PHYSICAL REVIEW D
PARTICLES AND FIELDS
THIRD SERIES, VOL. 2, NO. 8 15 OCTOBER 1970

Uniformly Accelerating Charged Mass in General Relativity*

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(Received 25 May 1970)

We discuss a type-{22} solution of the Einstein-Maxwell equations which represents the field of a uniformly accelerating charged point mass. It contains three arbitrary parameters m, e, and A, representing mass, charge, and acceleration, respectively. The solution is a direct generalization of the Reissner-Nordstrom solution of general relativity and the Born solution of classical electrodynamics. The external "mechanical" force necessary to produce the acceleration appears in the form of a timelike nodal two-surface extending from the particle's world line to infinity. This does not prevent us from regarding the solution as asymptotically flat and calculating the radiation pattern of its electromagnetic and gravitational waves. We find as well a maximal analytic extension of the solution and discuss its properties. Except for an extra "outer" Killing horizon due to the accelerated motion, the horizon structure closely resembles the Reissner-Nordstrom case.

I. INTRODUCTION

In this paper we discuss a three-parameter family of solutions to the source-free Einstein-Maxwell field equations. These solutions will be referred to collectively as the charged C metric, following the terminology of Ehlers and Kundt.[1] The C metric is a particularly interesting solution for many reasons. Aside from the Kerr metric, it is the only metric with fewer than three Killing vectors which it has been possible to analyze fully. It bridges the gap between the Schwarzschild and Reissner-Nordstrom solutions, which are completely understood, and two large classes of solutions which are familiar but not as well understood, the Weyl[2] and Robinson-Trautman[3] solutions. It provides new examples of many items of current interest: Killing horizons, trapped surfaces, incomplete geodesics, non-simply-connected topologies, as well as gravitational (and electromagnetic) radiation and null hypersurface boundaries at conformal infinity. In addition to all of this, the C metric has a clear and unambiguous physical interpretation as the combined gravitational and electromagnetic field of a uniformly accelerating charged mass. In contrast to Misner's description[4] of Taub-NUT space as a "counterexample to almost anything," we might describe the charged C metric as an example of almost everything.

The vacuum C metric has a long and interesting history, which we will briefly review. It was first discovered by Levi-Civita[5] in 1918 as the "soluzioni oblique" case of his static degenerate metrics. Levi-Civita found the class of metrics having a timelike Killing vector ξ^a orthogonal to a three-space whose Ricci curvature tensor is of the form[6]

$$^3R^a{}_b = \alpha\eta^a\eta_b + \beta\delta^a{}_b. \tag{1}$$

It turns out this is equivalent to the requirement that the field be static and that the Weyl tensor be type {22}. In fact, the two repeated principal null vectors l^a and n^a of the Weyl tensor are just the null linear combinations of ξ^a and η^a, modulo normalization conditions.[7] The intrinsic geometry of these vector fields will play an important part in our analysis.

The C metric was rediscovered by Newman and Tamburino[8] in 1961, by Robinson and Trautman[3] in 1961, and again by Ehlers and Kundt[1] in 1963, but no

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[1] J. Ehlers and W. Kundt, in *Gravitation, an Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[2] H. Weyl, Ann. Physik **54**, 117 (1917). See also T. Levi-Civita, Atti Accad. Nazl. Lincei, Rend. **27** (1), 3 (1918); **27** (2), 183 (1918); **28** (1), 3 (1919).
[3] I. Robinson and A. Trautman, Proc. Roy. Soc. (London) **A265**, 463 (1962).
[4] C. Misner, in *Lectures in Applied Mathematics*, edited by J. Ehlers (Interscience, New York, 1967), Vol. 8, p. 160.
[5] T. Levi-Civita, Atti Accad. Nazl. Lincei., Rend. **27**, 343 (1918).
[6] Although the four-dimensional Ricci tensor vanishes in vacuum, the three-dimensional Ricci tensor $^3R_{ab}$ does not vanish (except in flat space).
[7] Usually one of l^a or n^a, l^a say, is chosen to be parallelly propagated: $l^c\nabla_c l^a = 0$. The normalization of n^a is then fixed by requiring $l^a n_a = 1$.
[8] E. Newman and L. Tamburino, J. Math. Phys. **2**, 667 (1961).

interpretation for it was suggested. Robinson and Trautman were the first to recognize one of the interesting features of the C metric, however[3]: "The Riemann tensor contains the r^{-1} term which seems characteristic of radiation. The metric, however, admits a hypersurface-orthogonal Killing field. The solution might, therefore, be described as both static and radiative." The apparent paradox in this remark is dispelled when we note that the metric is by no means static near infinity because the Killing vector ξ^a becomes spacelike there.

II. METRIC

One obstacle to better understanding of the C metric has been an unfortunate choice of parametrization. The result is that the straightforward limits of the solution do not lead to familiar metrics. The metric is usually given in the form[1]

$$ds^2=(\tilde{x}+\tilde{y})^{-2}(Fd\tilde{t}^2-F^{-1}d\tilde{y}^2-G^{-1}d\tilde{x}^2-Gd\tilde{z}^2), \quad (2)$$

where

$$F=F(\tilde{y}), \quad G=G(\tilde{x}),$$

and F and G are related by

$$F(\tilde{y})=-G(-\tilde{y}). \quad (3)$$

With these conditions, the vacuum field equations are satisfied by any cubic polynomial of the form

$$G(\tilde{x})=a_0+a_1\tilde{x}+a_2\tilde{x}^2+a_3\tilde{x}^3. \quad (4)$$

The constants $a_0, \ldots, a_3$, however, are not all significant. To see this, consider the effect of the coordinate transformation

$$\begin{aligned} t&=c_0\tilde{t}, & z&=c_0\tilde{z}, \\ x&=Ac_0\tilde{x}+c_1, & y&=Ac_0\tilde{y}-c_1. \end{aligned} \quad (5)$$

In general, this will give a metric with the same form as Eq. (2) but with an over-all constant conformal factor A^{-2} and also with a different set of parameters a_i. (Thus, any one of the C-metrics can be mapped conformally on another.) We might fix $A=1$ and use Eq. (5) to set one of the a's to zero and one to unity. Levi-Civita[5] chose to put $a_3=1$ and $a_2=0$. In this form the metric has no flat space limit, since it turns out that the Weyl tensor does not depend on the remaining parameters.

A better choice, as will become apparent, is to use the entire freedom in Eq. (5) to set $a_1=0$ and $a_0=-a_2=1$. The new metric is

$$ds^2=A^{-2}(x+y)^{-2}(Fdt^2-F^{-1}dy^2-G^{-1}dx^2-Gdz^2), \quad (6)$$

with

$$F=-1+y^2-2mAy^3, \quad G=1-x^2-2mAx^3. \quad (7)$$

The charged version[9] of Eq. (6) satisfying the source-free Einstein-Maxwell equations is obtained simply by replacing F and G by quartics:

$$G=1-x^2-2mAx^3-e^2A^2x^4, \quad F=-G(-y). \quad (8)$$

Thus, the charged metric has just three arbitrary parameters, m, e, and A. We assume from now on that m, e, and A are positive.

The coordinates (t,y,x,z) are adapted to the timelike Killing vector $\xi^a=A\delta_0{}^a$, the nondegenerate eigenvector $\eta^a=\delta_1{}^a$ of the three-space Ricci tensor, and the spacelike Killing vector $\zeta^a=\delta_3{}^a$. The form of Eq. (6) shows explicitly that the solution is static, time reversible, and axially symmetric. Thus, it is a Weyl solution. By analogy with the method used by Finkelstein and others[10] on the Schwarzschild metric, we may seek instead null coordinates adapted to one of the two principal null vectors l^a or n^a. Define retarded coordinates u and r by

$$\begin{aligned} Au&=t+\int^y F^{-1}dy, \\ Ar&=(x+y)^{-1}. \end{aligned} \quad (9)$$

The metric then assumes the form

$$ds^2=Hdu^2+2dudr+2Ar^2dudx-r^2(G^{-1}dx^2+Gdz^2), \quad (10)$$

where

$$\begin{aligned} H&=-A^2r^2G(x-A^{-1}r^{-1}) \\ &=-A^2r^2G(x)+ArG'(x)+(1+6mAx+6e^2A^2x^2) \\ &\qquad -2(m+2e^2Ax)r^{-1}+e^2r^{-2}. \end{aligned} \quad (11)$$

The vector l^a generates a family of null hypersurfaces $N^+(u)$ on which $u=$const, and r is an affine parameter along l^a. The C metric is, therefore, a Robinson-Trautman solution. We have

$$\begin{aligned} l_a&=\nabla_a u=(1,0,0,0), \\ l^a&=\partial x^a/\partial r=(0,1,0,0), \end{aligned} \quad (12)$$

and

$$n^a=(1,\tfrac{1}{2}H,0,0). \quad (13)$$

We could just as well have defined an advanced null coordinate v by

$$Av=t-\int^y F^{-1}dy, \quad (14)$$

resulting in

$$ds^2=Hds^2-2dvdr-2Ar^2dvdx-r^2(G^{-1}dx^2+Gdz^2) \quad (15)$$

and a family of $v=$const null hypersurfaces $N^-(v)$ generated by n^a.

