy eigenstates were obtained by Husimi.[5] Now we shall derive formulas for the transition amplitudes between coherent states. As a matter of fact, they are essentially obtained in Ref. 5, but some supplementary transformations must be made in order to arrive at our formulas (we give the results for the N-dimensional oscillator):

$$T_{\alpha}{}^{\gamma}=\prod_{k=1}^{N}(\xi_k)^{-1/2}\exp\left\{\frac{1}{2}\left[\alpha_k{}^2\frac{\eta_k{}^*}{\xi_k}+\alpha_k\gamma_k{}^*\frac{2}{\xi_k}-(\gamma_k{}^*)^2\frac{\eta_k}{\xi_k}-|\alpha_k|^2-|\gamma_k|^2\right]\right\}. \quad (44)$$

Having the transition amplitudes between coherent states, we may obtain all the rest by simple differentiation. In the case of the quantum oscillator, this differentiation is especially simple if we again recall the generating function of Hermite polynomials.[15] For the transitions $|\mathbf{n};i\rangle\rightarrow|\gamma;f\rangle$ and $|\alpha;i\rangle\rightarrow|\mathbf{m};f\rangle$, we get immediately

$$T_{\mathbf{n}}{}^{\gamma}=\prod_{k=1}^{N}i^{n_k}\left[\frac{(\eta_k{}^*/2\xi_k)^{n_k}}{n_k!\xi_k}\right]^{1/2}\times\exp\left\{-\frac{1}{2}\left[|\gamma_k|^2+(\gamma_k{}^*)^2\frac{\eta_k}{\xi_k}\right]\right\}\times H_{n_k}\left[\frac{-i\gamma_k{}^*}{(2\xi_k\eta_k{}^*)^{1/2}}\right], \quad (45)$$

$$T_{\alpha}{}^{\mathbf{m}}=\prod_{k=1}^{N}\left[\frac{(\eta_k/2\xi_k)^{m_k}}{m_k!\xi_k}\right]^{1/2}\exp\left[\frac{1}{2}\left(\frac{\eta_k{}^*}{\xi_k}\alpha_k{}^2-|\alpha_k|^2\right)\right]\times H_{m_k}\left[\frac{\alpha_k}{(2\xi_k\eta_k)^{1/2}}\right]. \quad (46)$$

The function $T_{\alpha}{}^{\gamma}\exp[\frac{1}{2}(|\alpha|^2+|\gamma|^2)]$ is known to be the generating function for Legendre polynomials. Indeed (see 11.5.1 in Ref. 15)

$$\exp\left[\alpha^2\frac{\eta^*}{2\xi}+\alpha\gamma^*\frac{1}{\xi}-(\gamma^*)^2\frac{\eta}{2\xi}\right]=\sum_{p=0}^{\infty}\frac{1}{p!}\left[\alpha^2\frac{\eta^*}{2\xi}+\alpha\gamma^*\frac{1}{\xi}-(\gamma^*)^2\frac{\eta}{\xi 2}\right]^p$$
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\frac{\alpha^{2p}}{2^p}\exp(-ip\varphi_\xi)\sum_{q=-p}^{p}\left(-\frac{\gamma^*}{\alpha}\right)^{p+q}Y_p{}^q(\zeta), \quad (47)$$

where

$$\zeta=(-\cos\tfrac{1}{2}\theta,\ \sin\tfrac{1}{2}\theta\cos\varphi_\eta,\ -\sin\tfrac{1}{2}\theta\sin\varphi_\eta),\quad \cos\tfrac{1}{2}\theta=1/|\xi|;$$

or, in another form ($p+q=m, p-q=n$), the right-hand side of (47) is

$$\sum_{m,n=0}^{\infty}\frac{\alpha^n(\gamma^*)^m}{m!}\exp[i\tfrac{1}{2}(m-n)\varphi_\eta-i\tfrac{1}{2}(m+n)\varphi_\xi]\times P_{(m+n)/2}{}^{(m-n)/2}(\cos\tfrac{1}{2}\theta), \quad (48)$$

where m,n are of the same parity. So

$$T_{\mathbf{n}}{}^{\mathbf{m}}=\langle\mathbf{m};f|\mathbf{n};t\rightarrow\infty\rangle=\prod_{k=1}^{N}\left(\frac{n_k!}{m_k!\xi_k}\right)^{1/2}\times\exp[i\tfrac{1}{2}(m_k-n_k)\varphi_{\eta_k}-i\tfrac{1}{2}(m_k+n_k)\varphi_{\xi_k}]\times P_{(m_k+n_k)/2}{}^{(m_k-n_k)/2}(\cos\tfrac{1}{2}\theta_k). \quad (49)$$

This matrix element is an integral of the type

$$\int_{-\infty}^{+\infty}H_m(\alpha x)H_n(\beta x)e^{-\alpha x^2}dx,$$

well known in the literature (see, for example, Ref. 16).

In another form for the one-dimensional oscillator, this amplitude was obtained by Husimi.[5] For the transition probabilities, we have

$$W_{\mathbf{n}}{}^{\mathbf{m}}=\prod_{k=1}^{N}\frac{n_k!}{m_k!|\xi_k|}|P_{(m_k+n_k)/2}{}^{(m_k-n_k)/2}(\cos\tfrac{1}{2}\theta_k)|^2. \quad (50)$$

This is a generalization of the results for the one-dimensional oscillator.[5] The possibility of expressing the transition probability of the one-dimensional oscillator in terms of Legendre polynomials was noted in Ref. 17.

D. $U(N,1)$ Symmetry

It is well known[18] that the N-dimensional oscillator with equal and constant frequencies has as its invariance group the compact group $U(N)$. All the wave functions corresponding to an energy level realize an irreducible representation of this group. The dynamical symmetry which collects all the levels into one irreducible infinite-dimensional representation for such an oscillator is known to be the $U(N,1)$ group.[19] One can easily generalize these two statements for the N-dimensional oscillator with time-dependent frequencies. Indeed, if we take N^2 operators

$$T_{i,k}=A_i{}^{\dagger}A_k,\quad i,k=1,\ldots,N \quad (51)$$

they commute as the generators of the group $U(N)$:

$$[T_{i,k},T_{m,n}]=T_{i,n}\delta_{k,m}-T_{m,k}\delta_{i,n}. \quad (52)$$

These operators commute with the operator $i\partial/\partial t-H$. So the wave functions belonging to the main quantum number $M=n_1+n_2+\cdots+n_N$ realize the irreducible representation of the group $U(N)$ with the highest weight $(M,0,\ldots,0)$ as well as for a stationary oscillator. One can construct the additional operators, commuting

[16] P. A. Lee, J. Math. Phys. **46**, 215 (1967).

[17] V. S. Popov and A. M. Perelomov, Zh. Eksperim. i Teor. Fiz. **56**, 1375 (1969) [Soviet Phys. JETP **29**, 738 (1969)].

[18] E. Hill and H. Jauch, Phys. Rev. **57**, 641 (1940).

[19] A. O. Barut, Phys. Rev. **139**, 1433 (1965).

with $i\partial/\partial t - H$:

$$T_{N+1,i}=A_i(\sum_{j=1}^{N} A_j^\dagger A_j)^{1/2},$$
$$T_{k,N+1}=-A_k^\dagger(\sum_{j=1}^{N} A_j^\dagger A_j+1)^{1/2}, \quad (53)$$
$$T_{N+1,N+1}=-\sum_{j=1}^{N} A_j^\dagger A_j-1\,.$$

The $(N+1)^2$ operators $T_{\alpha,\beta}$, $\alpha,\beta=1,2\ldots,N+1$, defined by Eqs. (51) and (53), form the Lie algebra of the noncompact group $U(N,1)$:

$$[T_{\alpha,\beta},T_{\gamma,\delta}]=T_{\alpha,\delta}\delta_{\beta,\gamma}-T_{\gamma,\beta}\delta_{\alpha,\delta}\,. \quad (54)$$

So, all the solutions of Schrödinger equation (19) realize one ladder representation of the noncompact group $U(N,1)$. This is the same representation as for a stationary oscillator. We will say that the dynamical group of the N-dimensional oscillator is the $U(N,1)$ group.

