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Pair Production in Vacuum by an Alternating Field

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We discuss the creation of pairs of charged particles in an alternating electric field. The dependence on the frequency is computed and found negligible. We obtain a formula for the field intensities required in order to observe the effect $E \gtrsim m\omega_0 c/e \sinh(\hbar\omega_0/4mc^2)$.

I. INTRODUCTION

SOME experimentalists working on intense optical lasers have raised the question of testing nonlinear effects of vacuum quantum electrodynamics. In this article we investigate the possibility of observing the creation of pairs of charged particles (electrons and positrons) in oscillating electric fields. One might think of the collective effects of millions ($\sim 2mc^2/\hbar\omega_0$) of photons concentrated in a small volume and materializing their energy. On the other hand, Schwinger[1] long ago computed the effect in a pure static field. It turns out that the estimates based on his calculation are totally adequate, and lead to the present inobservability of that phenomenon. The only possibility would be to increase the available maximal fields by four orders of magnitude. While this rules out the observation of the absorptive aspects of nonlinearities, it might be interesting in the future to look for dispersive effects.

In order to reach the aforementioned conclusion, it is necessary to estimate how much the production rate depends on the frequency ω_0 of the field. This appears also as a challenging theoretical exercise, since it will allow us to describe the transition between two extreme domains. The first corresponds to a vanishing frequency and, as we shall discuss below, yields a singular expression for the rate proportional to $e^{-E_0/E}$. The second is the case of very low intensities where the field induces a weak perturbation in the vacuum state. As usual in such a circumstance, we foresee that the response can be expanded in powers of the perturbation. It is clear that only very high powers will come into play, since a number of the order of $2mc^2/\hbar\omega_0$ quanta is required to create a pair. Hence one expects a rate of the order of $(E/E_1)^{4mc^2/\hbar\omega_0}$.

The magnitude of the fields E_0 and E_1 appearing in the above expressions can be understood as follows. In a static situation, the work of the field on a typical distance of the problem (in our case the Compton wavelength of the electron, $\hbar/mc$) and on a unit charge e should provide the energy $2mc^2$; hence $eE\hbar/mc \sim mc^2$ or $E/E_0 \sim 1$, with $E_0 = m^2c^3/e\hbar$. At the opposite end in perturbation theory, we expand in powers of the vector potential E/ω_0, times the coupling constant e, divided by a momentum coming from the electron propagator $\sim mc$. Thus $E_1 \sim \omega_0 mc/e$. Let us add a remark on the singular behavior for the static case. In the zero-frequency limit, the $e^{-E_0/E}$ behavior of the rate can be understood as a quantum-mechanical barrier effect. This is analogous to ionization, where for an electron with binding energy V_0 the rate is approximately given by the square of the wave function at the exit of the potential barrier. This can be roughly estimated as

$$\exp\left(-2\int_0^{V_0/eE} dx[2m(V_0-eEx)]^{1/2}\right) = \exp\left(-\tfrac{4}{3}(2mV_0)^{1/2}\frac{V_0}{eE}\right).$$

In the present case the pairs might be thought as bound in vacuum with binding energy $V_0 \sim 2mc^2$.

We present in Sec. II a summary of the main theoretical results of Schwinger[1] needed for the sequel. They give a formal basis but require some elaboration to be used for practical estimates. Section III is devoted to the discussion of a number of simplifying approximations leading to tractable expressions. One wants to elaborate a model which retains the main features of the general case but nevertheless allows one to obtain final expressions in closed form. Instead of estimating

[1] J. Schwinger, Phys. Rev. **82**, 664 (1951); **93**, 615 (1954).

the effect for an arbitrary electromagnetic field varying in space and time, we will content ourselves with a pure electric field oscillating with a frequency ω_0. In spite of the slightly unrealistic character of this assumption, we expect that it retains the main features of a more general situation. It is, of course, well known that specific anomalies can occur; for instance, there is no pair creation in a plane-wave field.[1] Therefore, if the rate predicted by the theory were more favorable, one should pay more attention to the particular geometrical characterization. At the present stage, however, we present rather an order-of-magnitude estimate.

On the other hand, we are confronted with a nontrivial mathematical problem. We observe that under our assumptions we have to solve a slightly unfamiliar scattering problem in which the role of the usual configuration variable is played by time. This comes about as follows. We have to study the time evolution of a system where a pair is created. Now an antiparticle can be thought of as a wave packet moving backward in time. Thus we have a scattering wave which for large positive time has only positive frequencies, while for t large and negative it has some negative-frequency amplitude. As shown below, it is precisely this backward amplitude that we must try to find. Conditions are such that the quasiclassical approximation is valid: We have a rapidly oscillating phase with a frequency of the order of the energy gap as compared to a slowly varying amplitude, the rate of variation of which is of the order of eE/mc. The dimensionless ratio of these two quantities, $eE\hbar/m^2c^3$, is assumed to be small compared to unity. This justifies the use of a version of the WKB approximation which resembles some work done in the context of the ionization problem.[2] The work is now reduced to the evaluation of some tricky integrals by the steepest descent method which lead to the final formula (48). This formula yields the required smooth interpolation between the static and perturbative regimes. Finally, in Sec. IV we discuss our result and show that the effect can only be observed for field intensities $E \gtrsim E_c$, with

$$eE_c\hbar/m^2c^3 \sim \frac{\hbar\omega_0/mc^2}{\sinh(\hbar\omega_0/4mc^2)}.$$

This estimate proves that, in the foreseeable future, the frequency plays a negligible role, and that the experimentally obtainable fields are four orders of magnitude too small.

