Unitarity Effects in the Veneziano Model

STEPHEN HUMBLE
Rutherford High Energy Laboratory, Chilton, Didcot, Berkshire, England
(Received 25 May 1970)

It is assumed that the Veneziano model in some ways represents a good approximation to the scattering amplitude and as such already contains the effects of numerous coupled, inelastic channels. This hypothesis is used to calculate the absorption parameters for pion-pion scattering below 2 GeV using a partial-wave dispersion relation, where some simple forms for the phase shifts are assumed. The results are compared with other analyses and extended to the case of pion-kaon scattering. Some implications of these results are discussed, and the problem of the presence or absence of the Pomeranchukon is outlined.

I. INTRODUCTION

SINCE the introduction of the Veneziano model,[1] numerous attempts have been made to produce a modified version which satisfies unitarity.[2–5] The result of this work has been various unitarizing schemes which apparently give good agreement with the data. However, none of the methods so far proposed are free from one or more serious failings such as violation of crossing symmetry or analyticity, the appearance of infinities, and frequently even the inability to satisfy unitarity.

In view of these shortcomings it is worthwhile stressing that the unitarization procedures are largely independent of the Veneziano amplitude and would unitarize any suitable function of two variables. Their ability to satisfy the data therefore must lie in the fact that the physical scattering amplitude (which contains all unitarity corrections) apparently has nearly linearly rising trajectories and fairly narrow resonances, just as suggested by the Veneziano model. The unitarization schemes are successful not because they change the model so much, but because they change it so little. Thus except perhaps for the Pomeranchukon we may suppose that the Veneziano amplitude is itself a good approximation to the scattering amplitude and, as such, it already contains the effects of all coupled processes both in the direct and crossed channels. The question we might then ask is not how to unitarize this model but what unitarity, or inelastic, effects are already contained in it.

The purpose of this paper is to extend the predictive power of this representation in just this manner, and to estimate the inelastic effects which are inherent in the Veneziano amplitude for pion-pion scattering. This particular process is considered partly because of its simplicity, with equal masses and no spin, but also because of its implications for theoretical and phenomenological studies. For instance, it is usually assumed that the condition of purely elastic unitarity holds some way above the first inelastic threshold. However, since $\pi\pi$ is an annihilation channel coupling directly to the vacuum, one would expect inelastic effects to be of the utmost importance at high energies. It would be desirable therefore to be able to calculate such effects and to estimate where they might start to be dominant. To do this in terms of the Veneziano model, we must first define what we mean by its being a good approximation to the scattering amplitude. Clearly in the physical region the model cannot reproduce the detailed analytic structure, since it contains no branch-point singularities, only poles. However, away from this region the over-all effect of these poles could be very reasonable.[6] Starting from this assumption, we can calculate the low-energy inelastic effects using a partial-wave dispersion relation, where the left-hand cut contribution represents the general effect of the crossed-channel reactions and as such should be well approximated by the Veneziano model. In Sec. II we calculate these contributions in precisely this way and attempt to test their validity by comparison with other derivations. In Sec. III some fairly simple assumptions are made concerning the behavior of the partial-wave phase shifts, and the corresponding absorption parameters are determined. The results of this calculation and their implications in phenomenological studies are outlined in Sec. IV, and extended to include pion-kaon scattering. Finally in Sec. V some remarks are made concerning the presence or absence of the effects of the Pomeranchukon.

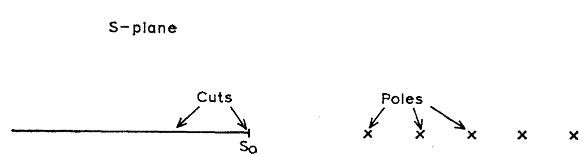

Fig. 1.

FIG. 1. s-plane singularities of the partial-wave Veneziano amplitude.

[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[2] C. Lovelace, in Proceedings of the Argonne Conference on $\pi\pi$ and πK Interactions (unpublished); CERN Report No. 1041 (unpublished).
[3] S. Humble, Nuovo Cimento Letters **2**, 541 (1969); D. G. Ravenhall and R. L. Schult, University of Illinois report, 1969 (unpublished).
[4] K. Kikhawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969).
[5] F. Arbab, Phys. Rev. **183**, 1207 (1969).

[6] The analogy with a photograph seems very apt: From a distance it looks like the real thing; only when we get close do we see the individual grains.