[9] W. Kinnersley, thesis, California Institute of Technology, 1969 (unpublished).
[10] D. Finkelstein, Phys. Rev. 110, 965 (1958); A. S. Eddington, Nature 113, 192 (1924).

III. FLAT-SPACE LIMIT

To gain insight into the nature of the solution, it is very helpful to look at the invariant components of the gravitational and electromagnetic fields. The curvature invariant[11] turns out to be

$$\begin{aligned}\Psi_2 &= -\tfrac{1}{2}C_{abcd}l^a n^b l^c n^d \\ &= -(m+2e^2Ax)r^{-3}+e^2r^{-4}.\end{aligned} \tag{16}$$

The electromagnetic field also has l^a, n^a as principal null vectors, and its invariant is

$$\begin{aligned}\Phi_1 &= \tfrac{1}{2}F_{ab}l^a n^b \\ &= \tfrac{1}{2}\sqrt{2}er^{-2}.\end{aligned} \tag{17}$$

The remaining Ψ_A's and Φ_A's of Newman and Penrose vanish.

Both Eqs. (16) and (17) have poles at $r=0$, and both fall off to zero asymptotically as $r\to\infty$. In this sense, we are justified in regarding r as a radial coordinate. The parameter m appears in Ψ_2 the way a mass would appear, and e resembles a charge.[12]

In order to understand the role played by the parameter A, the best case to examine is $m=e=0$, $A\neq 0$, because according to Eq. (16) this is the flat-space limit. What we must study in this limit is the geometry of the principal null vectors l^a and n^a by means of the coordinates intrinsically related to them.

Since G now reduces to

$$G=1-x^2, \tag{18}$$

a trigonometric substitution is suggested. We therefore let

$$x=\cos\theta,\quad z=\varphi. \tag{19}$$

In terms of these coordinates, the metric of Eq. (10) with $m=e=0$ becomes

$$\begin{aligned}ds^2 = (1-2Ar\cos\theta-A^2r^2\sin^2\theta)du^2+2dudr \\ -2Ar^2\sin\theta du d\theta-r^2(d\theta^2+\sin^2\theta d\varphi^2).\end{aligned} \tag{20}$$

Equation (20) is a flat-space metric closely related to ones discussed by Newman and Unti.[13] The transformation required to cast Eq. (20) into a Minkowski coordinate system is

$$\begin{aligned}\bar{t} &= (A^{-1}-r\cos\theta)\sinh Au+r\cosh Au, \\ \bar{z} &= (A^{-1}-r\cos\theta)\cosh Au+r\sinh Au, \\ \bar{x} &= r\sin\theta\cos\varphi, \\ \bar{y} &= r\sin\theta\sin\varphi,\end{aligned} \tag{21}$$

[11] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).

[12] These identifications are shown to be correct in the weak-field limit and several other important special cases. However, when either of the products mA or eA is large, the asymptotically defined Bondi-Sachs mass and charge differ from m and e. Since the relations are quite complicated, we have chosen throughout to keep m, e, and A as the parameters, in order that G may be written in a simple closed form.

[13] E. Newman and T. Unti, J. Math. Phys. **4**, 1467 (1963).

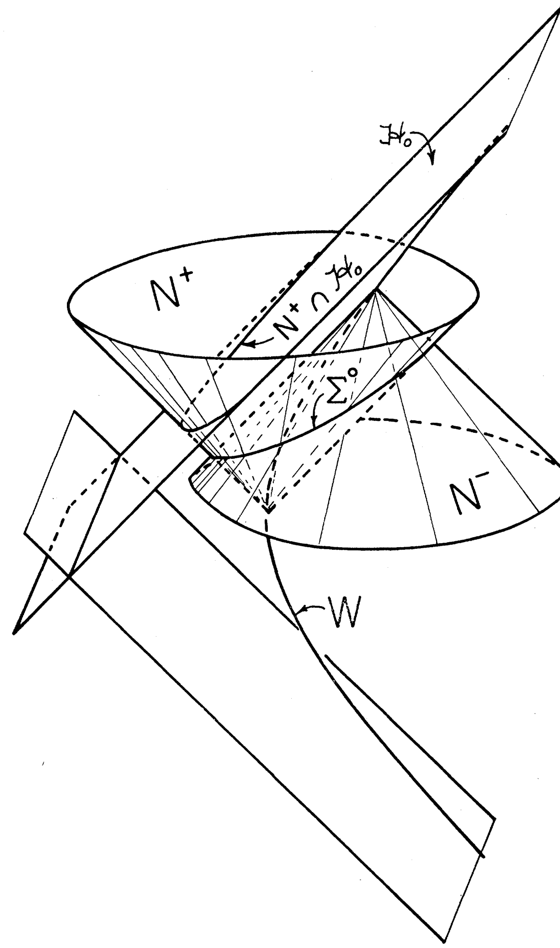

FIG. 1. Uniformly accelerating worldline W in Minkowski space. Typical advanced and retarded null cones $N^-(v)$, $N^+(u)$ are shown, together with their intersection $\Sigma^0(u,v)$. Intersection of N^+ with the horizon $\mathcal{H}_0(\bar{t}=\bar{z})$ is also shown.

which does in fact imply that

$$ds^2=d\bar{t}^2-d\bar{y}^2-d\bar{x}^2-d\bar{z}^2.$$

Provided $A>0$ and $r>0$, the (u,r,θ,φ) coordinates cover only the half-space $\bar{t}+\bar{z}>0$. The locus $r=0$ is a timelike curve W given by

$$\begin{aligned}\bar{t} &= A^{-1}\sinh Au, \\ \bar{z} &= A^{-1}\cosh Au, \\ \bar{x} &= \bar{y}=0.\end{aligned} \tag{22}$$

This is one branch of a hyperbola with constant acceleration A, parametrized in terms of its arc length u. W divides the $(\bar{t},\bar{z})$ plane into two regions: the region $\theta=0$, or $\bar{z}^2-\bar{t}^2<A^{-2}$, which we shall refer to as the north pole, and the region $\theta=\pi$, or $\bar{z}^2-\bar{t}^2>A^{-2}$, which we shall call the south pole.

We can eliminate θ and φ from Eq. (21) to obtain

$$(\bar{t}-A^{-1}\sinh Au)^2-(\bar{z}-A^{-1}\cosh Au)^2-\bar{x}^2-\bar{y}^2=0. \tag{23}$$

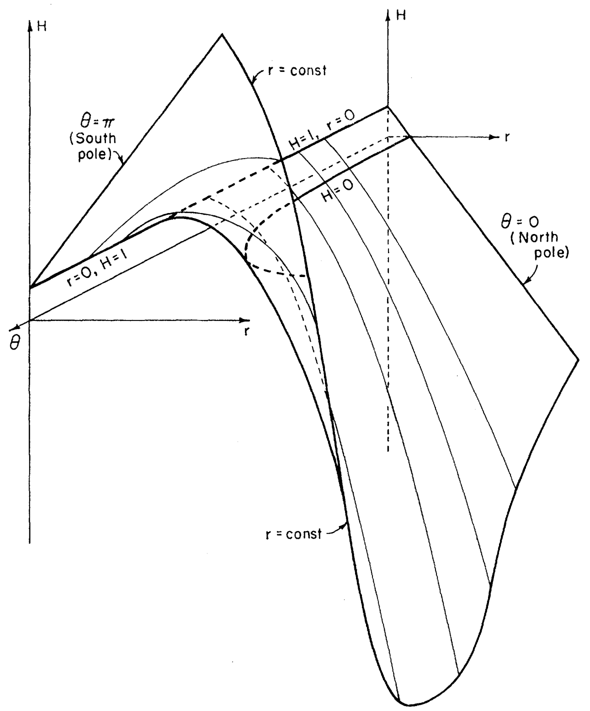

FIG. 2. Three-dimensional graph of the function $H(r,\theta)$ in the case of flat space. For all values of θ except one, H passes through zero producing the outer horizon. At $\theta=\pi$ the surface rises indefinitely.