II. THEORETICAL BACKGROUND

The general theory is due to Schwinger.[1] In this section we present a summary of the necessary results. The derivations are omitted except for the case of a constant field.

[2] A. M. Perelomov, V. S. Popov, and M. V. Terent'ev, Zh. Eksperim. i Teor. Fiz. **50**, 844 (1966); **51**, 309 (1966) [Soviet Phys. JETP **23**, 924 (1966); **24**, 207 (1967)].

The pair creation rate will be expressed in terms of the solution of a one-body Klein-Gordon or Dirac equation in the presence of a source. The particle current is coupled to the vector potential $A_\mu(x)$ of the external, c-number, electromagnetic field. At time $-\infty$, the state of the system is the vacuum $|0\rangle$; no particle or field is present. Then the electromagnetic field is turned on adiabatically and switched off at large positive times. Pair creation will have occurred if the modulus of the vacuum-to-vacuum S-matrix element is smaller than unity. More precisely, if we can write

$$|\langle 0|S|0\rangle|^2 = \exp\left(-\int d^4x\, w(x)\right), \qquad (1)$$

$w(x)$ will be interpreted as the probability of pair creation per unit volume and unit time.

It is now necessary to distinguish between fermions or bosons. The final results will be indexed by F or B according to whether they refer to Dirac or Klein-Gordon particles.

A. Dirac Particles

The S-matrix element is given in standard units, $\hbar = c = 1$, by

$$S_0(A) \equiv \langle 0|S|0\rangle = \langle 0|\mathcal{T}\exp\left[-ie\int d^4x\, j(x)\cdot A(x)\right]|0\rangle, \qquad (2)$$

where the current is expressed in terms of the free Dirac field $\Psi(x)$ as

$$j^\mu(x) = \tfrac{1}{2}[\bar{\Psi}_\alpha(x), \gamma^\mu{}_{\alpha\beta}\Psi_\beta(x)], \qquad (3)$$

and $\mathcal{T}$ denotes the time-ordering operator.

The first step is to establish the formal expression

$$S_0(A) = \det(G^{-1}G_0) = \exp[\mathrm{Tr}\ln(G^{-1}G_0)], \qquad (4)$$

where the propagators G_0 and G are

$$G_0 = \frac{1}{P - m + i\epsilon}, \quad G = \frac{1}{P - eA(x) - m + i\epsilon}. \qquad (5)$$

The symbols det and Tr refer to the product of the four-dimensional Dirac space and an ∞^4-dimensional space indexed by a space-time coordinate. The canonical operators P and X satisfy the commutation rules

$$[X_\mu, P_\nu] = ig_{\mu\nu}.$$

We introduce now the scattering operators satisfying

$$T = eA + eA\frac{1}{P - m + i\epsilon}T,$$

$$\bar{T} = eA + eA\frac{1}{P - m - i\epsilon}\bar{T}, \qquad (6)$$

$$(\bar{T} = \gamma^0 T^\dagger \gamma^0)$$

and the state-density operators

$$\rho_{(\pm)}=2\pi(\boldsymbol{P}+m)\theta_{\pm}(P_0)\delta(P^2-m^2). \qquad (7)$$

Making use of unitarity and of some algebraic identities, it is shown in Ref. 1 that

$$|S_0(A)|^2=\exp[-\mathrm{Tr}\ln(1-T\rho_{(+)}\bar{T}\rho_{(-)})]. \qquad (8)$$

Therefore, the pair-creation rate is

$$w_F(x)=\mathrm{tr}(x|\ln(1-T\rho_{(+)}\bar{T}\rho_{(-)})|x), \qquad (9)$$

in which tr refers to the trace on Dirac indices alone.

B. Klein-Gordon Particles

In the case of spin-zero bosons, we note the analogous formulas:

$$S_0(A)=[\mathrm{Det}\,(G^{-1}G_0)]^{-1}=\exp[-\mathrm{Tr}\ln(G^{-1}G_0)], \quad (4')$$

where

$$G_0=\frac{1}{P^2-m^2+i\epsilon}, \quad G=\frac{1}{(P-eA)^2-m^2+i\epsilon}. \quad (5')$$

Defining now

$$V=G_0^{-1}-G^{-1}=e[A(x)\cdot P+P\cdot A(x)]-e^2A^2(x),$$

the scattering operators are given by

$$T=V+V\frac{1}{P^2-m^2+i\epsilon}T,$$

$$\bar{T}=V+V\frac{1}{P^2-m^2-i\epsilon}\bar{T}, \qquad (6')$$

$$(\bar{T}=T^{\dagger})$$

and the density of states

$$\rho_{(\pm)}=2\pi\theta_{\pm}(P_0)\delta(P^2-m^2). \qquad (7')$$

The square modulus of the amplitude is

$$|S_0(A)|^2=\exp[+\mathrm{Tr}\ln(1-T\rho_{(+)}\bar{T}\rho_{(-)})]. \qquad (8')$$

And, finally,

$$w_B(x)=-(x|\ln(1-T\rho_{(+)}\bar{T}\rho_{(-)})|x). \qquad (9')$$

Expressions (9) and (9′) are interesting since they involve on-shell matrix elements. All the singularities of the perturbation expansion are explicitly extracted and, in particular, the thresholds are in evidence. Expressions (4) and (4′), which at first sight might seem more appropriate, do not have these properties. Nevertheless, for the case of a constant field, which is an exactly soluble model,[1] we shall start from (4) or (4′). A unitary transformation will reduce the problem to a one-dimensional harmonic oscillator. For the case of an oscillating field, studied in Sec. III, we shall be interested in recovering the constant field result at a zero-frequency limit. That is why we present this calculation in some detail.