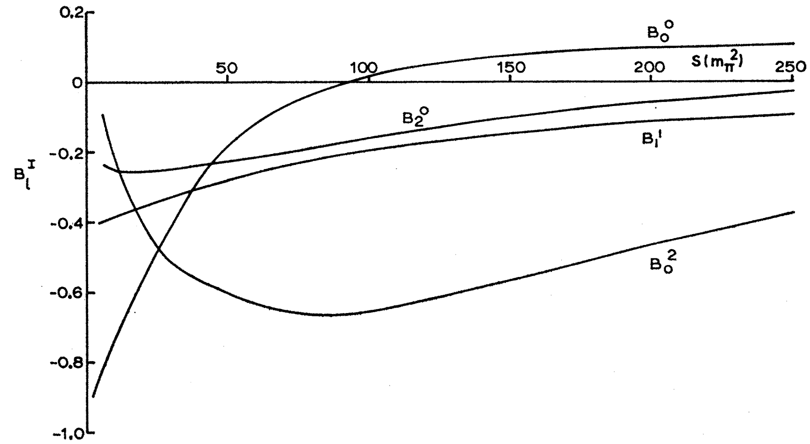

FIG. 2. Functions $B_l^I(s)$ determined from the Veneziano amplitude for $\pi\pi$ scattering.

II. CONTRIBUTION TO LEFT-HAND CUT

The Veneziano amplitude for pion-pion scattering with isospin I can be written as[2]

$$V_I(s,t,u)=\alpha_I B(s,t)+\beta_I B(s,u)+\gamma_I B(t,u)\,, \qquad (1)$$

where

$$\alpha_I=-f^2\begin{bmatrix}3\\2\\0\end{bmatrix},\quad \beta_I=-f^2\begin{bmatrix}3\\-2\\0\end{bmatrix},\quad \gamma_I=-f^2\begin{bmatrix}-1\\0\\2\end{bmatrix}, \qquad (2)$$

$$B(x,y)=\frac{\Gamma(1-a(x))\Gamma(1-a(y))}{\Gamma(1-a(x)-a(y))}, \qquad (3)$$

and $a(x)$ is the degenerate ρ-f_0 trajectory, assumed to be of the form

$$a(x)=a_0+a'x.$$

In general, the amplitude $V_I(s,t,u)$ has simple poles at values of the Mandelstam variables s, t, or $u>4m_\pi^2$; its partial-wave projection $V_l^I(s)$, therefore, will contain terms involving Legendre functions of the second kind, $Q_l(z_0)$, arising from the t- and u-channel poles. By considering the discontinuities of such terms it is seen that $V_l^I(s)$ has the s-plane singularities shown in Fig. 1, i.e., simple poles at $s>4m_\pi^2$ and cuts along the negative s axis. Taking the point of view that $V_l^I(s)$ is a good approximation to the partial-wave amplitude $A_l^I(s)$ at least outside the physical region, it should be possible to approximate the left-hand discontinuity of $A_l(s)$ by that of $V_l(s)$. That is, we can write

$$B_l^I\equiv\frac{1}{\pi}\int_L\frac{\mathrm{Im}A_l^I(s')}{s'-s}ds'\approx\frac{1}{\pi}\int_L\frac{\mathrm{Im}V_l^I(s')}{s'-s}ds', \qquad (4)$$

where L represents the range of the left-hand singularities. However, because of the analytic structure shown in Fig. 1, the last integral is just [$V_l^I(s)-$poles] and as such can be easily estimated at least for low values of s. Clearly it is impractical to subtract off the infinity of physical region poles associated with $V_l^I(s)$. Nevertheless, since we shall require a detailed knowledge of $B_l(s)$ only up to center-of-mass energies $\sqrt{s}\approx 2$ GeV, it is sufficient to subtract off the first ten poles. The resulting approximations to $B_l^I(s)$ are shown in Fig. 2. Subtracting off the next ten poles produces less than a 0.01% change in $B_l^I(s)$ over this range.

It is worth noting that this calculation of $B_l^I(s)$ is similar to the procedure outlined by Atkinson *et al.*[7] However, the motivation in the present paper is very different. We are not trying to impose *elastic* unitarity onto the Veneziano amplitude; rather we are attempting to deduce the effects of the full unitarity condition in the direct and crossed channels which are inherent in the assumption that the Veneziano model is a good representation of the scattering amplitude.

Since the basis of the calculation is the determination of $B_l^I(s)$, let us first compare it with some other derivations.