For a stationary oscillator this group does not commute with the Hamiltonian but it commutes with the operator $i\partial/\partial t-H$, being the invariance group of the Schrödinger equation with time.

V. MOTION OF CHARGED PARTICLE IN TIME-DEPENDENT ELECTROMAGNETIC FIELD

A. Invariants

We shall consider a particle of mass M and charge e moving in a classical electromagnetic field with a potential

$$\mathbf{A}(t)=\tfrac{1}{2}[\mathcal{H}(t),\mathbf{r}]\,,\quad \varphi=(e/2Mc^2)\chi(t)(x^2+y^2)\,, \quad (55)$$

where $\mathbf{r}$ is the position vector, $\mathcal{H}(t)$ is an axially symmetric magnetic field, and $\chi(t)$, $\mathcal{H}(t)$ are arbitrary continuous functions of time. The scalar potential φ corresponds to an axially symmetric, time-dependent uniform charge density equal to $-(e/2\pi Mc^2)\chi(t)$. We choose the axis z along the field $\mathcal{H}$; then $A_z=0$. The motion along the axis z is then trivial and we shall ignore it and treat only the motion in the xy plane. The Hamiltonian for such a system is

$$H=(1/2M)[(p_x-eA_x)^2+(p_y-eA_y)^2]+e\varphi\,,\qquad \hbar=c=1\,. \quad (56)$$

The spin-dependent part $-\mu\sigma_z\mathcal{H}$ is dropped, since what follows does not depend on it. The potential (55) obeys the Maxwell equations for any continuous functions $\mathcal{H}(t)$, $\chi(t)$ if only they do not change very fast in comparison with the speed of light so that we may ignore the radiation field. In practice this requirement is almost always fulfilled. The electromagnetic potential (55) is usually chosen for describing the electromagnetic field of a solenoid.[20]

[20] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1965).

By direct calculation one can verify that the following operators are invariants:

$$A(t)=\frac{1}{2e^{1/2}}\exp\left[\tfrac{1}{2}i\int^t\omega(\tau)d\tau\right]\times[\epsilon(t)(p_x+ip_y)-iM\dot{\epsilon}(t)(y-ix)],$$
$$B(t)=\frac{1}{2e^{1/2}}\exp\left[-\tfrac{1}{2}i\int^t\omega(\tau)d\tau\right]\times[\epsilon(t)(p_y+ip_x)-iM\dot{\epsilon}(t)(x-iy)], \quad (57)$$

where $\omega(t)=e\mathcal{H}(t)/M$, and $\epsilon(t)$ is any particular solution of the equation

$$\ddot{\epsilon}+\Omega^2(t)\epsilon=0\,,\quad \Omega^2(t)=\tfrac{1}{4}\omega^2(t)+(e^2/M^2)\chi(t)\,. \quad (58)$$

The operators (57) commute with the operator $i\partial/\partial t-H$ in consequence of $\dot{A}=\dot{B}=0$. In order to get the time-independent commutation relation of the operators A, $A^\dagger$ and B, $B^\dagger$ we choose the special solutions of (58),

$$\epsilon(t)=|\epsilon|\exp\left[i(e/M)\int^t|\varepsilon(\tau)|^{-2}d\tau\right]. \quad (59)$$

Equation (58) then produces[4]

$$\frac{d^2}{dt^2}|\epsilon|+\Omega^2(t)|\epsilon|-\left(\frac{e}{M}\right)^2\frac{1}{|\epsilon|^3}=0\,. \quad (60)$$

The following commutation relations hold:

$$[A,A^\dagger]=[B,B^\dagger]=e/|e|\,,\quad [A,B]=[A,B^\dagger]=0\,. \quad (61)$$

For oppositely charged particles, the lowering and raising operators A,B and $A^\dagger,B^\dagger$ change their places. For simplicity we suppose $e>0$. In Ref. 4 there was found one Hermitian invariant $K(t)$ which may be expressed in terms of our operators as

$$K(t)=A^\dagger(t)A(t)+\tfrac{1}{2}\,.$$

Because of the axial symmetry of the electromagnetic potential (55), the z component of the angular momentum is also an integral of motion and it too may be expressed in terms of our operators (57):

$$L_z=B^\dagger B-A^\dagger A\,.$$

In order to elucidate the physical meaning of the invariants $A(t)$ and $B(t)$, let us consider their limits (at $t\to-\infty$) for a constant magnetic field. Here we suppose that $\mathcal{H}(t)=\mathcal{H}_i$, $\varphi(t)=0$ if $t<0$, so that $\Omega(t)=\frac{1}{2}\omega_i$ if $t<0$. In the limit $t\to-\infty$, we choose as a solution of Eq. (58)

$$\epsilon(-\infty)=(2/\mathcal{H}_i)^{1/2}\exp(\tfrac{1}{2}i\omega_it)\,,$$
$$\dot{\epsilon}(-\infty)=\tfrac{1}{2}i\omega_i\epsilon(-\infty)\,. \quad (62)$$

The reason for such a choice will be given later. Under the initial conditions (62), the $t\to-\infty$ limit of the invariants (57) is given by

$$A_i=(\tfrac{1}{2}M\omega_i)^{1/2}\exp(i\omega_i t)[y-y_0-i(x-x_0)],\qquad B_i=(\tfrac{1}{2}M\omega_i)^{1/2}(x_0-iy_0),\quad (63)$$

where $x_0=\frac{1}{2}x+p_y/M\omega_i$, $y_0=\frac{1}{2}y-p_x/M\omega_i$ are the well-known[20] "coordinates" of the center of the orbit of a particle moving in a constant magnetic field $\mathcal{H}_i$.

It is apparent from Eq. (63) that the eigenvalue of the invariant B_i determines the coordinates of the center of the orbit in the xy plane, and the eigenvalue of the operator $A_i\exp(i\omega_i t)$ determines the current coordinates of the center of the packet. Hence the invariant A_i itself is connected with the coordinates of the initial point, where the motion has started from. Now it is obvious that the time-dependent invariants $A(t)$ and $B(t)$ are connected with the initial conditions of the motion. For classical motion there are four independent integrals of motion in consequence of the four degrees of freedom (motion in the xy plane), and the same situation holds in the quantum case.

B. Coherent States

In the same manner as coherent states were introduced in Sec. IV for a time-dependent oscillator, one may introduce coherent states for a charged particle moving in a time-dependent electromagnetic field of the type determined in Sec. V A. In this case we have two lowering operators $A(t)$ and $B(t)$ and the coherent states will carry two indices:

$$|\alpha,\beta;t\rangle=\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)]\times\sum_{n_1,n_2=0}^{\infty}\frac{\alpha^{n_1}\beta^{n_2}}{(n_1!n_2!)^{1/2}}|n_1,n_2;t\rangle,\quad (64)$$

where α,β are arbitrary constant complex numbers. Here

$$|n_1,n_2;t\rangle=\frac{(A^\dagger)^{n_1}(B^\dagger)^{n_2}}{(n_1!n_2!)^{1/2}}|0,0;t\rangle,\qquad A|0,0;t\rangle=B|0,0;t\rangle=0\quad (65)$$

are solutions of the Schrödinger equation $(i\partial/\partial t-H)\Psi=0$, and at the same time they are eigenstates of the invariants $K(t)$ and L:

$$K|n_1,n_2;t\rangle=(n_1+\tfrac{1}{2})|n_1,n_2;t\rangle,\qquad L_z|n_1,n_2;t\rangle=(n_2-n_1)|n_1,n_2;t\rangle.\quad (66)$$