C. Constant Field

Let us first use expression (4) for fermions. It can be written

$$\ln S_0=\mathrm{Tr}\ln\left[(\boldsymbol{P}-e\boldsymbol{A}-m)\frac{1}{\boldsymbol{P}-m+i\epsilon}\right]$$

$$=\mathrm{Tr}\ln\left[(\boldsymbol{P}-e\boldsymbol{A}+m)\frac{1}{\boldsymbol{P}+m-i\epsilon}\right].$$

To obtain the last equality, use has been made of the 4×4 matrix C such that $C\gamma_\mu C^{-1}=-\gamma_\mu{}^T$. Adding the two forms of $\ln S_0$, we get

$$\ln S_0=\tfrac{1}{2}\,\mathrm{Tr}\ln\left[[(\boldsymbol{P}-e\boldsymbol{A})^2-m^2]\frac{1}{P^2-m^2+i\epsilon}\right], \quad (10)$$

and from the identity

$$\ln x-\ln y=\int_0^\infty\frac{ds}{s}(e^{isy}-e^{isx}), \qquad (11)$$

which incorporates the correct $i\epsilon$ prescription,

$$-2\,\mathrm{Re}\ln S_0=\int d^4x\,w(x)=-\mathrm{Re}\int d^4x\int_0^\infty\frac{ds}{s}$$
$$\times\mathrm{tr}(x|e^{is(P^2-m^2)}-e^{is[(\boldsymbol{P}-e\boldsymbol{A})^2-m^2]}|x). \quad (12)$$

Hence the rate $w_F(x)$ reads

$$w_F(x)=\mathrm{Re}\,\mathrm{tr}\int_0^\infty\frac{ds}{s}$$
$$\times(x|e^{is[(P-eA)^2-m^2+(e/2)\sigma\cdot F]}-e^{is(P^2-m^2)}|x), \quad (13)$$

where $\sigma\cdot F\equiv\sigma^{\mu\nu}F_{\mu\nu}$, with the usual conventions

$$\sigma^{\mu\nu}=\tfrac{1}{2}i[\gamma^\mu,\gamma^\nu], \quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu.$$

In the case of bosons, (4′) yields directly

$$\ln S_0=-\mathrm{Tr}\ln\{[(P-eA)^2-m^2](P^2-m^2+i\epsilon)^{-1}\}, \quad (10')$$

and, after some manipulations similar to the ones above,

$$w_B(x)=-2\,\mathrm{Re}\int_0^\infty\frac{ds}{s}$$
$$\times(x|e^{is[(P-eA)^2-m^2]}-e^{is(P^2-m^2)}|x). \quad (13')$$

In fact, if one wants to compute the imaginary parts, instead of the real ones, of the integrals (13) and (13′), one has to remove a logarithmic divergence at $s=0$. This is done by a subtraction at $s=0$, the logarithmic divergence being absorbed, as shown by Schwinger, into a renormalization of the fields and charges. But for the real part the calculations are straightforward.

Pair creation does not occur in a pure magnetic field. This is clear since a constant field cannot transfer energy to a charged particle. It is the acceleration due to the electric field that enables particles to leak through the $2m$ potential barrier. Hence the physically relevant case can be taken to be the one of a pure electric field. We then choose the gauge as

$$A(x)=(0,0,0,-Ex_0). \quad (14)$$

The following identities hold:

$$\mathrm{tr}\,\exp(\tfrac{1}{2}ies\sigma\cdot F)=4\cosh(seE) \quad (15)$$

and

$$(P-eA)^2=(P^0)^2-P_\perp{}^2-(P^3+eEX^0)^2$$
$$=\exp\left(-i\frac{P^0P^3}{eE}\right)[(P^0)^2-P_\perp{}^2-e^2E^2(X^0)^2]\times\exp\left(i\frac{P^0P^3}{eE}\right). \quad (16)$$

Denoting by ω and $\mathbf{p}$ the eigenvalues of P_0 and $\mathbf{P}$, respectively, we thus have

$$(x|\exp\{is[(P-eA)^2-m^2]\}|x)=(2\pi)^{-4}\int\int d\omega d\omega'\times\int d^3p\exp\left\{i\left[(\omega-\omega')\left(t+\frac{p_3}{eE}\right)-s(p_\perp{}^2+m^2)\right]\right\}\times(\omega|\exp[is(P_0{}^2-e^2E^2X_0{}^2)]|\omega'). \quad (17)$$

The integrals over p_3 and $p_\perp$ are readily done, and we find

$$(x|\exp\{is[(P-eA)^2-m^2]\}|x)=\frac{\pi eE}{is}(e^{-ism^2})\int_{-\infty}^{+\infty}\frac{d\omega}{(2\pi)^3}\times(\omega|\exp[is(P_0{}^2-e^2E^2X_0{}^2)]|\omega). \quad (18)$$