The left-hand cut in partial-wave amplitude arises from the singularities in the crossed channels, and we can write

$$\mathrm{disc}A_l^I(s)_{s<0}=\frac{1}{s-4}\int_4^{-(s-4)}dt'\,P_l\left(1+\frac{2t'}{s-4}\right)$$
$$\times\beta_{II'}[\mathrm{Re}A_t^{I'}(t',s)+(-1)^I\,\mathrm{Re}A_u^{I'}(u't')], \qquad (5)$$

where A_t and A_u are the absorptive parts of the amplitude in the t and u channels, respectively. The Veneziano amplitude can again be used to approximate these functions in the following ways. Phenomenologically[8] it has sometimes proved useful to lose some degree of crossing symmetry in the Veneziano model and to give the trajectories in the physical region an imaginary part, i.e.,

$$a(t)=a_0+a't+b\theta(t-4m_\pi^2)(4m_\pi^2-t)^{1/2}. \qquad (6)$$

[7] D. Atkinson *et al.*, Trieste Report No. ICTP/69/34 (unpublished).

[8] B. Peterson and N. A. Tornqvist, Nucl. Phys. **B13**, 629 (1969).

With this modification to Eqs. (1)–(3), the absorptive parts A_t and A_u in Eq. (5) can be calculated in terms of the discontinuities across the t- and u-channel singularities. Alternatively, in the respective physical regions it is possible to relate $A_t^I(t,s)$ and $A_u^I(u,t)$ to the t- and u-channel amplitudes through the unitarity condition. These functions can then be determined in the regions required for Eq. (5) by analytic continuation or by a partial-wave decomposition. In this case the unitarization procedure of Lovelace[2] is used to calculate $\mathrm{Im}A_l^I(t)$ and $\mathrm{Im}A_l^I(u)$ for $t,u>4m_\pi^2$, $l\leqslant 4$ with both $\pi\pi$ and $K\bar{K}$ intermediate states.

It is found that both of these methods provide very similar results for the discontinuity of $A_l^I(s)$ given by Eq. (5) at least for low negative values of s. In calculating $B_l^I(s)$ from Eq. (4) using these solutions, it is found that the integrals diverge, which was to be expected. However, if we introduce a cutoff which effectively parametrizes our ignorance of the shorter-range forces, values for $B_l(s)$ can be obtained which have the same shape and are within 6% of those shown in Fig. 2. We believe this provides *some* degree of corroboration for the values of $B_l^I(s)$ given by the straightforward Veneziano representation. We shall now use these functions to investigate the necessary inelastic effects in low-energy pion-pion scattering. Further corroboration for our Veneziano model approximation to $B_l^I(s)$ will be sought by a comparison of the effect with results from other calculations.

III. INELASTIC EFFECTS IN PION-PION SCATTERING

The pion-pion partial-wave amplitude $A_l^I(s)$ is normalized so that

$$A_l^I(s)=[s/(s-4m_\pi^2)]^{1/2}\times\{\eta_l^I(s)\exp[2i\delta_l^I(s)]-1\}/2i, \quad (7)$$

where δ_l^I is the phase shift and η_l^I, the absorption parameter, provides a measure of the inelastic effects present in a partial wave, i.e., $\eta=1$ corresponds to purely elastic scattering while $\eta=0$ represents complete absorption.

To determine these absorption parameters, we shall primarily consider a once-subtracted dispersion relation for the partial-wave amplitude such that

$$\mathrm{Re}A(s)=A(4)+\frac{s-4}{\pi}\int_L\frac{\mathrm{Im}A(s)ds'}{(s'-s)(s'-4)}+\frac{s-4}{\pi}P\int_4^\infty\frac{\mathrm{Im}A(s')ds'}{(s'-s)(s'-4)}, \quad (8)$$

where, for brevity, we have dropped the l, I indices, and s is written in units of (pion mass)^2. Then, using Eq. (7), this can be written as an integral equation for η in terms of $B(s)$ and $\delta(s)$:

$$\frac{1}{2}\left(\frac{s}{s-4}\right)^{1/2}\eta\sin2\delta=A(4)+B(s)-B(4)+\frac{s-4}{\pi}P\int_4^\infty\frac{1-\eta\cos2\delta}{2(s'-4)(s'-s)}\left(\frac{s'}{s'-4}\right)^{1/2}ds', \quad (9)$$