This relation states that the surfaces $N^+(u)$ defined by $u=$const are null cones with their vertices on W. Likewise, it can be shown that Eq. (23) holds as well with u replaced by v. The $N^+(u)$ are future-directed ("retarded") null cones and the $N^-(v)$ are advanced null cones.

The situation is illustrated in Fig. 1, where two typical cones are shown, together with their intersection $\Sigma^0(u,v)$. The family of two-surfaces $\Sigma^0(u,v)$ on which $u=$const, $v=$const provides us with a natural set of "wavefronts" whose definition is clearly time symmetric. The surfaces Σ^0 are *not*, however, surfaces of constant r. We may define two other families whose claim to the title of wavefront is equally strong. The retarded wavefronts $\Sigma^+(u,r)$ on which $u=$const, $r=$const, and the advanced wavefronts $\Sigma^-(v,r)$ on which $v=$const, $r=$const, are also intrinsically defined. In the present case, $m=e=0$, all three wavefronts are perfectly spherical.

Although Minkowski space itself has many Killing vectors, the metric Eq. (20) distinguishes just two of these, namely, the rotation in the $(\bar{x},\bar{y})$ plane and the Lorentz transformation (velocity boost) in the $(\bar{t},\bar{z})$ plane. In the full C metric, this interpretation of the Killing vectors still survives in the flat asymptotic region, and hence the term "static" is definitely a misnomer. We can read off the norm of the Killing vector ξ^a directly from Eq. (20) [cf. Eq. (10)]:

$$\begin{aligned}\xi^a\xi_a &= H\\ &=1-2Ar\cos\theta-A^2r^2\sin^2\theta\\ &=A^2(\bar{z}^2-\bar{t}^2).\end{aligned}\tag{24}$$

The behavior is shown in Fig. 2, where we have plotted H as a function of r and θ. As r is allowed to increase, keeping θ constant, ξ^a starts off as a timelike vector; it becomes null where H passes through zero, and remains spacelike from there to $r=\infty$. A single exception is the ray along the south pole $\theta=\pi$, where ξ^a remains timelike for all values of r. Thus, for sufficiently large r, all but one of the null geodesic generators of N^+ reach a region where ξ^a is spacelike.

Figure 1 explains why this is so. The horizon of ξ^a, where $H=0$, is the null hyperplane $\bar{t}=\bar{z}$, or equivalently

$$r^{-1}=A(1-\cos\theta).\tag{25}$$

We have drawn the intersection of the horizon with $N^+(0)$. Equation (25) shows that this intersection is a paraboloid, and that the ray $\theta=\pi$ is missing from the intersection.

We now return to the full C metric. The discussion will parallel closely the preceding treatment of the flat-space geometry, for we will find that most of the features discussed above will persist.

IV. WAVEFRONTS

The surfaces $N^\pm$, Σ^0, and $\Sigma^\pm$ are defined just as before, as the surfaces on which certain of the intrinsic coordinates u, v, and r are constant. Since the metric of a null surface is degenerate, we cannot study the intrinsic geometry of N^+ without introducing on it a foliation, or slicing into spacelike two-surfaces. The properties of N^+ are the same no matter how you slice it, so we may choose as slices either Σ^0 or Σ^+.

The spacetime metric restricted to Σ^+ is

$$ds^2=-r^2(G^{-1}dx^2+Gdz^2)\tag{26}$$

with r held constant, while for Σ^0 it is

$$ds^2=-A^{-2}(x+y)^{-2}(G^{-1}dx^2+Gdz^2)$$

with y constant. We prefer to work with Σ^+ since its metric is somewhat simpler, although we must reemphasize that the results are independent of this choice.

Let us therefore consider the metric of Eq. (26), recalling that

$$G=1-x^2-2mAx^3-e^2A^2x^4.$$

To obtain the correct signature for the metric, we must restrict x to a range for which G is positive. At least one allowed range always exists[14] since $G(0)=1$. Let

[14] The choice of a_i's made in Eq. (7) guaranteed this. Hence strictly speaking the choice was a small (but reasonable) specialization.

the two roots of G which are the end points of this range be x_1 and x_2 with $x_2<0<x_1$. In case $e=0$, it is possible that x_2 is infinite, but x_1 is always finite since $G(x)<0$ for sufficiently large x.

We may cast Eq. (26) into a canonical form by defining two new coordinates on Σ^+:

$$\theta=\int_x^{x_1} G^{-1/2}dx\,, \qquad \varphi=\kappa z\,, \tag{27}$$

where $\kappa=\frac{1}{2}G'(x_1)$, and φ ranges from 0 to 2π. We also define

$$\rho(\theta)=\kappa^{-1}[G(x(\theta))]^{1/2}. \tag{28}$$

These transformations reduce the metric of Σ^+ to

$$ds^2=-r^2[d\theta^2+\rho^2(\theta)d\varphi^2]. \tag{29}$$

Using Eq. (29), it is a simple matter to visualize the wavefronts Σ^+ by embedding them isometrically as surfaces of revolution in Euclidean three-space. This has been done for a representative case in Fig. 3. At the north pole, $\rho(0)=0$ and $\rho'(0)=1$ so that (θ,φ) are locally plane polar coordinates, and the surface is smooth there. The south pole, however, will not usually be so fortunate. We proceed to discuss the situation at $x=x_2$.

Incorporating the transformations of Eqs. (27) and (28) into Eq. (10) yields

$$ds^2=Hdu^2+2dudr+2Ar^2\rho dr d\theta-r^2(d\theta^2+\rho^2d\varphi^2). \tag{30}$$

A small geodesic circle about the south pole $\theta_2=\theta(x_2)$ on Σ^+ has circumference

$$\begin{aligned}C&=2\pi r\rho\\ &=2\pi r\rho'(\theta_2)(\theta-\theta_2)+O((\theta-\theta_2)^2).\end{aligned}$$

From Eq. (30), we see that, since $\rho(\theta_2)=0$, the geodesic radius of this loop will be solely in the $d\theta$ direction. The radius is, therefore, approximately $r(\theta_2-\theta)$, and the ratio of the acual circumference to its corresponding Euclidean value is

$$\epsilon(\theta)=-\rho'(\theta_2)+O(\theta-\theta_2).$$

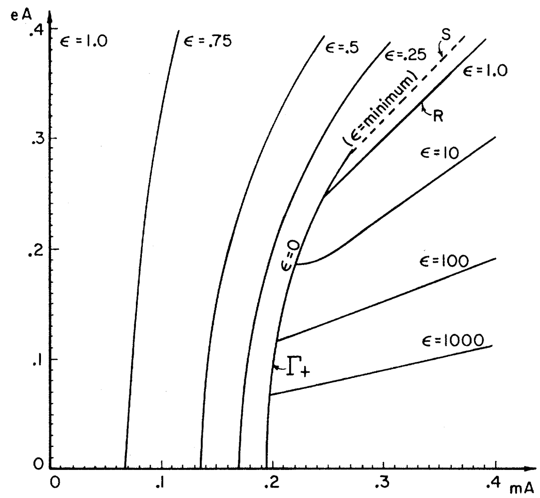

FIG. 4. Function $\epsilon=\epsilon(\theta_2)$ for different values of mA and eA. Whenever $\epsilon\neq 1$, the space-time must have a singular nodal two-surface.

Taking the limit as our loop shrinks down on the south pole gives $\epsilon(\theta_2)=-\rho'(\theta_2)$. Consequently, the wavefront Σ^+ possesses a node at the south pole, unless $\epsilon(\theta_2)=1$. In Fig. 4, we show how $\epsilon(\theta_2)$ behaves as the two dimensionless parameters mA and eA are varied.