The last integral over ω is the trace of the evolution operator of a "harmonic oscillator" with a pure imaginary frequency. We compute this trace by summing over the discrete levels:

$$\int_{-\infty}^{+\infty}d\omega(\omega|\exp[is(P_0{}^2-e^2E^2X_0{}^2)]|\omega)=\sum_0^\infty\exp[-(2n+1)seE]=\frac{1}{2\sinh(seE)}.$$

Using this result in (13), we get

$$w_F=-\frac{1}{(2\pi)^2}\int_0^\infty\frac{ds}{s^2}\left[eE\coth(eEs)-\frac{1}{s}\right]\sin(sm^2), \quad (19a)$$

where the $1/s$ term is the zero-field subtraction which appears in (13). For bosons, a similar expression holds with the spin factor $4\cosh(eEs)$ of (15) replaced by -2. Hence

$$w_B=\frac{1}{2(2\pi)^2}\int_0^\infty\frac{ds}{s^2}\left[\frac{eE}{\sinh(eEs)}-\frac{1}{s}\right]\sin(sm^2). \quad (19b)$$

For this constant-field case, w_F and w_B are, as expected, x independent. The integrals (19a) and (19b) are easily performed by contour integration, and the final formulas read

$$w_F=\frac{\alpha E^2}{\pi^2}\sum_1^\infty\frac{1}{n^2}\exp\left(-\frac{n\pi m^2}{eE}\right), \quad (20a)$$

$$w_B=\frac{\alpha E^2}{2\pi^2}\sum_1^\infty\frac{(-1)^{n-1}}{n^2}\exp\left(-\frac{n\pi m^2}{eE}\right). \quad (20b)$$

We conclude this section with two remarks about Eqs. (20a) and (20b).

(i) Even with the most intense laser beams presently available, m^2/eE is a large number (see Sec. IV), and the terms with $n>1$ are exponentially small as compared with the first term. Therefore, the ratio between the rates for charged fermions or bosons is essentially a factor of 2 due to spin.

(ii) The oscillation in sign for the bosons shows that the successive terms cannot be interpreted as the probability for the creation of 1, 2, ..., n pairs.

III. ALTERNATING FIELD

The aim of this section is to find practical estimates for the rate of pair creation in an oscillating field. The smallness of two typical parameters enables one to approximate in a manageable way the previous theoretical formulas valid in an arbitrary field. We require, of course, that the final expression agrees, in the zero-frequency limit, with the constant-field result (20). On the other hand, in weak fields we should recover the predictions of perturbation theory. This means that one has to find an interpolation between a power law in E and an exponential in E^{-1}. Let us then introduce our simplifying assumptions.

First we limit ourselves to frequencies $\omega_0\ll m$. Furthermore, eE is assumed to be small compared to m^2. Thus terms in the expansion of the logarithm in (9) and (9′) with thresholds at energies corresponding to the creation of 2, 3,... pairs can be safely ignored. This is also supported by remark (i) at the end of Sec. II, where we observed that in the static case under the above assumption, only the leading term in the probability could be retained. Consequently, we replace the logarithm in the expression for the probability by its first term. Then, as we learned from Sec. II, the spins of the final pairs contribute essentially to a counting factor. Therefore, we study the creation of charged bosons. Furthermore, we limit ourselves to an oscillating field constant throughout space. We expect that the space variation of the field might produce similar effects.

They are neglected for the sake of simplicity in this order-of-magnitude calculation.

The vector potential is then chosen as

$$A(x)=(0,0,0,A(t)).$$

We assume $A(t)$ to be periodic with frequency ω_0 and adiabatically damped for large times. A straightforward application of (9′) gives for the probability per unit volume

$$\frac{dW}{dV}=\frac{1}{2\pi}\int\frac{d^3p}{(2\omega)^2}|(-\omega|T_p|\omega)|^2,\quad \omega=(\mathbf{p}^2+m^2)^{1/2}. \tag{21}$$

The T_p matrix occurring in (21) is the solution of the integral equation:

$$T_p=V_p+V_p\frac{1}{P_0{}^2-\omega^2+i\epsilon}T_p, \tag{22}$$

where

$$V_p=-2ep_3A(X_0)+e^2A^2(X_0) \tag{23}$$

and

$$[X_0,P_0]=i.$$

In these expressions, $\mathbf{p}$ appears as a parameter. The space-translational invariance has reduced the problem to a one-dimensional Lippmann-Schwinger equation. Time plays the role of the usual configuration variable and P_0 of its conjugate momentum. The matrix element $(-\omega|T_p|\omega)$ then corresponds to the backward scattering amplitude. In turn, this amplitude is related to the asymptotic behavior of the solutions of the differential equation:

$$[+d^2/dt^2+\omega^2+V_p(t)]\psi(t)=0. \tag{24}$$

This equation is simply the Klein-Gordon equation for our one-dimensional problem. We look now for the solution which behaves asymptotically as

$$\begin{aligned} &t\to-\infty, \quad \psi(t)\sim e^{-i\omega t}+be^{i\omega t}\\ &t\to+\infty, \quad \psi(t)\sim ae^{-i\omega t}. \end{aligned} \tag{25}$$

The T-matrix elements are then related to a and b by normalization factors

$$\begin{aligned} (-\omega|T_p|\omega)&=(i\omega/\pi)b,\\ (\omega|T_p|\omega)&=(i\omega/\pi)(a-1). \end{aligned} \tag{26}$$