where the functions $B(s)$, given by Eq. (4), are determined from the Veneziano representation as described in Sec. II. The phase shifts, however, cannot be given directly by the Veneziano amplitude and we must use the values suggested by recent phase-shift analyses or assume some simple forms for them. For instance, it is worth noting that Lovelace's identification of the Veneziano amplitudes as K-matrix elements,[2] such that

$$A_l^I(s)\approx V_l^I(s)/[1-i\rho_l(s)V_l^I(s)]=\left(\frac{s}{s-4}\right)^{1/2}(\exp i\bar{\delta}_l^I)\sin\bar{\delta}_l^I, \quad (10)$$

where $\rho_l(s)$ is a phase-space factor,[2] produces phase shifts apparently in good agreement with experiment below the first inelastic threshold. It is plausible that these elastic model phase shifts $\bar{\delta}_l^I$ are still a fair approximation to the physical ones, δ_l^I, for some distance above this threshold even in the presence of inelasticity, although at higher energies it would seem unlikely that this would be the case.[9] To allow for this ignorance of δ at higher energies, we rewrite Eq. (9) in the form

$$\frac{1}{2}\left(\frac{s}{s-4}\right)^{1/2}\eta\sin2\delta-A(4)-B(s)+B(4)-\frac{s-4}{\pi}P\times\int_4^{\Lambda_1}\frac{1-\eta\cos2\delta}{2(s'-s)(s'-4)}\left(\frac{s'}{s'-4}\right)^{1/2}ds'=\lambda X(s),\quad s<\Lambda_1 \quad (11)$$

where

$$X(s)=\frac{s-4}{\pi}\int_{\Lambda_1}^\infty\left(\frac{s'}{s'-4}\right)^{1/2}\frac{ds'}{(s'-4)(s'-s)}$$

and

$$0<\lambda<1;$$

that is, we give the integrand in Eq. (9) λ times its maximum value above some point $\Lambda_1\approx 2$ GeV.

The values of $\eta_l^I(s)$ are now calculated from Eq. (11) by assuming that $\delta_l^I(s)$ is indeed well approximated by $\bar{\delta}_l^I(s)$ for $s<\Lambda_1$. Instead of solving this equation by matrix-inversion techniques, which build up large errors because of the discontinuity at Λ_1, a five-parameter function is assumed for η. An extensive search on the space of these parameters reveals which values of $\eta(s)$ are compatible with $\bar{\delta}(s)$ and $B(s)$. If the left-hand side of Eq. (11) lies within the limits of the right-hand

[9] Generally the phase shift $\bar{\delta}$ will increase by π as it passes through each resonance position. Analyses of meson-baryon scattering, however, suggest that phase shifts as a rule lie between $\pm2\pi$. Such evidence as there is indicates that this is likely to hold for meson-meson scattering also.