The coherent states $|\alpha,\beta;t\rangle$ are eigenstates of the time-dependent invariants $A(t)$ and $B(t)$:

$$A|\alpha,\beta;t\rangle=\alpha|\alpha,\beta;t\rangle,\quad B|\alpha,\beta;t\rangle=\beta|\alpha,\beta;t\rangle.\quad (67)$$

The physical meaning of the eigenvalues α and β has been discussed in Sec. V A. There are two unitary displacement operators

$$D(\alpha)=\exp(\alpha A^\dagger-\alpha^*A),\quad D^{-1}(\alpha)AD(\alpha)=A+\alpha,\qquad D(\beta)=\exp(\beta B^\dagger-\beta^*B),\quad D^{-1}(\beta)BD(\beta)=B+\beta,\quad (68)$$

which commute with each other: $[D(\alpha),D(\beta)]=0$. The coherent states $|\alpha,\beta;t\rangle$ may be constructed explicitly by acting upon the vacuum

$$|0,0;t\rangle=\left(\frac{e}{\pi}\right)^{1/2}\epsilon^{-1}\exp\left(i\frac{M\dot\epsilon}{2\epsilon}\zeta\zeta^*\right),\quad \zeta=x+iy\quad (69)$$

with operators (68):

$$|\alpha,\beta;t\rangle=D(\alpha)D(\beta)|0,0;t\rangle=\left(\frac{e}{\pi}\right)^{1/2}\epsilon^{-1}\exp\left(i\frac{M\dot\epsilon}{2\epsilon}\zeta\zeta^*\right)\times\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)+(e^{1/2}/|\epsilon|)(\beta\zeta e^{-i\gamma_-}+i\alpha\zeta^*e^{-i\gamma_+})-i\alpha\beta e^{-i(\gamma_++\gamma_-)}],\quad (70)$$

where

$$\gamma_\pm(t)=(e/M)\int^t|[\epsilon(\tau)|^{-2}\pm\mathcal{H}(\tau)]d\tau.$$

The eigenstates $|n_1,n_2;t\rangle$ of the operators K and L_z are orthonormalized:

$$\langle n_1,n_2;t|m_1,m_2;t\rangle=\delta_{n_1,m_1}\delta_{n_2,m_2}.$$

Then relation (64) yields

$$\langle\alpha,\beta;t|\alpha',\beta';t\rangle=\exp[\alpha^*\alpha'+\beta^*\beta'-\tfrac{1}{2}(|\alpha|^2+|\alpha'|^2+|\beta|^2+|\beta'|^2)],\quad (71)$$

so that the coherent states $|\alpha,\beta;t\rangle$ are normalized but not orthogonal.

One may introduce coherent states with respect to one of the operators A or B. Let us consider the eigenstates of the invariants K and B:

$$K|n_1,\beta;t\rangle=(n_1+\tfrac{1}{2})|n_1,\beta;t\rangle,\qquad B|n_1,\beta;t\rangle=\beta|n_1,\beta;t\rangle.\quad (72)$$

It is easy to verify that the functions

$$|n_1,\beta;t\rangle=D(\beta)|n_1,O;t\rangle=\left(\frac{e/\pi}{n_1!}\right)^{1/2}i^{n_1}\epsilon^{-1}\left[\frac{e^{1/2}}{|\epsilon|}\zeta^*-\beta e^{-i\gamma_-}\right]^{n_1}\times\exp\left[i\frac{M\dot\epsilon}{2\epsilon}\zeta\zeta^*+\frac{e^{1/2}}{|\epsilon|}\beta\zeta e^{-i\gamma_-}-in_1\gamma_+-\tfrac{1}{2}|\beta|^2\right]\quad (73)$$

obey requirements (72) and their scalar product is given by

$$\langle n_1,\beta;t|k_1,\beta';t\rangle=\delta_{n_1,k_1}\exp[-\tfrac{1}{2}(|\beta|^2+|\beta'|^2)+\beta^*\beta'].\quad (74)$$

The coherent states $|\alpha,n_2;t\rangle$ may be constructed in the same way: $|\alpha,n_2;t\rangle=D(\alpha)|0,n_2;t\rangle$. The coherent states (73) are the generating functions for the states $|n_1,n_2;t\rangle$, and one can obtain the explicit form of $|n_1,n_2;t\rangle$ by expanding them in powers of the number β, as well as by differentiation of $|\alpha,\beta;t\rangle$ with respect to α and β simultaneously:

$$|n_1,n_2;t\rangle=i^{n_1}(-1)^p\left[\frac{p!e/\pi}{(p+|n_1-n_2|)!}\right]^{1/2}\left(\frac{e}{|\epsilon|^2}\rho^2\right)^{|n_1-n_2|/2}\times\exp\left[i\frac{M\dot{\epsilon}}{2\epsilon}\rho^2+i(n_2-n_1)\varphi-in_1\gamma_+-in_2\gamma_-\right]\times L_p^{|n_1-n_2|}\left(\frac{e}{|\epsilon|^2}\rho^2\right),\quad(75)$$

where $p=\frac{1}{2}(n_1+n_2-|n_1-n_2|)$, φ and ρ are polar coordinates, and $L_p{}^s(x)$ are Laguerre polynomials.

The coherent states (70) form a complete set:

$$(1/\pi^2)\int|\alpha,\beta;t\rangle\langle\alpha,\beta;t|d^2\alpha d^2\beta=1,\quad(76)$$

and for the arbitrary state $|f\rangle$ we have the expansion

$$|f\rangle=(1/\pi^2)\int|\alpha,\beta;t\rangle f(\alpha^*,\beta^*)\times\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)]d^2\alpha d^2\beta,\quad(77)$$

where

$$f(\alpha^*,\beta^*)=\langle\alpha,\beta;t|f\rangle\exp[\tfrac{1}{2}(|\alpha|^2+|\beta|^2)].$$

For the states $|n_1,n_2;t\rangle$ we get the following expansion in terms of coherent states:

$$|n_1,n_2;t\rangle=\frac{1}{\pi^2}\int\frac{(\alpha^*)^{n_1}(\beta^*)^{n_2}}{(n_1!n_2!)^{1/2}}\times\exp[\tfrac{1}{2}(|\alpha|^2+|\beta|^2)]|\alpha,\beta;t\rangle d^2\alpha d^2\beta.\quad(78)$$

This expansion is the reverse of the expansion (64).

Employing the explicit form of the coherent state $|\alpha,\beta;t\rangle$, the Green's function of the Schrödinger equation for a charged particle in a varying electromagnetic field may be found as[12]

$$G(x_2,y_2,t_2;x_1,y_1,t_1)=\frac{1}{\pi^2}\int|\alpha,\beta;t_2\rangle\langle\alpha,\beta;t_1|d^2\alpha d^2\beta=\frac{e/2\pi i}{|\epsilon_1\epsilon_2|\sin\gamma}\exp\left\{i\left[\frac{M}{4e}\left(R_2{}^2\frac{d|\epsilon|^2}{dt_2}-R_1{}^2\frac{d|\epsilon|^2}{dt_1}\right)+\tfrac{1}{2}\cot\gamma(\mathbf{R}_1-\mathbf{R}_2)^2+[\mathbf{R}_1,\mathbf{R}_2]_z\right]\right\},\quad(79)$$

where

$$\gamma=\frac{e}{M}\int_{t_1}^{t_2}|\epsilon(\tau)|^{-2}d\tau,\quad R_i=\frac{e^{1/2}}{|\epsilon_i|}(x_i{}^2+y_i{}^2)^{1/2},$$

and

$$\varphi_{Ri}=\tan^{-1}(y_i/x_i)-\gamma_-(t_i),\quad i=1,2.$$

This function is a generalization of the Green's function of the Schrödinger equation for a charged particle in a constant magnetic field.[1,21] The Feynman integral method was used by Batalin and Fradkin[22] to find the Green's functions of relativistic particles in constant fields of a general type. Equation (64) shows that in the coherent state $|\alpha,\beta;t\rangle$ we have Poisson distributions for the quantum numbers n_1 and n_2:

$$|\langle\alpha,\beta;t|n_1,n_2;t\rangle|^2=\frac{|\alpha|^{2n_1}|\beta|^{2n_2}}{n_1!n_2!}\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)].\quad(80)$$