Returning now to the null hypersurface itself, we see that N^+ will also have a nodal line or "crease" at the south pole when $\epsilon(\theta_2)\neq 1$. The situation is more serious than that, however, since the above argument shows that the entire spacetime inherits the crease in the form of a nodal timelike two-surface extending from the singularity at $r=0$ to infinity in the south-pole direction. When $\epsilon(\theta_2)\neq 1$, the spacetime is genuinely conical near $x=x_2$, and this effect cannot be removed.[15]

In Fig. 5, we exhibit the various possibilities which arise for the shape of Σ^+. The (mA,eA) plane is divided into five regions by the curves $\Gamma_\pm$, $\Delta_\pm$, and straight lines R, S, and T. Their equations are

$$\begin{aligned}\Gamma_\pm&:\ 54m^2A^2=1+36e^2A^2\pm(1-12e^2A^2)^{3/2},\\ \Delta_\pm&:\ 108m^2A^2=1+72e^2A^2\pm(1-36e^2A^2)(1-9e^2A^2)^{1/2},\\ R&:\ e^2=m^2,\\ S&:\ 9m^2=8e^2,\\ T&:\ 3m^2=2e^2.\end{aligned} \tag{31}$$

Γ is the locus of points where two roots of G coincide. On S two extrema coincide, and on T two inflections coincide. Δ is the locus where a root coincides with an

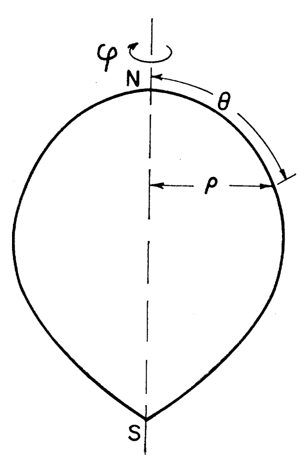

FIG. 3. Embedding diagram for the spacelike two-surface Σ^+. Surface is axially symmetric with the north pole at the top. Embedding is aided by transforming to an arc-length coordinate θ defined by

$$\theta=\int_x^{x_1} G^{-1/2}dx,$$

an azimuthal coordinate $\varphi=\kappa z$, where $\kappa=-\frac{1}{2}G'(x_1)$, and a radial function $\rho(\theta)=\kappa^{-1}\times G^{1/2}(x(\theta))$. In terms of these, the two-metric reduces to $-d\sigma^2=d\theta^2+\rho^2(\theta)d\varphi^2$. Typically, Σ^+ will have a sharp node at the south pole.

[15] Changing the value of the constant κ could make the south pole regular only at the expense of the north pole. There is no way to make both end points nonsingular simultaneously. If we had allowed z an unrestricted range, the wavefronts would have been singular at both poles.

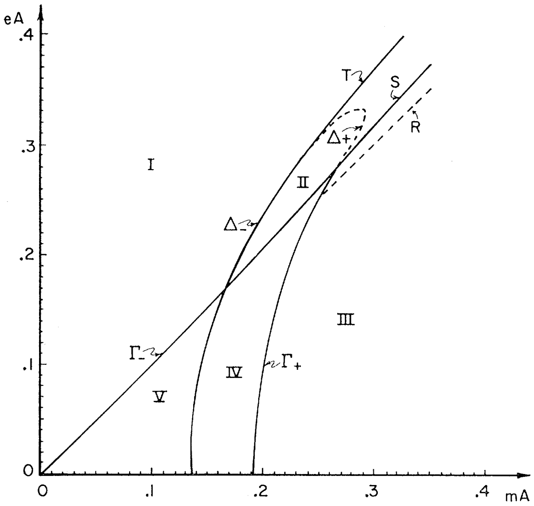

FIG. 5. The (mA, eA) plane divided into various regions by curves $\Gamma_\pm$ and $\Delta_\pm$, and by straight lines R, S, and T. In regions I, II, and III, G has two roots, while in IV and V it has four. Σ^+ is a teardrop in region I, a teardrop with a negatively curved neck in region II, and a dumbbell in III. In regions IV and V, two distinct solutions are possible for given m, e, and A since two allowed ranges for x exist. In region V, the Gauss curvature of Σ^+ is everywhere positive, while in IV it is negative near the south pole. Solutions with no node occur along the vertical axis $m=0$ and along the dashed portion of R.

inflection. In region I, Σ^+ will be teardrop shaped. In region II, it will be a teardrop with a portion near the south pole having negative curvature. In region III, Σ^+ will be a dumbbell. In regions IV and V, G has four real roots. There are in fact two permissible ranges for x in these regions, both yielding teardrop surfaces. The Gauss curvature of Σ^+ is again negative near the south pole in region IV. Along Γ^+, Σ^+ will have an infinite tail.

There are a number of special cases which deserve mention. $A=0$ comprises the Minkowski, Schwarzschild, and Reissner-Nordstrom metrics. We have $G=1-x^2$, $x=\cos\theta$, $\rho=\sin\theta$, and the Gauss curvature $K=-\frac{1}{2}G''$ of Σ^+ is constant. Σ^+ is, therefore, a sphere in this case.

For the vacuum C metrics, $e=0$. If $mA<3^{-3/2}$, G has three roots and Σ^+ is compact with a node at the south pole.[16] When $mA=3^{-3/2}$, two roots of G coincide and the wavefront develops an infinitely long tail. If $mA>3^{-3/2}$, G has only one root and Σ^+ flares completely open.

For $m=0$, G is an even function of x and no node is present. Σ^+ is symmetrical and resembles a prolate spheroid.

The case $|e|=m$ is the only other one in which Σ^+ can be free of nodes. $G=1-(x+mAx^2)^2$. If $mA=(12)^{-1/2}$, Σ^+ again has an infinitely long tail. When mA exceeds this value, however, the surface recontracts bringing a second lobe with it so that Σ^+ is shaped like a symmetrical dumbbell.

We conclude that, save for the two exceptions $mA=0$ and $eA=mA>(12)^{-1/2}$, the spacetime always contains a nodal timelike two-surface.

V. HORIZONS

The natural prototype for any discussion of horizons is the null hypersurface $r=2m$ in the Schwarzschild solution. There are several definitions of the word "horizon" currently in use, but their common intent is to point to hypersurfaces in other space-times which share certain features of this Schwarzschild sphere. Penrose[17] defines an "event horizon" as the boundary of the asymptotic region from which timelike curves may escape to infinity. An event horizon is always a null hypersurface, and frequently there may be more than one present. For example, the completed Reissner-Nordstrom solution[18] has many separate asymptotic regions (when $m>e$) and all surfaces labeled by $r=m\pm(m^2-e^2)^{1/2}$ are event horizons. Event horizons obviously can exist even in space-times which lack Killing vectors, but finding them is difficult because it requires complete knowledge of the integrated geodesics. Symmetries make the job much easier by restricting attention to local behavior of the geodesics.

A "Killing horizon" according to Carter[19] is a null hypersurface, invariant under all isometries of the space-time, whose null generator is also a Killing vector. Carter has shown in particular that if ξ^a is a hypersurface-orthogonal Killing vector which commutes with all others, then the surfaces where $\xi^a\xi_a=0$ are Killing horizons. In our case, we have just two independent commuting Killing vectors, with components in the coordinates of Eq. (6) given by $\xi^a=A\,\delta_0{}^a$ and $\zeta^a=\delta_3{}^a$. The vector ζ^a is everywhere spacelike. This makes the

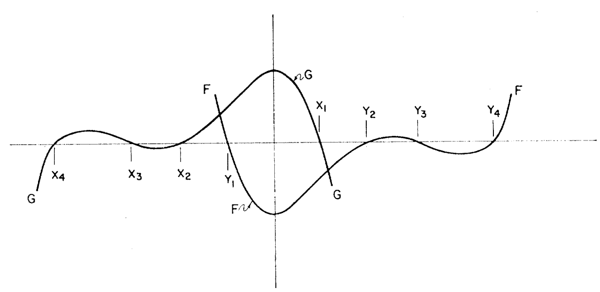

FIG. 6. Quartic curves $G(x)$ and $F(y)$ plotted on the same axes to show the relation between their roots. Intervals x_1 to x_2 and x_3 to x_4 are "allowed ranges" for x. Horizons occur at y_1, y_2, y_3, and y_4. The root $y_1=-x_1$ corresponds to $r=\infty$. Outer horizon occurs at $y=y_2$. We see that in this case two inner horizons are present at $y=y_3$ and y_4.

[16] In all cases with $A\neq0$, x and ρ may be expressed as elliptic functions (with appropriate modulus) of θ.