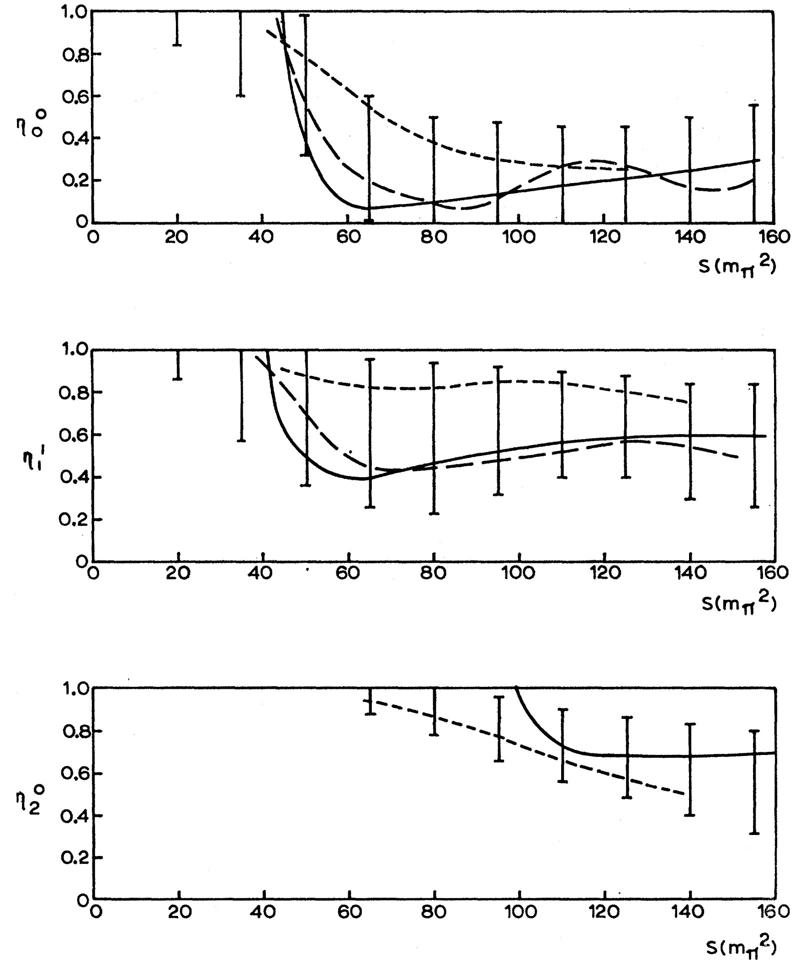

FIG. 3. $\pi\pi$ absorption parameters. The error bars are given by Eq. (11) and the solid lines represent the solutions of Eq. (15). The phenomenological values of Ref. 15 are indicated by the dotted lines and the broken lines are the values calculated from Eq. (19).

side as λ is varied from 0 to 1 for all $s<\Lambda_1$, then η is taken to be an acceptable solution. The range of these solutions is shown in Fig. 3 for $\eta_0{}^0$, $\eta_1{}^1$, and $\eta_2{}^0$. In the case of $I=2$ amplitudes, it is found that this calculation is compatible with no large inelastic effects in the region considered.

While the above analysis provides some order-of-magnitude results for the absorption parameters, it does not give any very definite values for them. However, such values can be found if we make a slightly more stringent assumption about the phase shift $\delta_l{}^I(s)$. At low energies let us suppose that we still have $\delta\approx\bar{\delta}$, but this time we shall assume that after the first resonance δ approaches π and stays close to that value thereafter. This again appears to be very plausible, by analogy with meson-baryon phase shifts, for instance. Also, from the above calculation it appears that η is close to unity for some distance above the first resonance position. If we represent this situation by

$$\left.\begin{aligned}\delta_l{}^I(s)&=\bar{\delta}_l{}^I(s)\\ \eta_l{}^I(s)&=\ 1\end{aligned}\right\},\quad s<\Lambda_2 \tag{12}$$

and

$$\delta_l{}^I(s)=\pi\,,\quad s>\Lambda_2$$

where Λ_2 is taken as $1\frac{1}{2}$ times the resonance width above the first resonance position,[10] then Eq. (9) becomes

$$\frac{s-4}{\pi}\int_{\Lambda_2}^{\infty}\frac{1-\eta(s)}{2(s'-s)(s'-4)}\left(\frac{s'}{s'-4}\right)^{1/2}+Y(s)=0\,, \tag{13}$$

where

$$Y(s)=A(4)+B(s)-B(4)+\frac{s-4}{\pi}P\int_4^{\Lambda_2}\frac{1-\cos2\delta}{(s'-s)(s'-4)}\left(\frac{s'}{s'-4}\right)^{1/2}. \tag{14}$$

This can be inverted to give

$$\eta(s)=1+\frac{2}{\pi}(s-\Lambda_2)^{1/2}\left(\frac{s-4}{s}\right)^{1/2}\times\int_{\Lambda_2}^{\infty}\frac{Y(s')ds'}{(s'-\Lambda_2)^{1/2}(s'-s)}. \tag{15}$$

[10] The phase shifts are taken to go smoothly to π at this point.

The resulting values for the absorption parameters for the $I=0$ partial waves are shown in Fig. 3. For the $I=2$ nonresonating partial waves we fix Λ_2 at about 1 GeV and put $\delta_l^2=0$ for $s>\Lambda_2$. In this case we find from Eq. (15) that the contributions to η_l^2 coming from $B(s)$ and the integral over the phase shift in Eq. (14) tend to cancel. This is contrary to the situation in the $I=0$, 1 amplitudes where these contributions reinforce one another. The resulting $I=2$ inelastic cross section therefore is found to be small in the region considered, which is perhaps not surprising if at such energies inelastic cross sections, like elastic ones, are dominated by direct-channel resonances.

The important contribution to the integral in Eq. (15) comes from the lower values of $Y(s')$. Thus the fact that $B(s)$ is only determined up to about 2 GeV is not critical. Giving $B(s)$ some asymptotic behavior such as constant or slowly decreasing produces very little change in the values of η. In fact, in the numerical calculation we let $B(s)$ tend to a constant.