In order to elucidate the physical interpretations of $|\alpha,\beta;t\rangle$, let us consider their limit as $t\to-\infty$. In this limit under the choice (62), our coherent states coincide with the initial coherent states $|\alpha,\beta;i\rangle$, which were treated in Ref. 9, for a charged particle in a constant magnetic field $\mathcal{H}_i$. We note that our states $|\alpha,\beta;t\rangle$ and their limits $|\alpha,\beta;t\to-\infty\rangle$ are in the Schrödinger representation, while in Ref. 9 coherent states are time independent and correspond to the Heisenberg picture. The state $|\alpha,\beta;t\to-\infty\rangle=|\alpha,\beta;i\rangle$ is time dependent and obeys the Schrödinger equation $i\partial/\partial t\psi=H\psi$;

$$|\alpha,\beta;i\rangle=(M\omega_i/2\pi)^{1/2}\exp(-\tfrac{1}{2}i\omega_it-\tfrac{1}{4}M\omega_i\zeta\zeta^*)\times\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)+(\tfrac{1}{2}M\omega_i)^{1/2}(\beta\zeta+i\alpha\zeta^*e^{-i\omega_it})-i\alpha\beta e^{-i\omega_it}].\quad(81)$$

To within a constant factor, the invariant $K(-\infty)$ coincides with the Hamiltonian H_i:

$$H_i=\omega_iK(-\infty)=\omega_i(A_i^\dagger A+\tfrac{1}{2}),\quad(82)$$

where A is given by (63). Thus the eigenstates $|n_1,n_2;i\rangle$ coincide with the well-known energy and the L_z eigenstates of a charged particle moving in a constant magnetic field.[20] In the classical case the particle performs simple motion along a circular orbit with the center at (x_0,y_0). Since $|\alpha,\beta;i\rangle$ is an eigenstate of the operators A and B [see Eq. (63)], it is clear that the expectation values of x and y coordinates carry out an harmonic motion with the amplitude $|\alpha|(2/M\omega_i)^{1/2}$. The coordinates of the center of the motion are $(2/M\omega_i)^{1/2}\,\mathrm{Re}\beta$ and $-(2/M\omega_i)^{1/2}\,\mathrm{Im}\beta$.

So the coherent states $|\alpha,\beta;i\rangle$ are the most classical states for a charged particle in a constant field $\mathcal{H}_i$.[9] The

[21] E. H. Sondheimer and A. H. Wilson, Proc. Roy. Soc. (London) **A219**, 173 (1951).

[22] I. Batalin and E. S. Fradkin (unpublished).

"semicoherent" states $|n_1,\beta;i\rangle$ provide the best classical approach for stationary energy eigenstates. In these states the coordinates x_0 and y_0 of the center of the packet orbit are determined simultaneously as exactly as is possible in accordance with the uncertainty principle while the coordinates x and y do not obey this condition. This takes place only in the states $|\alpha,\beta;i\rangle$, which are coherent with respect to both A_i and B_i operators. In a similar manner the packets $|\alpha,\beta;t\rangle$ describe the most classical states for a charged particle in an axially symmetric time-dependent electromagnetic field. Of course the coordinates x and y now carry out a more complicated form of motion, not just simple harmonic motion.

C. Transition Probabilities

1. Let us now suppose the electromagnetic field to be constant in the remote past and the remote future. More precisely, let us suppose

$$\begin{aligned} \mathcal{H}(t)&=\mathcal{H}_i, \quad \varphi(t)=0 \quad \text{if } t<0, \\ \mathcal{H}(t)&=\mathcal{H}_f, \quad \varphi(t)=0 \quad \text{as } t\to+\infty. \end{aligned} \tag{83}$$

Under these conditions, as $t\to\mp\infty$ there exist initial and final coherent states and Landau solutions and the problem of transitions between these states may be solved. The transition amplitude connecting an initial state $|i\rangle$ to a final one $|f\rangle$ is given by matrix element (35). We have already chosen the solution of the auxiliary classical equation (58) for the limit $t\to-\infty$ [see Eqs. (62)]. The reason for this choice is to get the condition[4] $[K(-\infty),H_i]=0$, so that in the limit $t\to-\infty$ the eigenstates of K and L_z would coincide with the Landau solutions for a constant field $\mathcal{H}_i$ and coherent states $|\alpha,\beta;t\to-\infty\rangle$, with the initial coherent states $|\alpha,\beta;i\rangle$. At the limit $t\to+\infty$, the time dependence of $\Omega(t)$ produces a more general solution of Eq. (58) and it is clear then that in general $[K(\infty),H_f]\neq0$, and $|n_1,n_2;\infty\rangle$ and $|\alpha,\beta;\infty\rangle$ may be expanded in terms of the corresponding final states. We shall use the following definition of the final operators:

$$\begin{aligned} A_f&=(2M\omega_f)^{-1/2} \\ &\quad\times\exp[i(\omega_f t+\varphi_0)][p_x+ip_y+\tfrac{1}{2}M\omega_f(y-ix)], \\ B_f&=(2M\omega_f)^{-1/2} \\ &\quad\times\exp(-i\varphi_0)[p_y+ip_x+\tfrac{1}{2}M\omega_f(x-iy)], \end{aligned} \tag{84}$$

where

$$\varphi_0=\frac{1}{2}\int_0^\infty[\omega(t)-\omega_f]dt.$$

This definition of A_f and B_f differs from the corresponding definition of A_i and B_i [see Eq. (63)] by the phase factor $\exp(\pm i\varphi_0)$, and the same definitions for the operators A_f and B_f were taken into account in Ref. 12. The final states (related to the constant field $\mathcal{H}_f$) are given by

$$\begin{aligned} |\gamma,\delta;f\rangle &=\left(\frac{M\omega_f}{2\pi}\right)^{1/2}\exp[-\tfrac{1}{2}(i\omega_f t+\tfrac{1}{2}M\omega_f\zeta\zeta^*+|\gamma|^2+|\delta|^2)] \\ &\quad\times\exp\{(\tfrac{1}{2}M\omega_f)^{1/2}[\delta\zeta e^{i\varphi_0}+i\gamma\zeta^* e^{-i(\omega_f\mu t+\varphi_0)}] \\ &\quad -i\gamma\delta e^{-i\omega_f t}\}, \end{aligned} \tag{85}$$

$$\begin{aligned} |m_1,m_2;f\rangle &=(-1)^p i^{m_1}\left[\frac{p!M\omega_f/2\pi}{(p+|m_1-m_2|)!}\right]^{1/2} \\ &\quad\times\exp[i(m_2-m_1)(\varphi+\varphi_0)] \\ &\quad\times\exp[-\tfrac{1}{4}M\omega_f\rho^2-i(m_1+\tfrac{1}{2})\omega_f t] \\ &\quad\times(\tfrac{1}{2}M\omega_f\rho^2)^{|m_1-m_2|/2}L_p^{|m_1-m_2|}(\tfrac{1}{2}M\omega_f\rho^2), \end{aligned} \tag{86}$$

where $p=\frac{1}{2}(m_1+m_2-|m_1-m_2|)$, and φ and ρ are polar coordinates.