[17] R. Penrose, in *Contemporary Physics* (IAEA, Vienna, 1969), Vol. 1, p. 545.

[18] J. C. Graves and D. R. Brill, Phys. Rev. **120**, 1507 (1960); B. Carter, Phys. Letters **21**, 423 (1966).

[19] B. Carter, J. Math. Phys. **10**, 70 (1969).

search for Killing horizons trivial since we merely need to find the zeros of $H=\xi^a\xi_a$. We will see later that they coincide with the event horizons.

The coordinate r has been restricted to the range $0<r<\infty$ by the presence of infinite curvature at $r=0$; x has been restricted to a closed interval $x_2\leq x\leq x_1$ to ensure the proper signature. From Eq. (9), we see, therefore, that y has a lower bound depending on x, namely, $y>y_1\equiv -x_1$ at the north pole, and $y>y_2\equiv -x_2$ at the south pole (see Fig. 6). Small y corresponds to large r. The zero at $y=y_2$ we will call the "outer horizon" $\mathcal{H}_0$. It always exists whenever Σ^+ is closed. It does not completely surround the source, since it extends to $r=\infty$ at the south pole. We expect it to correspond to the null hyperplane $\bar{t}=\bar{z}$ encountered in the discussion of flat space. A typical two-surface $N^+\cap\mathcal{H}_0$ will closely resemble the paraboloid of Fig. 1.

In regions IV and V, Fig. 5, G has four roots, and therefore, we will find horizons $\mathcal{H}_1$ and $\mathcal{H}_2$. On $\Gamma_\pm$ they coalesce and throughout regions I, II, and III no inner horizons are present. In the limit of small A, this transition takes place at $e=m$ as is well known from the Reissner-Nordstrom solution, since Γ_- approaches the origin with a slope of unity. When we take the limit $e=0$, $\mathcal{H}_2$ shrinks to join the singularity at the origin. Figures 2, 7, and 8 show H plotted as a function of r and θ for the flat case, the charge-free case, and the general case, respectively. The horizons are clearly shown, as well as the peculiar behavior near the south pole.

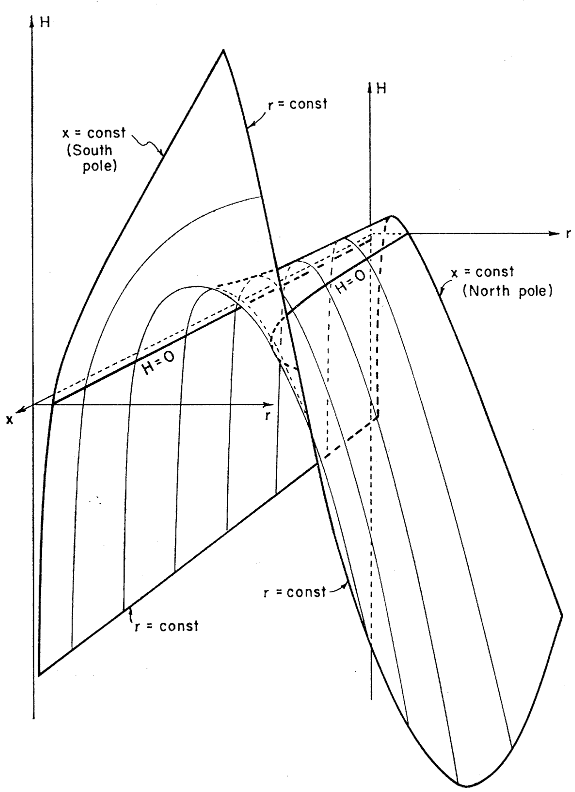

FIG. 7. Function $H(r,x)$ plotted for the case $e=0$. Surface bends down sharply near $r=0$, producing an inner "Schwarzschild" horizon.

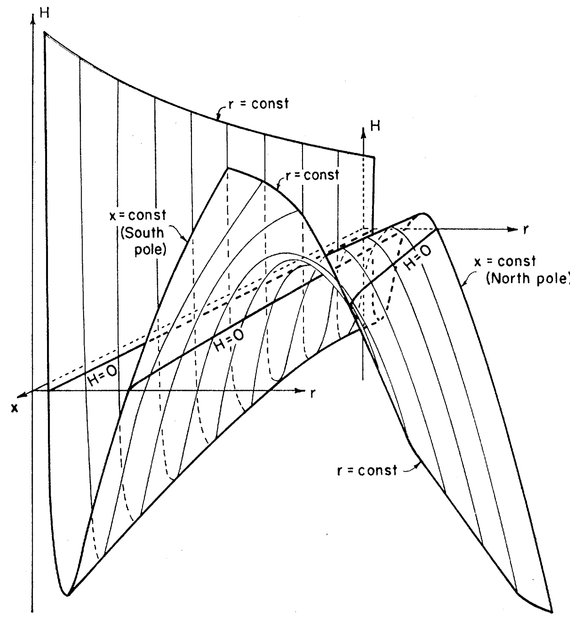

FIG. 8. Function $H(r,x)$ plotted for the charged case. Second inner horizon has appeared. Behavior of H for large r is essentially the same as in Figs. 2 and 7.

VI. MAXIMAL EXTENSION

Even when the flat-space limit of the C metric was being discussed, it was apparent that the solution was geodesically incomplete and that an extension would be necessary. The original time-symmetric coordinates of Eq. (6), being analogous to Schwarzschild coordinates, cover only the static region $\bar{z}^2>\bar{t}^2$. The advanced and retarded coordinate systems of Eqs. (10) and (15) each extend this patch across one horizon. Thus, the (u,r,x,z) coordinates covered the half-space $\bar{t}+\bar{z}>0$. By analogy with the Kruskal extension of the Schwarzschild metric, the natural remedy is to fill the remaining half-space with a second (time-reversed) copy. This means that instead of one accelerating particle we are actually dealing with two (see Fig. 9). In fact, this step has been shown to be necessary in the accelerating particle solution of classical electrodynamics, the Born solution. There the motivation was to avoid a singularity of the type $\delta(\bar{z}+\bar{t})$ as well as a violation of the field equations.[20] In order to make the field continuous across the interface, the second particle must carry an opposite charge. The possibility of a mutual interaction of these charges does *not* arise since nowhere do their fields overlap. The field of the first is retarded, while the field of the second is purely advanced. In the

[20] T. Fulton and F. Rohrlich, Ann. Phys. (N. Y.) 9, 499 (1960).

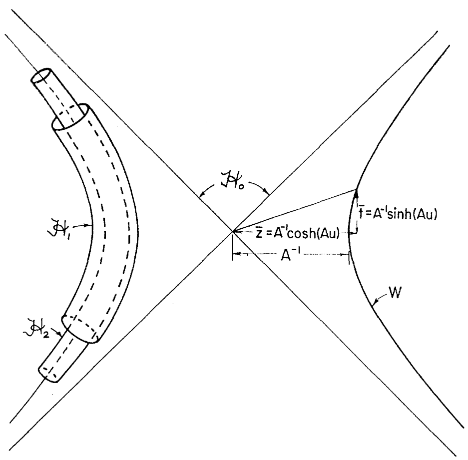

FIG. 9. Qualitative picture of the charged C metric extended to include both uniformly accelerating worldlines. Outer horizons $\mathcal{H}_0$ have the topology of two intersecting planes, while the inner horizons $\mathcal{H}_1$ and $\mathcal{H}_2$ completely surround either particle.

general relativistic solution, we will also need to discuss extensions across the inner horizons, and for this reason we now turn to a less intuitive approach.

We wish to perform the u and v extensions simultaneously. Consider again the principal null congruences l^a and n^a. Together they generate a family of "principal timelike two-surfaces" $T(x,z)$ given by $x=\text{const}$, $z=\text{const}$. In fact, $T(x,z)$ happen to be the orthogonal trajectories of the spacelike two-surfaces $\Sigma^0(u,v)$. We may seek an extension of T using the block-diagram technique.[21] This is possible in spite of the fact that T is not totally geodesic. The only geodesics which remain in T are the two families of null generators.