Clearly the values of δ and η given by Eq. (12) represent to some extent an idealized situation. As s increases above Λ_2 the higher resonances will generally produce anticlockwise circles in the Argand diagram. However, it is probable that these circles could be quite small and δ will not vary too much from its assumed value π. Such variation as there is will produce sharp fluctuations in the absorption parameters near the resonance energies but we believe that the over-all magnitudes of η given by Eq. (15) can still be a fair approximation to the physical situation.

Finally, let us mention one further method which has been used to investigate inelastic effects in partial waves, namely, the N/D equations. Since in this section we do not wish to make any assumption about which coupled channels are important, we cannot use the multichannel approach of Bjorken[11] and instead must consider the single-channel N/D equations of Frye and Warnock.[12] Here in principle the partial-wave amplitude can be determined by a knowledge of $B(s)$ and the absorption parameter $\eta(s)$. We believe we have a good approximation for the function $B(s)$; therefore it should be possible to determine the forms of $\eta(s)$ that are compatible with the ϵ or ρ resonance parameters, for example, given by the Veneziano model.

However, as always, there is the ambiguity of CDD (Castillejo, Dalitz, and Dyson) parameters whose presence is determined by the asymptotic behavior of the phase shift. Without such parameters it seems impossible to produce either an S- or P-wave resonance with any resemblance to the ϵ or ρ for any absorption parameters which are not pathological. This is perhaps not surprising if, as we believe, a resonating phase shift tends to π rather than zero. In this case it is sufficient to introduce one CDD parameter c such that the N/D equations become[13]

$$\mathrm{Re}D(s)=1+cs-\frac{s-4}{\pi}P\int_4^\infty \frac{2\,\mathrm{Re}N(s')}{1+\eta(s')}\times\left(\frac{s'-4}{s'}\right)^{1/2}\frac{ds'}{(s'-s)(s'-4)} \quad (16)$$

and

$$\frac{2\eta(s)}{1+\eta(s)}\mathrm{Re}N(s)=\bar{B}(s)(1+cs)+\frac{1}{\pi}P\int_4^\infty \frac{2\,\mathrm{Re}N(s')}{1+\eta(s')}\times\left(\frac{s'-4}{s'}\right)^{1/2}\frac{ds'}{s'-s}\left[\bar{B}(s')-\frac{s-4}{s'-4}\bar{B}(s)\right], \quad (17)$$

where

$$\bar{B}(s)=B(s)+\frac{1}{\pi}P\int_{s_1}^\infty \frac{1-\eta(s')}{2(s'-s)}\left(\frac{s'}{s'-4}\right)^{1/2}ds'. \quad (18)$$

One parameter is also sufficient if the N/D equations are used to impose unitarity only on a finite range of the right-hand cut, $4<s<s_1$, provided that $\delta(s_1)\leqslant\pi$. With this extra degree of freedom[14] it is possible, using absorption parameters very similar to those given by Eq. (15), to produce a ρ resonance of mass 760 MeV and width 130 MeV and an ϵ resonance at 700 MeV with a width of 300 MeV. Some of the higher resonances can also be anticipated by allowing some sharp structure in the corresponding $\eta(s)$. This has the effect of producing small resonancelike circles in the Argand diagram. However, the widths, elasticities, and even the masses of these resonances are not sufficiently well known to fix this detailed structure in $\eta(s)$. This, of course, is just a restatement of the problem of assuming the behavior of $\delta(s)$ which we have discussed above.

IV. DISCUSSION OF RESULTS

In Sec. III we used the Veneziano model to calculate the low-energy absorption parameters, using partial-wave dispersion and some simple assumptions about the phase shifts. In this section, as a test of our analysis, we compare these values of the absorption parameters with those obtained by other analyses. Unfortunately, the experimental values are difficult to obtain accurately since they involve the detailed study of several partial waves from pion-production data which are themselves not too well determined. Even so, one analysis by Oh *et al.*[15] has suggested the values shown in Fig. 3. These are within the range allowed by Eq. (11) but suggest

[11] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).
[12] G. Frye and R. Warnock, Phys. Rev. **130**, 478 (1963).
[13] D. H. Lyth, University of Lancaster report (unpublished).
[14] For s waves there is an extra subtraction constant which we take from the Veneziano model.
[15] B. Y. Oh *et al.*, Phys. Rev. Letters **23**, 331 (1969).