The invariants A and B may be expressed in terms of final operators (84) as follows:

$$\begin{aligned} A(\infty)&=\xi A_f+\eta B_f^\dagger, \\ B(\infty)&=\eta A_f^\dagger+\xi B_f, \end{aligned} \tag{87}$$

where the quantities ξ and η are the $t\to\infty$ limits of the functions

$$\begin{aligned} \xi(t)&=(2\mathcal{H}_f)^{-1/2}\left(\tfrac{1}{2}\mathcal{H}\epsilon-i\frac{M}{e}\dot\epsilon\right) \\ &\quad\times\exp\left[i\left(\varphi_0-\frac{1}{2}\int_0^t\omega(\tau)d\tau\right)\right], \\ \eta(t)&=(2\mathcal{H}_f)^{-1/2}\left(\tfrac{1}{2}i\mathcal{H}\epsilon-\frac{M}{e}\dot\epsilon\right) \\ &\quad\times\exp\left[-i\left(\varphi_0-\frac{1}{2}\int_0^t\omega(\tau)d\tau\right)\right]. \end{aligned} \tag{88}$$

The commutation relations (61) require

$$|\xi|^2-|\eta|^2=1, \tag{89}$$

which is clearly satisfied. The general solution of Eq. (58) as $t\to\infty$ may be written as

$$\begin{aligned} \epsilon_f&=(2/\mathcal{H}_f)^{1/2}\xi\exp(\tfrac{1}{2}i\omega_f t) \\ &\quad -i(2/\mathcal{H}_f)^{1/2}\eta\exp(-\tfrac{1}{2}i\omega_f t), \end{aligned} \tag{90}$$

and since the solutions of the Schrödinger equation are given in terms of the function $\epsilon(t)$, all the transition amplitudes are completely determined by the parameters ξ and η. We shall use also the compact (θ) and noncompact (δ) parameters,

$$\cos\theta=1-2|\eta/\xi|^2, \quad \cosh\delta=|\xi|^2+|\eta|^2. \tag{91}$$

It is very useful to calculate first the transition amplitude connecting coherent states $|\alpha,\beta;i\rangle$ and $|\gamma,\delta;f\rangle$,

$$T_{\alpha,\beta}{}^{\gamma,\delta}=\langle\gamma,\delta;f|\alpha,\beta;t\to\infty\rangle$$
$$=(1/\xi)\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2)]$$
$$\times\exp[(1/\xi)(\alpha\beta\eta^*+\beta\delta^*+\alpha\gamma^*-\gamma^*\delta^*\eta)]. \quad (92)$$

From the definition of the coherent states, it follows that the amplitude $T_{\alpha,\beta}{}^{\gamma,\delta}$ is the generating function of the rest of the amplitudes. For example, expanding the right-hand sides of Eq. (92) in powers of γ^* and δ^*, and comparing with the relation

$$T_{\alpha,\beta}{}^{\gamma,\delta}=\exp[-\tfrac{1}{2}(|\gamma|^2+|\delta|^2)]\times\sum_{m_1,m_2=0}^{\infty}\frac{(\gamma^*)^{m_1}(\delta^*)^{m_2}}{(m_1!m_2!)^{1/2}}T_{\alpha,\beta}{}^{m_1,m_2}, \quad (93)$$

we find

$$T_{\alpha,\beta}{}^{m_1,m_2}=\left(\frac{m_2!}{m_1!}\right)^{1/2}\left(-\frac{\eta}{\xi}\right)^{m_2}\left(\frac{\alpha}{\xi}\right)^{m_1-m_2}\times T_{\alpha,\beta}{}^{0,0}L_{m_2}{}^{m_1-m_2}\left(\frac{\alpha\beta}{\xi\eta}\right),\quad m_1\geqslant m_2$$
$$=\left(\frac{m_1!}{m_2!}\right)^{1/2}\left(-\frac{\eta}{\xi}\right)^{m_1}\left(\frac{\beta}{\xi}\right)^{m_2-m_1}\times T_{\alpha,\beta}{}^{0,0}L_{m_1}{}^{m_2-m_1}\left(\frac{\alpha\beta}{\xi\eta}\right),\quad m_1<m_2 \quad (94)$$

where $L_p{}^s(x)$ are Laguerre polynomials. Differentiating Eq. (92) with respect to α and β in a similar way, we obtain the amplitudes

$$T_{n_1,n_2}{}^{\gamma,\delta}=\left(\frac{n_2!}{n_1!}\right)^{1/2}\left(-\frac{\eta^*}{\xi}\right)^{n_2}\left(\frac{\gamma^*}{\xi}\right)^{n_1-n_2}\times T_{0,0}{}^{\gamma,\delta}L_{n_2}{}^{n_1-n_2}\left(-\frac{\gamma^*\delta^*}{\xi\eta^*}\right),\quad n_1\geqslant n_2$$
$$=\left(\frac{n_1!}{n_2!}\right)^{1/2}\left(-\frac{\eta^*}{\xi}\right)^{n_1}\left(\frac{\delta^*}{\xi}\right)^{n_2-n_1}\times T_{0,0}{}^{\gamma,\delta}L_{n_1}{}^{n_2-n_1}\left(-\frac{\gamma^*\delta^*}{\xi\eta^*}\right),\quad n_1<n_2 \quad (95)$$

where $T_{\alpha,\beta}{}^{0,0}$ and $T_{0,0}{}^{\gamma,\delta}$ are given by Eq. (92) for $\gamma=\delta=0$ and $\alpha=\beta=0$, respectively. The transition amplitudes for the "semicoherent" states may be got in the same manner.

The transition amplitudes connecting the energy and angular momentum eigenstates may be calculated by expanding, for example, $T_{\alpha,\beta}{}^{m_1,m_2}$ in powers of α and β. For this purpose we have used the differentiation formulas for the Laguerre polynomials[15]:

(a) $n_2\geqslant n_1,\quad m_i\geqslant n_i,\quad i=1,2$

$$T_{n_1,n_2}{}^{m_1,m_2}=\frac{(-1)^{m_1-n_1}}{\xi}\left(\frac{n_1!m_2!}{n_2!m_1!}\right)^{1/2}\frac{(\xi^*)^{n_1}\eta^{m_1-n_1}}{\xi^{m_2}}\times P_{n_1}{}^{(m_1-n_1,n_2-n_1)}\left(1-2\left|\frac{\eta}{\xi}\right|^2\right), \quad (96)$$

(b) $m_2\geqslant n_1,\quad m_i<n_i$

$$T_{n_1,n_2}{}^{m_1,m_2}=\frac{1}{\xi}\left(\frac{n_2!m_1!}{m_2!n_1!}\right)^{1/2}\frac{(\xi^*)^{m_1}(\eta^*)^{n_1-m_1}}{\xi^{n_2}}\times P_{m_1}{}^{(n_1-m_1,n_2-n_1)}\left(1-2\left|\frac{\eta}{\xi}\right|^2\right), \quad (97)$$

where $P_n{}^{(\alpha,\beta)}(x)$ are Jacobi polynomials and the conservation of the momentum L_z $(n_2-n_1=m_2-m_1)$ is taken into account. In Ref. 12 amplitudes (96) and (97) were expressed in terms of $\cosh\delta=|\xi|^2+|\eta|^2$. Both Eqs. (96) and (97) are related to the positive $L_z=n_2-n_1\geqslant 0$. In the case of negative L_z, one gets the same Eqs. (96) and (97) with the replacement of the indices $1\rightleftarrows 2$. For the transition probabilities, Eq. (96) yields

$$W_{n_1,n_2}{}^{m_1,m_2}=\frac{m_2!n_1!}{m_1!n_2!}R^{m_1-n_1}(1-R)^{n_2-n_1+1}\times|P_{n_1}{}^{(m_1-n_1,n_2-n_1)}|^2, \quad (98)$$

where $R=|\eta/\xi|^2<1$ [in terms of the parameters θ and δ we have $R=\sin^2(\frac{1}{2}\theta)=\tanh^2(\frac{1}{2}\delta)$]. The case (b), $m_i<n_i$, is described by the same Eq. (98) if we change $n_i\rightleftarrows m_i$, $i=1,2$.