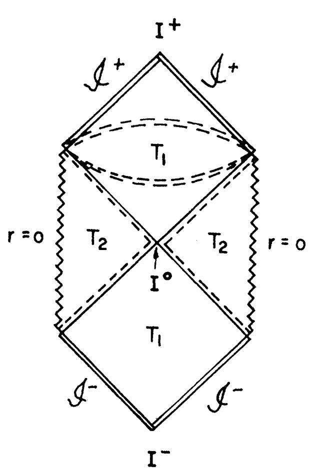

FIG. 10. Block diagram showing the extension of a solution lying in region I of Fig. 5. T_1 and T_2 are the two distinct types of coordinate patches (conformally compactified). Null boundaries at conformal infinity are represented by double lines, while the zigzag lines represent the infinite curvature encountered at $r=0$.

[21] M. Walker, J. Math. Phys. (to be published).

The horizons intersect T in null lines, dividing it into a collection of well-behaved coordinate patches or "blocks." The blocks are glued together along their horizon boundaries in all possible ways, keeping the timelike coordinate running vertically in each block and ensuring that Ψ_2 is smooth[22] across each horizon. The results of this procedure applied to the C metric are shown in Figs. 10–12 for the cases in which $G(x)$ has two, three, or four roots.

The most peculiar aspect of the diagrams is the appearance of the boundary at conformal infinity.[23] The various T surfaces differ somewhat and this requires some explanation. The timelike boundary points I^+ and I^- appear only on the T surface $x=x_1$. As we cross from one T to the next letting x decrease, conformal infinity bows down more and more. Finally at $x=x_2$, the spacelike boundary I^0 appears and the blocks become completely severed. This behavior is more closely related to the surfaces we have chosen to extend than the space-time itself. To understand what is happening, it is helpful to examine the T surfaces in the flat-space limit, where they become surfaces of constant φ and θ. From Eq. (21), $T(\theta,\varphi)$ is given by

$$\bar{t}^2-\bar{z}^2-[\bar{x}\sec\varphi+A^{-1}\cot\theta]^2=-A^{-2}\csc^2\theta. \quad (32)$$

This is the equation of a timelike elliptical hyperboloid in the three-space (t,z,x). It is a ruled surface generated by two families of null straight lines (the l^a and n^a congruences) and contains both branches of the hyperbola $\bar{t}^2-\bar{z}^2=-A^{-2}$, $\bar{x}=0$. For $\theta=0$ or π, T reduces to either the north pole or south pole region of the $(\bar{t},\bar{z})$ plane. With the exception of these two, all the surfaces become null surfaces as $\bar{t}\to\infty$. To reach the timelike future, e.g., along a path $\bar{x}$, $\bar{y}$, $\bar{z}=\text{const}$, one must cross all the T surfaces and eventually approach the north pole, which explains the appearance of I^+ there.

Extensions such as these are not unique since there is often great freedom in choosing the global topology.

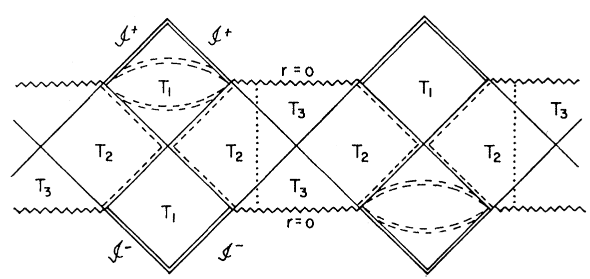

FIG. 11. Block diagram extension of the vacuum C metric ($e=0$). In this case, G has three roots and three different blocks are necessary: T_1, T_2, and T_3. Singularity is now in a region where both Killing vectors are spacelike.

[22] Most extensions which have been studied (including the present ones) are analytic. Analyticity is, however, a much more severe restriction than necessary. Compare H. Bondi, in *Lectures on GRT*, edited by S. Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, N. J., 1965), Vol. I, p. 420.

[23] R. Penrose, Proc. Roy. Soc. (London) **A284**, 159 (1965).

Usually one selects the simply-connected topology of the universal covering space as the natural choice. However, when identifications can be made in the manifold without destroying causality, it may be preferable to do so. In Fig. 13, we have wrapped up Fig. 11 by identifying all of the asymptotic regions. This amounts to regarding the two accelerating particles as opposite mouths of the same wormhole.

In certain regions, $\Sigma^0(u,v)$ may be trapped surfaces.[24] The trapping occurs because r decreases along all future-directed null geodesics orthogonal to Σ^0, leading eventually to the singularity at $r=0$. Trapped surfaces exist inside the inner horizon in the case with three roots and between the two inner horizons in case there are four roots.

For the C metric, it is rather easy to obtain the integrated form of all null geodesics. Let k^a be an affinely parametrized tangent vector to a null geodesic. First integrals[25] result from the presence of two Killing vectors, and also from the conformal Killing tensor $\Psi_2^{-2/3}l^{(a}n^{b)}$ which exists for all type-{22} solutions.[26] In the coordinates (u,r,x,z), we find the solution

$$k^a=(H^{-1}(E+R),\ -R-AP,\ r^{-2}P, G^{-1}J_z)\,, \qquad (33)$$

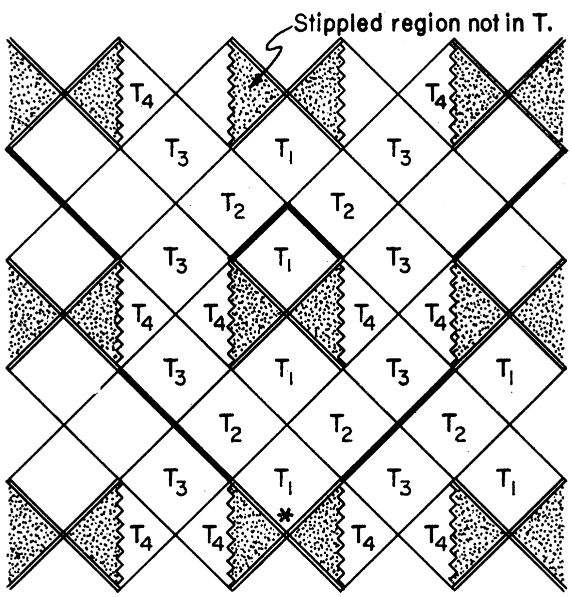

FIG. 12. When G has four roots, the extension is considerably more complex. An observer starting at the asterisk could reach any point in the region bordered by heavy lines but no others. The extension we show is not simply connected. The universal covering space would require an infinite number of sheets (and hence an infinite number of journal pages to draw).

[24] R. Penrose, Phys. Rev. Letters **14**, 57 (1965).

[25] First integrals of the null geodesic equations may also be obtained using the fact that the metric, Eq. (6), is conformally decomposable [see J. A. Schouten, *Ricci Calculus* (Springer, Berlin, 1954), p. 287], and that null geodesics are conformally invariant. The theorem of Ref. 26 has not been proved in the Einstein-Maxwell case but is probably still true.

[26] M. Walker and R. Penrose, Commun. Math. Phys. (to be published).

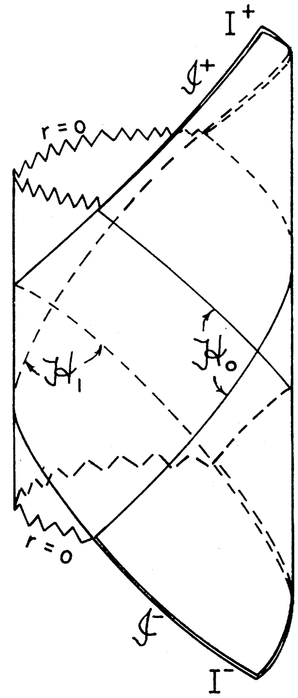

FIG. 13. Here the dashed lines of Fig. 11 have been identified and the surface has been wrapped around like a cylinder. Topology is not $S^2\times S^1\times R$, however, since on the nodal surface $\mathcal{J}$ pinches in to coincide with $\mathcal{H}_0$ and severs the diagram.

where

$$R=\pm(E^2-r^{-2}HJ^2)^{1/2},$$
$$P=\pm(GJ^2-J_z^2)^{1/2},$$

and E, J_z, and J are constants of the motion. The constants E and J_z are associated with the two Killing vectors, while J comes from the conformal Killing tensor. Physically they are the energy and angular momentum of a test particle traveling along the geodesic. When $J=J_z=0$, the geodesic lies entirely in T.

Equation (33) may be used to demonstrate that none of the null geodesics in our extended solutions can be further extended. There are just three ways in which a null geodesic can leave its coordinate patch within a finite parameter length:

(a) One of the components of k^a may become infinite. This cannot happen to the z component because P must be real, and $G\to 0$ requires $J_z=0$. The u component becomes infinite when $H=0$, and then the geodesic simply passes on to the next block.