We notice now that ω is a large energy, greater than m. This fact suggests that we look at the classical approximation. More precisely, if we define the "variable frequency"

$$\omega(t)=[\omega^2+V_p(t)]^{1/2} \equiv[m^2+p_\perp{}^2+[p_3-eA(t)]^2]^{1/2}, \tag{27}$$

the classical approximation can be applied if $\omega(t)$ varies slowly, i.e., if the dimensionless ratio $\dot\omega(t)/\omega^2(t)$ is much smaller than unity. This ratio is given by

$$\frac{\dot\omega}{\omega^2}=\frac{eE(t)[p_3-eA(t)]}{\{m^2+p_\perp{}^2+[p_3-eA(t)]^2\}^{3/2}},$$

and is bounded by

$$\left|\frac{\dot\omega}{\omega^2}\right|<\frac{eE}{m^2+p_\perp{}^2}<\frac{eE}{m^2},$$

in which E is the amplitude of the oscillating field. Therefore, the calculation will assume $eE/m^2\ll1$; all the existing laser beams satisfy this condition. The WKB method suggests that we look for a wave function of the form

$$\psi(t)=\alpha(t)e^{-i\chi(t)}+\beta(t)e^{i\chi(t)}, \tag{28}$$

with

$$\chi(t)=\int_0^t dt'\,\omega(t'). \tag{29}$$

The boundary conditions are

$$\alpha(-\infty)=1,\quad \beta(+\infty)=0. \tag{30}$$

The identification with the asymptotic behavior (25) yields

$$\begin{aligned} a&=\alpha(+\infty)\exp\left(-i\int_{-\infty}^{+\infty}dt[\omega(t)-\omega]\right),\\ b&=\beta(-\infty)\exp\left(-2i\int_{-\infty}^{+0}dt[\omega(t)-\omega]\right). \end{aligned} \tag{31}$$

In (28), $\psi(t)$ has been replaced by two unknown functions; we require that they fulfill the condition

$$\dot\alpha e^{-i\chi}+\dot\beta e^{i\chi}=0.$$

Equation (24) then gives

$$\dot\alpha e^{-i\chi}-\dot\beta e^{i\chi}=-[\dot\omega(t)/\omega(t)](\alpha e^{-i\chi}-\beta e^{i\chi}).$$

The following system is then completely equivalent to the original equation:

$$\begin{aligned} \dot\alpha&=-[\dot\omega(t)/2\omega(t)](\alpha-\beta e^{2i\chi}),\\ \dot\beta&=-[\dot\omega(t)/2\omega(t)](\beta-\alpha e^{-2i\chi}). \end{aligned} \tag{32}$$

From the adiabatic switching assumption, $\dot\omega(t)$ vanishes when $|t|\to\infty$; therefore, α and β tend to constants for large times. Furthermore, according to the previous hypothesis α and β vary slowly in time. The phase $e^{2i\chi}$ oscillates very rapidly as compared to the variation of α and β, since $\dot\chi=\omega(t)\ll|\dot\omega(t)/\omega(t)|$. As a first step, we can neglect these oscillating terms in (32). Taking into account the boundary conditions, one finds

$$\begin{aligned} \alpha^{(0)}(t)&=[\omega/\omega(t)]^{1/2},\\ \beta^{(0)}(t)&=0. \end{aligned} \tag{33}$$

In terms of the T matrix, this means

$$(\omega|T_p|\omega) =\frac{i\omega}{\pi}\left[\exp\left(-i\int_{-\infty}^{+\infty}dt\{[\omega^2+V_p(t)]^{1/2}-\omega\}\right)-1\right], \tag{34}$$

in which we recognize the familiar result of the classical approximation for forward scattering. But the other matrix element is zero. At high energy, this first approximation does not lead to any backward scattering, which translated in our terms means no pair creation. Therefore, it is necessary to go one step further. For this purpose we solve the second Eq. (32) where α is replaced by its approximate value $\alpha^{(0)}(t)$, given in (33). The solution reads

$$\beta(t)=-\int_t^\infty dt'\frac{\dot\omega(t')}{2\omega(t')}e^{-2i\chi(t')}. \qquad (35)$$

Using relation (31), we have $|b|=|\beta(-\infty)|$ and, from (26), $|(-\omega|T_p|\omega)|=(\omega/\pi)|\beta(-\infty)|$. This can be inserted in the expression for the probability [Eq. (21)]:

$$\frac{dW}{dV}=\int\frac{d^3p}{(2\pi)^3}\left|\int_{-\infty}^{+\infty}dt\frac{\dot\omega(t)}{2\omega(t)}e^{-2i\chi(t)}\right|^2. \qquad (36)$$

To convince oneself that the factors are correct, we can extract from (36) the lowest order in the field:

$$\frac{dW}{dV}=\frac{\pi\alpha}{3}\int_{2m}^\infty d\omega\left(1-\frac{4m^2}{\omega^2}\right)^{3/2}|E(\omega)|^2,$$

which agrees with the exact result from a direct calculation.[1] In order to define a probability per unit time, we assume now that the field and its vector potential essentially vanish outside the time interval $-\frac{1}{2}T$, $\frac{1}{2}T$. The pair creation rate w is then

$$w=\frac{dW}{dVdt}=\lim_{T\to\infty}\frac{1}{T}\int\frac{d^3p}{(2\pi)^3}\left|\int_{-T/2}^{T/2}dt\frac{\omega(t)}{2\omega(t)}e^{-2i\chi(t)}\right|^2. \qquad (37)$$