2. Now we shall discuss our formula for transition probabilities, (98). The probability $W_{n_1,n_2}{}^{m_1,m_2}$ was briefly discussed in Ref. 12. We obtained $W_{n_1,n_2}{}^{m_1,m_2}$ in terms of the transition $|0,0;i\rangle\to|0,0;f\rangle$: $W_{0,0}{}^{0,0}=1-R$. Our equations produce the following symmetries of the transition probabilities: $W_{n_1,n_2}{}^{m_1,m_2}=W_{n_2,n_1}{}^{m_2,m_1}$, which means that the transition probabilities do not depend on the sign of L_z, and $W_{n_1,n_2}{}^{m_1,m_2}=W_{m_1,m_2}{}^{n_1,n_2}$. From the general principles of quantum mechanics it follows that the second relation holds if $\mathcal{H}(-t)=\mathcal{H}(t)$, $\varphi(-t)=\varphi(t)$, while we have found it valid for any $\mathcal{H}(t)$ and $\varphi(t)$. This fact is a consequence of R being independent of time inversion, because R may be treated as a reflection coefficient of a particle from the one-dimensional effective potential, determined by $\Omega(t)$, and then, as is well known,[20] R does not depend on the time inversion. Indeed, one can observe that Eq. (58) is analogous to the one-dimensional Schrödinger equation if we replace t by x and $\Omega^2(t)$ by $k^2(x)$. Then, taking into account Eqs. (62) and (90), we find that $i\eta/\xi^*$ is the amplitude of the reflex wave. The corresponding effective potential

$U(x)$ is determined by the $\Omega(t)$ as follows:

$$\Omega^2(x)=\Omega_i^2+2M[U(-\infty)-U(x)].$$

Here we suppose that the mass of the scattering particle is again M in order to avoid new notation. Since $\Omega^2(t)$ may be positive and negative, we have for the energy of the scattering particle

$$E=\Omega_i^2/2M+U(-\infty)\gtrless U(x),$$

which is to be taken into account for the computation of the reflection coefficient R. This simple analogy permits us to use for the calculation of R (or $W_{0,0}{}^{0,0}=1-R$) the methods which are well worked out in quantum mechanics. We shall give some examples (without derivation).

In the case of $\Omega^2>0$ for the Eckart potential,[23]

$$R=\{\cosh[\pi l(\omega_i-\omega_f)]+\cos(\pi^2-8M\pi^2l^2B)^{1/2}\}\times\{\cosh[\pi l(\omega_i+\omega_f)]+\cos(\pi^2-8M\pi^2l^2B)^{1/2}\}^{-1},$$

where B and l are parameters of the potential. For the barrier $U(x)=U_0/\cosh^2\alpha x$ [20] and $\Omega^2<0$, we have

$$R=\cos^2[\tfrac{1}{2}\pi(1-8MU_0/\alpha^2)^{1/2}]\times\{\sinh^2(\pi\omega_i/2\alpha)+\cos^2[\tfrac{1}{2}\pi(1-8MU_0/\alpha^2)^{1/2}]\}^{-1},\quad 8MU_0<\alpha^2.$$

The adiabatic approximation ($W_{n_1,n_2}{}^{n_1,n_2}\approx 1$) is valid if $R\ll 1$ and $n_1n_2R\ll 1$. One may come to this conclusion if we expand $W_{n_1,n_2}{}^{m_1,m_2}$ in series of the parameter R (in the following formulas $m_i\geqslant n_i$, $i=1,2$, and $L_z>0$),

$$W_{n_1,n_2}{}^{n_1,n_2}=1-(2n_1n_2+n_1+n_2+1)R+O(R),\tag{99}$$

$$W_{n_1,n_2}{}^{m_1,m_2}=\frac{m_1!m_2!}{n_1!n_2!}\frac{R^{m_1-n_1}}{[(m_1-n_1)!]^2}\times\left[1-\left(\frac{2n_1n_2}{1+m_1-n_1}+n_1+n_2+1\right)R+\cdots\right].\tag{100}$$

For high initial excitations ($n_1n_2k\leqslant 1$), the asymptotic form (99) is not valid. For $n_1,n_2\to\infty$ (and L_z fixed), we may use the asymptotic form of the Jacobi polynomials[24] and get

$$W_{n_1,n_2}{}^{m_1,m_2}\approx\frac{m_1!m_2!}{n_1!n_2!}\frac{\tfrac{1}{2}\theta\cot\tfrac{1}{2}\theta}{N^{2(m_1-n_1)}}|J_{m_1-n_1}(N\theta)|^2,\tag{101}$$

where $N=\tfrac{1}{2}(m_2+n_1+1)$ and $\sin^2\tfrac{1}{2}\theta=R$. Equation (100) holds for $m_i\to\infty$. In the asymptotic equations (99)–(101), the angular momentum L_z is fixed and small in comparison with n_2 and m_2. For large (and positive) L_z, one has

$$W_{n_1,n_2}{}^{m_1,m_2}=\binom{n_2}{n_1}\binom{m_2}{m_1}R^{m_1+n_1}(1-R)^{L_z+1}\times\left[1-\frac{2n_1m_2}{1+L_z}\cdot\frac{1-R}{R}+O\left(\frac{1}{R^2L_z{}^2}\right)\right].\tag{102}$$

[23] C. Eckart, Phys. Rev. **35**, 1303 (1930).

[24] G. Szego, *Orthogonal Polynomials*, revised edition (American Mathematical Society, New York, 1959).

This formula is not valid if $R\to 0$. From the theory of zeros of the Jacobi polynomials,[24] it follows that the number of zeros of the transition probability $W_{n_1,n_2}{}^{m_1,m_2}(R)$ in the interval $0\leqslant R<1$ is equal to the smallest of the numbers n_1, n_2, m_1, and m_2.

D. Adiabatic Approximation

If the frequency parameter $\Omega(t)$ is changing adiabatically from an initial value Ω_i to the final one Ω_f, i.e.,

$$\frac{1}{\Omega^2(t)}\frac{d\Omega(t)}{dt}\equiv\theta(t),\quad|\theta(t)|\ll 1\tag{103}$$

then to the first order of $\theta(t)$, the solution of Eq. (58) is given by[4]

$$\epsilon(t)=\left(\frac{M}{e}\Omega(t)\right)^{-1/2}\exp\left[i\int_0^t\Omega(\tau)d\tau\right].\tag{104}$$

In this limit $\eta=0$ and

$$\xi=\exp\left\{i\int_0^\infty[\Omega(\tau)-\tfrac{1}{2}\omega_f]d\tau\right\},\quad|\xi|=1$$

and, as is to be expected, the reflection coefficient vanishes. Equation (98) in its adiabatic approximation produces $W_{n_1,n_2}{}^{m_1,m_2}=\delta_{n_1,m_1}\delta_{n_2,m_2}$ in accordance with the adiabatic theory. The invariants $A(t)$ and $B(t)$ in this approximation have the following expressions as their $t\to\infty$ limits:

$$A(\infty)=A_f\exp\left\{i\int_0^\infty[\Omega(\tau)-\tfrac{1}{2}\omega_f]d\tau\right\},$$
$$B(\infty)=B_f\exp\left\{i\int_0^\infty[\Omega(\tau)-\tfrac{1}{2}\omega_f]d\tau\right\},\tag{105}$$

where A_f and B_f are given by Eq. (84). This means, in particular, that the initial coherent state $|\alpha,\beta;i\rangle$ with $t\to\infty$ remains coherent with respect to the final operators A_f and B_f; here $|\alpha|$ and $|\beta|$ are again the moduli of their eigenvalues, but the phases depend on the behavior of the electromagnetic field for all times. In the language of classical motion this means that the radius of the particle's orbit in the xy plane is changed in accordance with ω_f, the center of the orbit is moved along the circle with radius $|\beta|$, and the classical phase of the oscillations depends on the behavior of the electromagnetic potential for all times. At this limit the eigenstates $|n_1,n_2;t\to\infty\rangle$ differ from the Landau solutions by a phase factor.