(b) The geodesic may strike the singularity at $r=0$. The reality of both P and R in this case requires $J=J_z=0$. Hence such a geodesic must actually lie in T. At $r=0$, infinite curvature is encountered, and although a geodesic of this type will be incomplete, it will also be inextendable.

(c) The geodesic may try to depart across one of the end points $x=x_1$ or x_2. Again this can happen only if $J_z=0$. Such a geodesic must be continued as an incoming geodesic at an angle φ' which is diametrically opposite: $\varphi'=\varphi\pm\pi$. This is analogous to geodesics on a two-sphere which cross the pole. The presence of the node gives no trouble whatsoever in this respect.

Thus, we see that the only incomplete null geodesics in our extensions are inextendable and lie in T. To finish the proof that we have found maximal extensions, two further steps are necessary. One should check the behavior of timelike and spacelike geodesics, since in some cases the completion of null geodesics alone does not suffice.[27] Also, one should exhibit Kruskal-type coordinates which cover the corners where four blocks come together. We have not confirmed these points but have no reason to expect any difficulty in doing so.

VII. RADIATION

In the naive sense, we say that gravitational radiation is present whenever the curvature falls off asymptotically as r^{-1}. In order to give this idea precise meaning, we must specify in an invariant way both the coordinate system to be used and the tetrad basis in which the curvature components are to be calculated. A suitable choice in these matters has been given by Newman and Unti[28] (NU), to whom we refer the reader for further details. NU coordinates can be introduced in any space-time which is asymptotically flat: they are unique up to one of the BMS (Bondi-Metzner-Sachs) transformations. However, since they are given as an expansion in powers of r^{-1}, one should bear in mind that they will usually be applicable only outside some radius of convergence $r=r_0$.

The coordinate transformation we use will be of the form

$$\begin{aligned}\bar{u}&=U+O(r^{-1}), \quad \bar{r}=Rr+O(1),\\ \bar{x}&=X+O(r^{-1}), \quad \bar{\varphi}=\varphi,\end{aligned} \tag{34}$$

where U, R, and X are functions of u and x alone, and $\bar{u}$, $\bar{r}$, $\bar{x}$, and $\bar{\varphi}$ are the NU coordinates. We require that the components of the transformed metric obey the following NU conditions:

$$\begin{aligned}&g^{\bar{u}\bar{r}}=1+O(r^{-1}),\\ &g^{\bar{r}\bar{r}}=O(1),\\ &g^{\bar{r}\bar{x}}=O(r^{-1}),\\ &g^{\bar{x}\bar{x}}=r^{-2}R^{-2}(1-X^2)+O(r^{-3}),\\ &g^{\bar{\varphi}\bar{\varphi}}=r^{-2}R^{-2}(1-X^2)^{-1}+O(r^{-3}).\end{aligned} \tag{35}$$

Thus, the transformation has two tasks to accomplish: It must remove the $O(r)$ term in g^{rr} which is symptomatic of acceleration, and it must rescale the radial coordinate such that the $\bar{u}=$const, $\bar{r}=$const surfaces asymptotically become metric two-spheres.[29] Note that these surfaces are quite distinct from the surfaces Σ discussed earlier.

When the transformation is applied to the C metric, Eqs. (35) imply

$$\begin{aligned}&U_u+AGU_x=R^{-1}, \quad R_u+AGR_x=\tfrac{1}{2}AG_xR,\\ &X_u+AGX_x=0, \qquad R^2(1-X^2)=\kappa^{-2}G,\\ &\qquad R^{-2}(1-X^2)=GX_x^{\,2},\end{aligned} \tag{36}$$

where the subscripts are used to denote partial derivatives. These equations have the general solution

$$\begin{aligned}U&=(\kappa A^{-1}\operatorname{sech}\chi)\int_0^x G^{-3/2}dx+\alpha(\chi),\\ R&=\kappa^{-1}G^{1/2}\cosh\chi, \quad X=-\tanh\chi,\end{aligned} \tag{37}$$

where

$$\chi=\kappa Au-\kappa\int_0^x G^{-1}dx+C. \tag{38}$$

C is an arbitrary constant, and $\alpha(\chi)$ is an arbitrary function representing the BMS freedom. We choose α to make $U=0$ coincide with $u=0$, and set $C=0$.

The curvature components must also be referred to a new tetrad of basis vectors. The NU basis vectors $\bar{l}^a$, $\bar{n}^a$ are related to $\bar{u}$ and $\bar{r}$ in the same ways that l^a, n^a are related to u and r [see Eq. (12)]. In particular, we need the fact that

$$\bar{n}^a=\partial x^a/\partial\bar{u}+\tfrac{1}{2}\partial x^a/\partial\bar{r}+O(r^{-1}). \tag{39}$$

The rotation required from l^a, n^a to $\bar{l}^a$, $\bar{n}^a$ will be parametrized as the resultant of three successive operations: a null rotation about l^a, followed by a Lorentz boost in the l^a, n^a plane, followed by a null rotation about n^a. If the usual parameters[30] which describe these steps are a, λ, and b, respectively, then

$$\bar{n}^a l_a=\lambda^{-1}, \quad \bar{n}^a n_a=\lambda^{-1}a^2. \tag{40}$$

Furthermore, from the known way[30] in which Ψ_A and Φ_A transform under such rotations,

$$\bar{\Psi}_4=6a^2\lambda^{-2}\Psi_2, \quad \bar{\Phi}_2=2a\lambda^{-1}\Phi_1. \tag{41}$$

Combining Eqs. (34), (37), (39), and (40) with Eqs. (12) and (13), we can solve for the parameters

$$\begin{aligned}a&=\tfrac{1}{2}\sqrt{2}\kappa AR(1-X^2)^{1/2}r+O(1),\\ \lambda^{-1}&=R+O(r^{-1}).\end{aligned} \tag{42}$$

(The value of b may also be calculated but is apparently irrelevant for obtaining the radiation.) Equations (41)

[27] R. Geroch, thesis, Princeton University, 1966 (unpublished).

[28] E. Newman and T. Unti, J. Math. Phys. 3, 891 (1962).

[29] More precisely, we require that the conformally related metric $(\bar{r})^{-2}ds^2$ reduce to that of a unit sphere when $(\bar{r})^{-1}=0$. This is most easily seen from Eq. (6). If $(\bar{r})^{-1}=\kappa A(x+y)G^{-1/2}\operatorname{sech}(\kappa t)$ and $\sin\theta=\operatorname{sech}(\kappa t)$, then in the limit as $(\bar{r})^{-1}\to 0$, $(\bar{r})^{-2}$ times the metric of Eq. (6) becomes $-(d\theta^2+\sin^2\theta\, d\varphi^2)$. Since $t=Au-\int G^{-1}dx$ (at infinity) from Eq. (9), this approach gives $R=\kappa^{-1}G^{1/2}\cosh\chi$, in complete agreement with Eqs. (37) and (38). It also shows that the null hypersurface at conformal infinity has the correct form for an asymptotically flat space-time (see Ref. 23).

[30] A. I. Janis and E. T. Newman, J. Math. Phys. 6, 902 (1965).

and (42) imply

$$\bar{\Psi}_4=3\kappa^2A^2R^5(1-X^2)(m+2e^2Ax)\bar{r}^{-1}+O(\bar{r}^{-2}),$$
$$\bar{\Phi}_2=\kappa AR^3(1-X^2)^{1/2}e\bar{r}^{-1}+O(\bar{r}^{-2}), \tag{43}$$

demonstrating that both gravitational and electromagnetic radiation are present.

To get a better understanding of the radiation, one would like to have Eq. (43) expressed entirely in terms of the unaccelerated coordinates $\bar{u}$, $\bar{r}$, and $\bar{x}$. If possible, one would also like to perform a resolution into the various multipole contributions. It does not appear feasible to carry out either of these steps in closed form as long as we insist on working in the exact theory. In the framework of linearized gravitational theory, the situation is much simpler, and we now turn in that direction.