During the time interval $-\frac{1}{2}T$, $\frac{1}{2}T$, we take

$$A(t)=(E/\omega_0)\cos\omega_0 t. \qquad (38)$$

Thus

$$\omega(t)=[m^2+p_\perp^2+(p_3-(eE/\omega_0)\cos\omega_0 t)^2]^{1/2}$$

and

$$\chi(t)=\int_0^t dt'\omega(t')=\Omega t+\Phi(t).$$

The function $\Phi(t)$ is periodic as the field itself with frequency ω_0; and Ω is a renormalized frequency,

$$\Omega=\int_0^{2\pi}\frac{dx}{2\pi}\left[m^2+p_\perp^2+\left(p_3-\frac{eE}{\omega_0}\cos x\right)^2\right]^{1/2}.$$

Let us expand the periodic function $[\dot\omega(t)/2\omega(t)]e^{-2i\Phi(t)}$ as a Fourier series:

$$\frac{\dot\omega(t)}{2\omega(t)}e^{-2i\Phi(t)}=\sum_{-\infty}^{+\infty}c_n e^{in\omega_0 t}.$$

Using this expansion, we can proceed to the large-T limit in Eq. (37):

$$\lim_{T\to\infty}\frac{1}{T}\left|\int_{-T/2}^{T/2}\frac{\dot\omega(t)}{2\omega(t)}e^{-2i\Phi(t)}e^{-2i\Omega t}\right|^2 =2\pi\sum_n\delta(n\omega_0-2\Omega)|c_n|^2.$$

Since the ratio $\Omega/\omega_0\gtrsim m/\omega_0\gg 1$, the discrete summation over n can be replaced by an integral, and the probability per unit time and unit volume derived from (37) reads

$$w=\int\frac{d^3p}{(2\pi)^2}\sum_n\delta(n\omega_0-2\Omega)|c_n|^2 \cong\omega_0\left(\frac{eE}{2\omega_0}\right)^2\int\frac{d^3p}{(2\pi)^2}|c|^2. \qquad (39a)$$

We have set $(\frac{1}{2}eE)c_{2\Omega/\omega_0}=c$, with c given by

$$c=\int_{-\pi}^{+\pi}\frac{dx}{2\pi}\frac{\sin x\,[p_3-(eE/\omega_0)\cos x]}{m^2+p_\perp^2+[p_3-(eE/\omega_0)\cos x]^2}\times\exp\left\{\frac{2i}{\omega_0}\int_0^x dx'\left[m^2+p_\perp^2+\left(p_3-\frac{eE}{\omega_0}\cos x'\right)^2\right]^{1/2}\right\}. \qquad (39b)$$

Let us now discuss the mathematics required to extract useful information from (39).

The expression for c contains a very rapidly oscillating phase factor with frequencies of order m/ω_0. Its evaluation requires the application of the steepest-descent method in the complex x plane. This will be done with some care since in the neighborhood of the point of interest we have both a branch point and a pole. In Eq. (39b) the function to be integrated is periodic, with period 2π due to the condition that $2\Omega/\omega_0$ is an integer (this condition is, of course, irrelevant from the physical point of view since $2\Omega/\omega_0$ is so large). Furthermore, in the strip $-\pi\leq\mathrm{Re}x\leq+\pi$, there are four branch points which originate from the zeros of $\{m^2+p_\perp^2+[p_3-(eE/\omega_0)\cos x]^2\}^{1/2}$. We denote them by x_0, $\bar x_0$, $-x_0$, and $-\bar x_0$, where x_0 is such that both its real and imaginary parts are positive (see Fig. 1):

$$p_3-(eE/\omega_0)\cos x_0=i(m^2+p_\perp^2)^{1/2},\quad 0<\mathrm{Re}x_0<\tfrac{1}{2}\pi,\quad 0<\mathrm{Im}x_0. \qquad (40)$$

We have assumed $p_3>0$, since $|c|^2$ is an even function of p_3.

From the periodicity condition we can compute c by allowing the end points of the integration path to be $-\pi+i\lambda$ and $\pi+i\lambda$, with λ real and arbitrary. In this process of deformation, we are prevented from pushing the contour to infinity by the poles and branch cuts. In the upper strip, the cuts are taken along equal modulus curves of the exponential in such a way that along any other direction the function decreases very rapidly.

In the vicinity of the singularities the exponential is stationary. Therefore, the main contributions to the integral come from this region.