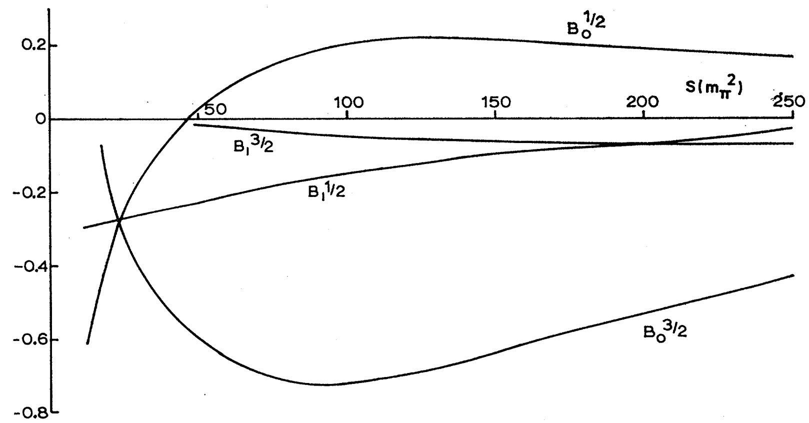

FIG. 4. Functions $B_l^I(s)$ determined from the Veneziano amplitude for πK scattering.

lower values for η_2^0 and larger values for η_0^0 and η_1^1 than those given by Eq. (15).[16]

To obtain a further comparison, we can make some estimate for these parameters by calculating the cross sections for the processes $\pi\pi\to K\bar{K}$, $\pi\pi\to\pi\omega$, and $\pi\pi\to 4\pi$ which, it is often supposed, are the more important contributions below 2 GeV. For the first two processes, we consider the exchange of the lowest-mass particles plus the contribution from direct-channel bound states and resonances,[17] with the couplings and widths given either by experiment, where known, or by the Veneziano-pole residues. The four-pion amplitude is approximated by a single-pion-exchange graph alone.[18] With the normalization corresponding to Eq. (7), the absorption parameters are related to these amplitudes by

$$1-[\eta_l^I(s)]^2=4\sum_\alpha |A_l^I(s)_{\pi\pi\to\alpha}|^2, \tag{19}$$

where $\alpha=K\bar{K}$, $\pi\omega$, or 4π as allowed by isospin. The resulting values for η_l^I are again shown in Fig. 3.

It will be noticed that these values are not too different from those obtained from Eq. (15). The oscillations in η_l^I correspond to the higher resonances which we have effectively ignored in that equation. However, we believe the results of Eq. (15) can still represent a very good over-all approximation to η_l^I. This view is further reinforced by comparison of our results with those of Griss,[19] who has used finite-energy sum rules to study Regge behavior in a "unitarized" Veneziano model. His solutions, particularly those with complex trajectories, have values for η_l^I which are similar to ours. It is also interesting to note that several of these solutions have the phase shifts tending to π as we have already assumed.

One important consequence of our results is that below 1 GeV the absorption parameters are apparently not too different from unity, so that the amplitudes satisfy fairly closely the condition of elastic unitarity. Above 1 GeV, however, this is no longer the case and it appears that inelastic effects are very important at least for the "nonexotic" channels. It should be stressed that any theoretical or phenomenological analysis of pion-pion scattering must allow for this contingency if it hopes to explain the details of this process above 1 GeV.

It is also worth noting that the above remarks are not confined to $\pi\pi$ scattering alone but apply equally well to πK scattering. To see this, we calculate the low-energy generalized Born terms $B_l^I(s)$ for the πK scattering in the manner described in Sec. II.

The Veneziano amplitude for this process is

$$\bar{V}_I(s,t,u)=\bar{\alpha}_I\bar{B}(s,t)+\bar{\beta}_I\bar{B}(u,t)\,, \tag{20}$$

where

$$\bar{\alpha}_I=-f^2\begin{pmatrix}3\\0\end{pmatrix},\quad \bar{\beta}_I=-f^2\begin{pmatrix}-1\\2\end{pmatrix}, \tag{21}$$

$$\bar{B}(x,y)=\frac{\Gamma(1-g(x))\Gamma(1-a(y))}{\Gamma(1-g(x)-a(y))}, \tag{22}$$

and $g(x)$ is the K^* trajectory.[2] The resulting functions $B_l^I(s)$ are shown in Fig. 4. The corresponding absorption parameters are obtained from assumptions similar to those leading to Eq. (15) and are given in Fig. 5. It will be seen that the parameters $\eta_l^{1/2}$, $l=0$, 1 are considerably less than unity above 1 GeV, and it would therefore seem important also to consider inelastic effects in analyses of πK scattering above this energy. However, for the $I=\frac{3}{2}$ partial waves it is found that the contributions of $\eta_l^{3/2}$ coming from $B_l^{3/2}$ and $\delta_l^{3/2}$ in Eq. (15) tend to cancel just as in the case of the $I=2$ partial waves. We would thus expect inelastic effects to be somewhat smaller for these processes, although detailed results are dependent on the assumed behavior of the phase shift.