To obtain $\bar{\Psi}_4$ and $\bar{\Phi}_2$ correct to first order in mA and eA, it is sufficient to perform the coordinate transformation in zeroth order (i.e., flat space). There, NU coordinates are merely the familiar null retarded coordinates based upon a straight worldline. Given the relation between (u, r, x,φ) and Minkowski coordinates as in Eq. (21), it is quite easy to verify Eq. (37) in the flat-space limit and to show moreover that

$$R=(1+2AUX+A^2U^2)^{-1/2}. \tag{44}$$

Thus, dropping the bars, Eq. (43) simplifies to

$$\Psi_4=3mA^2(1-X^2)(1+2AUX+A^2U^2)^{-5/2}r^{-1},$$
$$\Phi_2=eA(1-X^2)^{1/2}(1+2AUX+A^2U^2)^{-3/2}r^{-1}. \tag{45}$$

[Note that e contributes to Ψ_4 in Eq. (43) even when $m=0$, but that this occurs only in second order.] The form of these quantities for a general retarded multipole is[30]

$$\Psi_4=\frac{2^l(l-2)!}{(2l)!}P_l^2(X)\left(\frac{\partial}{\partial U}\right)^{l+2}a_l(U)r^{-1},$$
$$\Phi_2=-2^l\frac{(l-1)!}{(2l)!}P_l^1(X)\left(\frac{\partial}{\partial U}\right)^{l+1}b_l(U)r^{-1}, \tag{46}$$

where a_l and b_l are the multipole moments. The resolution is obtained immediately when we realize that the expressions in Eq. (45) are just the generating functions for $P_l^m(X)$:

$$h_m(c,X)=\frac{(2m-1)!!(-c)^m(1-X^2)^{m/2}}{(1-2cX+c^2)^{(2m+1)/2}}$$
$$=\sum_{l=m}^{\infty}c^lP_l^m(X). \tag{47}$$

The form of the expansion depends on the size of AU. For $|AU|<1$,

$$\Psi_4=\sum mA^2(-AU)^{l-2}P_l^2(X),$$
$$\Phi_2=\sum eA(-AU)^{l-1}P_l^1(X). \tag{48}$$

A comparison with Eq. (46) shows that a_l, b_l must be polynomials in u of order $2l$. If we conjecture that all the integration constants are zero, the result is

$$a_l=(-2)^{-l}mA^lU^{2l},$$
$$b_l=-(-2)^{-l}eA^lU^{2l}. \tag{49}$$

This conjecture is confirmed by a lengthier calculation of Ψ_0 and Φ_0.

Solutions of this form occupy a very special position in linearized theory. A general linearized field is a superposition of advanced and retarded waves. In some cases, the distinction is irrelevant. Any static solution, for example, may equally well be regarded as an advanced or a retarded field. We will call such solutions "hermaphroditic." If we ask what is the condition that a linearized multipole field be hermaphroditic, the answer is that the moment must be a polynomial in U of order $2l$ or less. Note that this is quite a different requirement than that of time symmetry. The oddest property of hermaphroditic solutions is that they obey both the incoming and outgoing Sommerfeld radiation conditions, yet both incoming and outgoing radiation may be present.

Another apparent paradox is that Eq. (49) can be shown to hold for $l=0$ and 1 as well. The reason this is at all strange is that in the usual linearized analysis[30] the gravitational dipole moment can only be linear in U, not quadratic as it is here. The disparity lies in the fact that our NU metric will be singular even though the Ψ_A's are not. The shear σ^0, which appears at $O(r^{-3})$ in $g^{\bar{x}\bar{x}}$ and $g^{\bar{\varphi}\bar{\varphi}}$, contains a dipole term[31]

$$\sigma^0=mA^2U(1+2\cos\theta)\tan^2(\tfrac{1}{2}\theta). \tag{50}$$

Hence both it and the news function $\dot{\sigma}^0$ have poles at $\theta=\pi$, contrary to the usual assumptions. One might suppose this behavior is due in some way to the node; yet in the linearized theory, one is working in flat space and the node is absent.

For $|AU|>1$, one can show from Eq. (45) that all moments are proportional to U^{-1}. In this domain the solution is not hermaphroditic.

VIII. DISCUSSION

The evidence we have presented shows that the charged C metric is the general relativistic analog of the Born solution[32] encountered in classical electrodynamics. Most of the difficulties of interpretation are inherited from this classical solution rather than from relativity. As we have seen, they largely persist in the linearized C metric in which the fields are merely painted onto a flat space-time.

Uniformly accelerated motion has sometimes been confused with "runaway" motion, and we wish to

[31] Dipole here means that the term is a solution to Legendres' equation for $l=1$, $m=2$.

[32] T. Fulton and F. Rohrlich, Ann. Phys. (N. Y.) 9, 499 (1960).

point out that our solution is not an example of the latter. For a runaway solution, the applied external force is zero, whereas in uniformly accelerated motion it is the self-force which vanishes. The self-force, according to Dirac's analysis[33] is a Lorentz force which arises from the particle's interaction with its own nonsingular advanced-minus-retarded field. For hermaphroditic solutions, this difference is zero, and therefore, so is the self-force. Thus, we expect an external mechanical force "$F=mA$" will be needed to accelerate the inertial mass of our particle. By "external force" of course we mean any force which does not arise from interaction with either the electromagnetic or gravitational fields. In our solution the manifestation of the force is a nodal timelike two-surface. It is similar to the "struts" or lines of stress which have been invoked in other Weyl solutions.

Our solution exhibits a clear resemblance to the accelerating solutions of Bonnor and Swaminarayan[34] (BS). We would like, however, to emphasize as well the differences between them. The metrics they present have two pairs of particles rather than one, although BS point out that this feature is not essential. What is essential, we feel, is the meaning of the term "point particle." Whereas our singularities in every way resemble the Schwarzschild singularity, the BS singularities are of the type first studied by Curzon.[35] This implies that they are not surrounded by horizons and that rather than being monopole particles they have a complex multipole structure. The presence of four particles corresponds to the way in which the nonaccelerating Curzon metric can be made to accommodate several "particles" strung along the symmetry axis and held apart by struts. A further point of difference is that the BS solutions are of type $\{1111\}$, whereas ours is of type $\{22\}$. One is tempted to conjecture from this parallel that there would also exist Curzon-like analogs to the other type-$\{22\}$ solutions, in particular Kerr-NUT space.[36]

[33] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
[34] W. B. Bonnor and N. S. Swaminarayan, Z. Physik **177**, 240 (1964).
[35] H. E. J. Curzon, Proc. Math. Soc. (London) **23**, xxix (1924); **23**, 477 (1925).
[36] M. Demiański and E. T. Newman, Bull. Acad. Polon. Sci. **14**, 653 (1966).

Bičák[37] has derived the radiation pattern for the BS solution in the limit in which the particle pairs are quite close. His results and ours are again strikingly similar. In his multipole analysis, however, the quadrupole term is absent. The particles investigated by Bičák have equal and *opposite* masses, so we might well expect a cancellation of this sort to occur.

None of the BS solutions are simultaneously free of both nodes and negative mass. We do find several node-free cases, as mentioned in Sec. IV. The ones for $mA=0$ we qualitatively understand since no external force is necessary in this case. The ones for which $mA=eA > (12)^{-1/2}$ we do not understand. We suspect that for such strong accelerations the particle interpretation somehow breaks down.

The charged C metric does provide an interesting counterexample to the widely held view that the radiative Robsinson-Trautman (RT) solutions cannot represent the field of a bounded source. There is no doubt that the type-$\{4\}$ RT solutions must have angular singularities in the sense that Ψ_4 has poles. For RT solutions of other types, the situation is not as clear. We have exhibited a three-parameter family of charged RT solutions in which the singularity is much milder and several cases in which the radiation is accompanied by *no angular singularity at all.*

We plan to discuss further the asymptotic properties of the C metric using the conformal techniques of Penrose, as well as analyze some of its interesting singular limits in a future paper.

We conclude with the remark that our success with the C metric leads us to hope that the remaining uninterpreted type-$\{22\}$ solutions[38] may yield to a similar analysis, thereby leading to a deeper understanding of physical implications of general relativity.

Note Added in Proof. The C metric is also a counterexample to the claim that all algebraically special fields have zero Newman-Penrose constants.

ACKNOWLEDGMENTS

We would like to extend our thanks to Professor Roger Penrose and Dr. M. Demiański and Dr. E. Couch for very helpful private discussions.

[37] J. Bičák, Proc. Roy. Soc. (London) **A302**, 201 (1968).
[38] W. Kinnersley, J. Math. Phys. **10**, 1195 (1969).