In order to find the best contour, we must study in detail the topology of the surface of the exponential. This is sketched in Fig. 1. In the neighborhood of x_0 the integrand behaves like $(x-x_0)^{-1}\exp[(x-x_0)^{3/2}]$. The contour must avoid x_0 and end in the valleys following the two steepest descent lines. These lines make an angle $\frac{2}{3}\pi$. The contour is then made of two parts, Γ_1 around x_0, and Γ_2 around $-\bar{x}_0$. Furthermore, if c_1 and c_2 are their respective contributions, $c_2=-\bar{c}_1$, and therefore

$$c=2i\,\mathrm{Im}c_1\,,$$

$$c_1\simeq\exp\left\{\frac{2i}{\omega_0}\int_0^{x_0}dx\left[m^2+p_\perp^2+\left(p_3-\frac{eE}{\omega_0}\cos x\right)^2\right]^{1/2}\right\}\frac{\omega_0}{2eE}$$

$$\times\int_{\Gamma_1}\frac{dx}{2\pi x}\exp\left\{\frac{4i}{3\omega_0}x^{3/2}\left[2i(p_\perp^2+m^2)^{1/2}\frac{eE}{\omega_0}\sin x_0\right]^{1/2}\right\}.$$

Through a change of scale, the last integral can be written

$$\int_{\tilde{\Gamma}_1}\frac{dy}{2\pi y}\exp(y^{3/2})\,.$$

The contour $\tilde{\Gamma}_1$ is depicted in Fig. 2. Through the mapping $u=y^{3/2}$, $\tilde{\Gamma}_1$ is transformed into a positive circuit around the origin with end points at $-\infty$. Hence

$$\int_{\tilde{\Gamma}_1}\frac{dy}{2\pi y}\exp(y^{3/2})=\tfrac{2}{3}\oint\frac{du}{2\pi u}e^u=\tfrac{2}{3}i\,.$$

We define now the real quantities A and B through

$$-A+iB=\frac{2i}{\omega_0}\int_0^{x_0}dx\left[m^2+p_\perp^2+\left(p_3-\frac{eE}{\omega_0}\cos x\right)^2\right]^{1/2}, \quad (41)$$

where A is positive. With this notation and the evaluation of c, we obtain the final expression

$$w=\tfrac{1}{9}\omega_0\int\frac{d^3p}{(2\pi)^2}e^{-2A}\cos^2 B\,. \quad (42)$$

The crucial point in (42) is the exponential decrease of e^{-2A}, which indicates, as expected, that apart from inessential threshold factors the pairs tend to be emitted with small momenta.

This allows us to estimate (42) as follows.

(i) p_3 is set equal to zero, the range of the p_3 integration being of order of magnitude $2eE/\omega_0$ as suggested by the classical equation of motion.

(ii) $\cos^2 B$ is replaced by its average value, $\frac{1}{2}$.

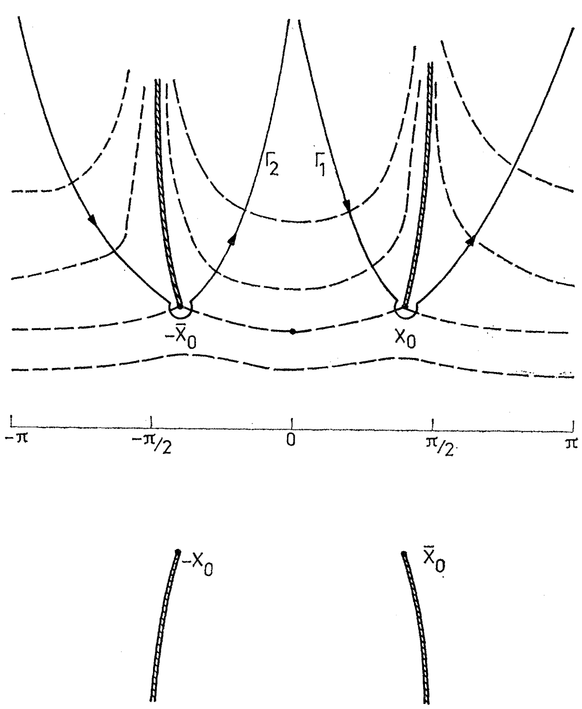

FIG. 1. Sketch of the equal-modulus lines of the function $\exp\{(2i/\omega_0)\int_0^x dx'[m^2+p_\perp^2+(p_3-(eE/\omega_0)\cos x')^2]^{1/2}\}$ needed in Eq. (39a). The integration contour consists of Γ_1 and Γ_2.

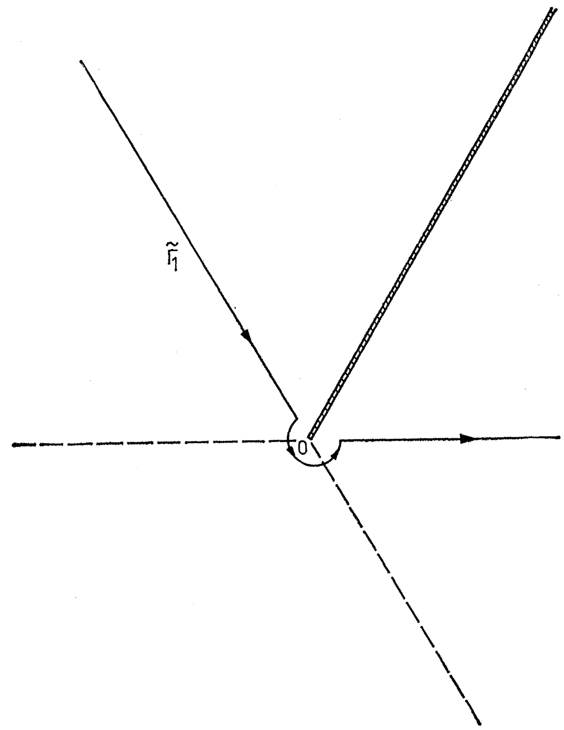

FIG. 2. Contour on integration $\tilde{\Gamma}_1$ around the branch point in the complex y plane.