[16] It is widely believed that the ρ' (if it exists) is very inelastic, suggesting a rather larger value for $1-\eta_1^1$ than that of Ref. 15.

[17] The details of this calculation are similar to those given by P. W. Coulter and G. L. Shaw, Phys. Rev. **138**, B1273 (1965). See also J. Fulco, G. Shaw, and D. Wong, *ibid.* **137**, B1242 (1965).

[18] The four-pion cross section is found to be relatively small.

[19] M. L. Griss, University of Illinois report, 1970 (unpublished).

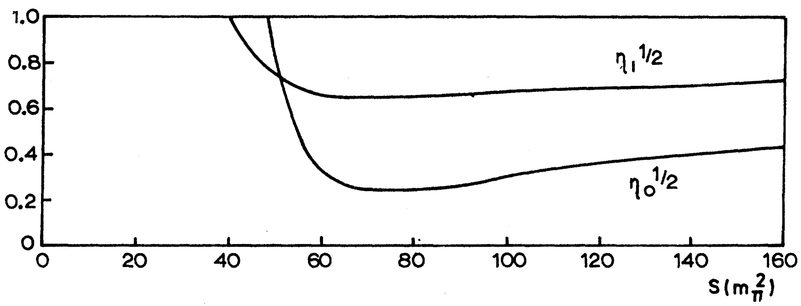

FIG. 5. πK absorption parameters determined from assumptions similar to those leading to Eq. (15).

V. CONCLUSIONS

We have investigated in this paper some of the consequences of assuming that the Veneziano model is a good approximation to the scattering amplitude, and as such already contains many of the effects of unitarity. Clearly this assumption is likely to be better away from the physical region where the model cannot reproduce the detailed analytic structure of the amplitude. Consequently we have had to rely on physical intuition to provide us with approximate values of the phase shifts in this region. Nevertheless, making only very simple assumptions for these phase shifts, the resulting absorption parameters for $\pi\pi$ and also for πK scattering turn out to be very reasonable. Indeed it should be stressed that even values $0 \leqslant \eta \leqslant 1$ were by no means assured. Moreover, although there is some apparent discrepancy with one phenomenological analysis, the shape of these parameters seems to be in fair over-all agreement with some other estimates for them.

We would suggest that the source of this agreement comes largely from the positions and residues of the poles in the Veneziano model which combine to give the contributions to the integrals over the left-hand cuts. Such contributions, however, are controlled by the unitarity condition in the crossed channels. Therefore, agreement reinforces the idea that the pole parameters in the Veneziano model already incorporate the effects of unitarity.

In conclusion let us mention one further effect which is usually associated with unitarity, namely, the Pomeranchukon. It will be seen that in our calculation of the absorption parameters we have restricted ourselves to fairly low s values, where the Pomeranchukon is likely to be negligible, and, in the calculation of $B_l^I(s)$, simple estimates suggest the Pomeranchukon contribution should be extremely small. Nevertheless it is very likely that it is the high-energy, Pomeranchukon controlled, unitarity condition which produces the almost linearly rising trajectories and corresponding residues that are approximated so well by the Veneziano representation. In this sense it is possible that the *effects* of the Pomeranchukon are also contained in the Veneziano model, even though it does not have the corresponding J-plane singularity or correct asymptotic behavior.[19] It must be noted, however, that the only way to test this possibility in the present approach would involve going to large s values and calculating numerous partial waves, each of which would require a knowledge of the phase shift and asymptotic behavior of the function $B_l^I(s)$. Thus we are unable to say at present whether the Veneziano amplitude represents an approximation to the physical scattering amplitude with or without the Pomeranchukon. Nevertheless we should emphasize that such considerations are negligible for the calculations presented in this paper.

ACKNOWLEDGMENT

The author wishes to thank Dr. D. H. Lyth for a useful discussion on the N/D method.