There exists still one more case when the reflection coefficient vanishes. Let $\Omega^2(t)=g^{-4}(t)-\ddot{g}(t)/g(t)$, where $g(t)$ is an arbitrary function which obeys the conditions $g(\mp\infty)=(\tfrac{1}{2}\omega_{i,f})^{-1/2}$. Then Eq. (58) has an exact solution,

$$\epsilon(t)=g(t)\exp\left(i\int_0^t g^{-2}(\tau)d\tau\right),$$

and, thus, $R\equiv 0$.

VI. ADIABATIC INVARIANTS AND THEIR EVOLUTION

A. N-Dimensional Oscillator

We have discussed the problem of the adiabatic invariants in quantum mechanics in Sec. II. Now we shall illustrate the statement that there are $2n$ adiabatic invariants corresponding to the exact ones (n is the number of the degrees of freedom).

For the N-dimensional quantum oscillator, we have constructed N non-Hermitian exact invariants $A_k(t)$, $k=1,2,\ldots,N$, which have as their $t\rightarrow-\infty$ limits (for constant frequencies $\Omega_k{}^i$) N operators $A_k{}^i$, $k=1,\ldots,N$. For the limit $t\rightarrow\infty$ (constant frequencies $\Omega_k{}^f$) there are N operators $A_k{}^f$. The difference of the operators $A_k{}^i$ is determined as

$$\Delta_{A_k}=\frac{\langle t\rightarrow\infty|A_k{}^f|t\rightarrow\infty\rangle-\langle i|A_k{}^i|i\rangle}{\langle i|A_k{}^i|i\rangle},\quad(106)$$

where $|t\rightarrow\infty\rangle$ is the $t\rightarrow\infty$ limit of the initial state $|i\rangle$. The quantities Δ_{A_k} may be easily computed if we express $A_k{}^f$ in terms of the operators $A_k(\infty)$ and $A_k{}^\dagger(\infty)$ [the reversed formulas of Eq. (39)]:

$$A_k{}^f=\xi_k{}^*A_k(\infty)-\eta_kA_k{}^\dagger(\infty).\quad(107)$$

If the initial state is (a) coherent state $|\boldsymbol{\alpha};i\rangle$, then

$$\Delta_{A_k}=\xi^*-1-\eta_k(\alpha_k{}^*/\alpha_k);\quad(108)$$

if it is (b) energy eigenstate $|\mathbf{n};i\rangle$, then

$$\langle\mathbf{n};i|A_k{}^i|\mathbf{n};i\rangle=\langle\mathbf{n};\infty|A_k{}^f|\mathbf{n};\infty\rangle=0.\quad(109)$$

In the adiabatic approximation we have to put[4]

$$\epsilon_k=\Omega_k{}^{-1/2}\exp\left(i\int_0^t\Omega_k(\tau)d\tau\right);$$

then $\eta_k=0$,

$$\xi_k=\exp\left(i\int_0^\infty[\Omega_k(\tau)-\Omega_k{}^f]d\tau\right),$$

and Eq. (108) yields

$$\Delta_{A_k}=\exp\left(i\int_0^\infty[\Omega_k{}^f-\Omega_k(\tau)]d\tau\right)-1.\quad(110)$$

It is clear, then, that if we define new final operators

$$(A_k{}^f)'=A_k{}^f\exp\left(i\int_0^\infty[\Omega_k(\tau)-\Omega_k{}^f]d\tau\right),\quad(111)$$

we shall get $\Delta_{A_k}{}^i=0$, which means that there are N non-Hermitian adiabatic invariants $A_k{}^i$ which correspond to the exact invariants $A_k(t)$. One may construct N Hermitian (quadratic) adiabatic invariants $(A_k{}^i)^\dagger A_k{}^i$ which correspond to the exact Hermitian invariants $A_k{}^\dagger A_k$, $k=1,\ldots,N$. The evolution of these quadratic adiabatic invariants is given as follows (this is a trivial generalization of the results of Ref. 17 for the one-dimensional case): (a) If the initial state is coherent $|\boldsymbol{\alpha};i\rangle$, then

$$\Delta_{A_k{}^\dagger A_k}=2|\eta_k|^2+\frac{|\eta_k|^2}{|\alpha_k|^2}-2\,\mathrm{Re}\left(\xi_k\eta_k\frac{\alpha_k{}^*}{\alpha_k}\right);\quad(112)$$

(b) if the initial state is an energy eigenstate $|\mathbf{n};i\rangle$, then

$$\Delta_{A_k{}^\dagger A_k}=|\eta_k|^2(2+1/n_k).\quad(113)$$

In classical mechanics the quantity E/Ω is an adiabatic invariant, and in the quantum case its analogous adiabatic invariant (for the N-dimensional oscillator) is

$$I=\sum_{k=1}^N(A_k{}^i)^\dagger A_k{}^i+\tfrac{1}{2}.$$

The evolution of I for the initial energy eigenstate is

$$\Delta_I=\sum_{k=1}^N 2|\eta_k|^2(2n_k+1)\Big[\sum_{k=1}^N(2n_k+1)\Big]^{-1}.\quad(114)$$

For $N=1$ we get the results of Ref. 17 for the one-dimensional quantum oscillator $\Delta_I=2|\eta_k|^2$, i.e., Δ_I does not depend on the quantum number of the stationary energy eigenstate.

B. Charged Particle in Electromagnetic Field

We have considered charged-particle motion in an electromagnetic field of special type (55). For the motion in the xy plane we have constructed two non-Hermitian exact invariants $A(t)$ and $B(t)$, linear with respect to the coordinates and momenta [Eq. (57)]. In accordance with the discussion in Sec. II, there must be two linear adiabatic invariants which correspond to the exact ones. These adiabatic invariants are A_i and B_i. Indeed, let us calculate their evolution [by a formula quite analogous to (106)]. At first we reverse transformations (87)

$$\begin{aligned}A_f&=\xi^*A(\infty)-\eta B^\dagger(\infty),\\ B_f&=-\eta A^\dagger(\infty)+\xi^*B(\infty).\end{aligned}\quad(115)$$

For the evolution of the operators A_i and B_i we get, if the initial state is (a) a coherent state $|\alpha,\beta;i\rangle$,

$$\Delta_A=\xi^*-1-\eta(\beta^*/\alpha),\quad\Delta_B=\xi^*-1-\eta(\alpha^*/\beta);\quad(116)$$

if it is (b) a Landau solution $|n_1,n_2;i\rangle$,

$$\begin{aligned}\langle n_1,n_2;\infty|A_f|n_1,n_2;\infty\rangle&=\langle n_1,n_2;i|A_i|n_1,n_2;i\rangle=0,\\ \langle n_1,n_2;\infty|B_f|n_1,n_2;\infty\rangle&=\langle n_1,n_2;i|B_i|n_1,n_2;i\rangle=0.\end{aligned}\quad(117)$$

In the adiabatic limit $\eta=0$,

$$\xi=\exp\left(i\int_0^\infty [\Omega(\tau)-\tfrac{1}{2}\omega_f]d\tau\right).$$

It is apparent from Eq. (116) that if we define new final operators

$$A_f'=A_f\exp\left(i\int_0^\infty [\Omega(\tau)-\tfrac{1}{2}\omega_f]d\tau\right),$$
$$B_f'=B_f\exp\left(i\int_0^\infty [\Omega(\tau)-\tfrac{1}{2}\omega_f]d\tau\right), \tag{118}$$

we shall get for any initial state in the adiabatic limit $\Delta_{A'}=\Delta_{B'}=0$, and thus, under this condition, we have two linear adiabatic invariants connected with the exact ones $A(t)$ and $B(t)$. The evolution of the quadratic adiabatic invariants $A_i^\dagger A_i$ and $B_i^\dagger B_i$, which correspond to the exact Hermitian invariants $A^\dagger(t)A(t)$ and $B^\dagger(t)B(t)$, may be calculated in a similar manner: (a) for the state $|\alpha,\beta;i\rangle$,

$$\Delta_{A^\dagger A}=|\eta|^2\frac{|\alpha|^2+|\beta|^2+1}{|\alpha|^2}-2\,\mathrm{Re}\left(\xi\eta\frac{\beta^*}{\alpha}\right),$$
$$\Delta_{B^\dagger B}=|\eta|^2\frac{|\alpha|^2+|\beta|^2+1}{|\beta|^2}-2\,\mathrm{Re}\left(\xi\eta\frac{\alpha^*}{\beta}\right); \tag{119}$$

(b) for the state $|n_1,n_2;i\rangle$,

$$\Delta_{A^\dagger A}=|\eta|^2\frac{n_1+n_2+1}{n_1},\quad \Delta_{B^\dagger B}=|\eta|^2\frac{n_1+n_2+1}{n_2}. \tag{120}$$

It is obvious from Eqs. (119) and (120) that in the adiabatic limit ($\eta=0$) for any initial state $\Delta_{A^\dagger A}=\Delta_{B^\dagger B}=0$. All evolutions depend on the quantum numbers of the initial states. The adiabatic invariant $I=A_i^\dagger A_i+\frac{1}{2}$ corresponds to the classical adiabatic invariant E/ω. The evolution of I in the state $|n_1,n_2;t\rangle$ is

$$\Delta_I=2|\eta|^2\frac{n_1+n_2+1}{2n_1+1}, \tag{121}$$

and we see that Δ_I depends on the quantum numbers n_1 and n_2 (in the case of the one-dimensional quantum oscillator, such a dependence does not hold).

VII. CHARGED PARTICLE WITH VARYING MASS

All the results of the previous sections may be easily generalized for the case of a charged particle with varying mass.

The Schrödinger equation for a particle with time-dependent mass may be reduced to the equation for a particle with constant mass M_0 by the following transformation of time:

$$t'=\int^t \frac{M_0}{M(\tau)}d\tau. \tag{122}$$

We then consider a charged particle of mass M_0 and charge e moving in an electromagnetic field with a potential

$$\mathbf{A}'=\mathbf{A}(t(t')),\quad \varphi'=\frac{1}{M_0}M(t(t'))\varphi(t(t')),$$

and thus all the results of the previous sections hold, related to this electromagnetic potential. The case of a particle of varying mass $M(t)$ moving in a constant magnetic field $\mathcal{H}$, $\varphi=0$ is very simple. One can easily check that the solution of the Schrödinger equation in this case is

$$|n_1,n_2;t\rangle=\exp[i\alpha_{n_1}(t)]|n_1,n_2\rangle, \tag{123}$$

where

$$\alpha_{n1}=-(n_1+\tfrac{1}{2})\int^t \frac{e\mathcal{H}}{M(\tau)}d\tau$$

and $|n_1,n_2\rangle$ are eigenfunctions of the Hamiltonian H and of L_z for a particle of constant mass M_0 in a constant magnetic field $\mathcal{H}$. The effect of varying mass is reduced to a phase factor.

VIII. CONCLUDING REMARKS

It should be noted that the known connection of coherent states with magnetic translations[25] makes it possible to obtain the Bloch wave functions for a charge in a time-dependent magnetic field and to introduce a quasimomentum representation for time-dependent Hamiltonians. The mentioned correspondance of exact invariants and adiabatic invariants shows that the adiabatic invariants may be treated from a group-theoretical point of view. For the N-dimensional oscillator one will have the noncompact group $U(N,1)$, and for a charge in a magnetic field one has the group $U(2,1)$,[26] both constructed from adiabatic invariants.

On the basis of these symmetries, one can interpret the formulas for the transition amplitudes from a group-theoretical viewpoint. The constructed coherent states are very convenient in density-matrix calculations for the systems under consideration.

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APPENDIX: TRANSITION AMPLITUDES AND D FUNCTIONS

Using the definition of the $D_{m,m'}{}^j$ function,[27]

$$D_{m,m'}{}^j(\cos\theta)=\left[\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}\right]^{1/2}(\cos\tfrac{1}{2}\theta)^{m+m'}\times(-\sin\tfrac{1}{2}\theta)^{m-m'}P_{j-m}{}^{(m-m',m+m')}(\cos\theta)\,, \quad (124)$$

one can obtain the following expressions for the transition amplitudes between the energy and L_z eigenstates of a charged particle in a magnetic field:

(a) $L_z\leqslant 0$, $m_i\leqslant n_i$, $i=1,2$

$$T_{n_1,n_2}{}^{m_1,m_2}=(-1)^{m-m'}\cos\tfrac{1}{2}\theta\times\exp\{i[(m'-m)\varphi_\eta-(2j+1)\varphi_\xi]\}\times D_{m,m'}{}^j(\cos\theta)\,, \quad (125)$$

where $j=\frac{1}{2}(m_1+n_2)$, $m=\frac{1}{2}(n_1-m_2)$, $m'=\frac{1}{2}(m_1-n_2)$, φ_η and φ_ξ are phases of the parameters η and ξ, and θ is defined by Eq. (91);

(b) $L_z\leqslant 0$, $m_i>n_i$, $i=1,2$

$$T_{n_1,n_2}{}^{m_1,m_2}=\cos\tfrac{1}{2}\theta\times\exp\{i[(m-m')\varphi_\eta-(2j+1)\varphi_\xi]\}\times D_{m,m'}{}^j(\cos\theta)\,, \quad (126)$$

where $j=\frac{1}{2}(m_1+n_2)$, $m=\frac{1}{2}(m_1-n_2)$, $m'=\frac{1}{2}(n_1-m_2)$. The cases of positive L_z may be obtained from (125) and (126) by the replacement $n_1\rightleftarrows n_2$, $m_1\rightleftarrows m_2$. These expressions correspond to the known result due to Schwinger[28] for the generating function of $D_{m,m'}{}^j$ functions. This generating function can be connected with the transition amplitude (92) for coherent states $|\alpha,\beta;i\rangle$ and $|\gamma,\delta;f\rangle$.[29]

If one puts into (92) $\alpha=\beta$, $\gamma=\delta$, and $\xi=2\xi'$ and then compares the amplitude $T_{\alpha,\alpha}{}^{\gamma,\gamma}$ with the amplitude (48) (for the case $N=1$), it is seen that one can obtain the relation [we take into account Eq. (52)]

$$P_m{}^n(2\cos\tfrac{1}{2}\theta)=(m+n)!\sum_{l=n-m}^{m-n}\frac{\delta_{c_l,c_{m-n}}D_{(|l|+n)/2,(|l|-n)/2}{}^{m/2}(\cos\theta)}{[(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}l)!(\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}l)!(\frac{1}{2}m-\frac{1}{2}n+\frac{1}{2}l)!(\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}l)!]^{1/2}}, \quad (127)$$

where

$$\delta_{c_l,c_{m-n}}=1 \quad \text{if } l,m-n \text{ are odd, odd or even, even}$$
$$=0 \quad \text{if } l,m-n \text{ are odd, even or even, odd.}$$

From this discussion we conclude that the problems under consideration have an additional compact symmetry, corresponding to the three-dimensional rotations.

[27] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, N. J., 1957).

[28] I. Schwinger, *Quantum Theory of Angular Momentum* (Academic, New York, 1965).

[29] M. H. Johnson and B. A. Lippmann, Phys. Rev. **76**, 828 (1949).