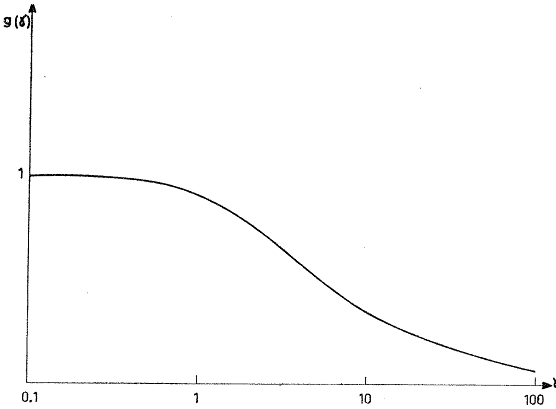

FIG. 3. The curve $g(\gamma)=(4/\pi)\int_0^1 dy[(1-y^2)/(1+\gamma^2y^2)]^{1/2}$.

These simplifications amount to multiplying the result by a numerical factor. From Eq. (40) we get

$$x_0=\tfrac{1}{2}\pi+i\sinh^{-1}[(\omega_0/eE)(m^2+p_\perp^2)^{1/2}]. \qquad (43)$$

Then, the significant quantity A is expressed as

$$A=\frac{2}{\omega_0}\int_0^{\sinh^{-1}[(\omega_0/eE)(m^2+p_\perp^2)^{1/2}]}dx \times\left[m^2+p_\perp^2-\frac{e^2E^2}{\omega_0^2}(\sinh x)^2\right]^{1/2}.$$

Making the change of integration variable $y=(eE/\omega_0)\times(m^2+p_\perp^2)^{1/2}\sinh x$, we obtain

$$A=\frac{2(m^2+p_\perp^2)}{eE}\int_0^1 dy\left[\frac{1-y^2}{1+y^2\omega_0^2(m^2+p_\perp^2)/e^2E^2}\right]^{1/2}. \qquad (44)$$

Let us define the function

$$g(z)=\frac{4}{\pi}\int_0^1 dy\left(\frac{1-y^2}{1+z^2y^2}\right)^{1/2}, \qquad (45)$$

where the normalization is chosen in such a way that $g(0)=1$. Using formula (44) for A and replacing $\cos^2B$ by $\frac{1}{2}$ as discussed above, we obtain

$$w\simeq\frac{eE}{36\pi}\int_0^\infty dp^2\exp\left[-\frac{\pi(m^2+p^2)}{eE}g\left(\frac{\omega_0}{eE}(m^2+p^2)^{1/2}\right)\right],$$

or, with $m^2+p^2=(eE/\omega_0)^2u^2$,

$$w=\frac{(eE)^3}{18\pi\omega_0^2}\int_\gamma^\infty du\, u\exp\left[-\frac{\pi eE}{\omega_0^2}u^2g(u)\right]. \qquad (46)$$

In this last expression appears the essential parameter

$$\gamma=m\omega_0/eE, \qquad (47)$$

which is the ratio of the two small quantities $(\omega_0/m)/(eE/m^2)$. Up to now we have not made any assumption on its size. Under the integral sign in (46) the function $u^2g(u)$ is monotonically increasing:

$$\frac{eE}{\omega_0^2}u^2g(u)\geq\frac{eE}{\omega_0^2}\gamma^2g(\gamma)=\frac{m^2}{eE}g(\gamma)\gg1.$$

This allows us to perform the integration and leads to the final result

$$w=\frac{\alpha E^2}{2\pi}\frac{1}{g(\gamma)+\frac{1}{2}\gamma g'(\gamma)}\exp\left[-\frac{\pi m^2}{eE}g(\gamma)\right], \qquad (48)$$

where the smooth function g has been given in (45).

IV. DISCUSSION OF RESULT

In this section we discuss our final formula (48) and make some numerical estimates using existing values for the field and frequency.

The parameter γ describes the transition from the high-field, low-frequency limit ($\gamma\ll1$: constant-field case), to the low-field, perturbative regime ($\gamma\gg1$). The curve $g(\gamma)$ is represented in Fig. 3. One readily derives from (45) that

$$\begin{aligned}\gamma\ll1,\quad & g(\gamma)=1-\tfrac{1}{8}\gamma^2+O(\gamma^4),\\ \gamma\gg1,\quad & g(\gamma)=(4/\pi\gamma)\ln(2\gamma)+O(1/\gamma).\end{aligned} \qquad (49)$$

In general $g(\gamma)\simeq(4/\pi\gamma)\sinh^{-1}\gamma$ is an almost uniform approximation except for a factor of order $\frac{1}{4}\pi$ for small γ. Hence

$$\begin{aligned}\gamma\ll1,\quad & w\simeq\frac{\alpha E^2}{2\pi}\exp\left(-\frac{\pi m^2}{eE}\right)\\ \gamma\gg1,\quad & w\simeq\frac{\alpha E^2}{8}\left(\frac{eE}{2m\omega_0}\right)^{4m/\omega_0}.\end{aligned} \qquad (50)$$

The value for small γ agrees indeed with the calculation of Sec. II apart from an inessential factor π, while for large γ (or $eE/2m\omega_0\ll1$) we find an amplitude of probability proportional to the potential raised to a power equal to the minimum number of photons necessary to produce a pair, namely, $2m/\omega_0$. The general formula (48) interpolates between these two extreme situations.

The most optimistic data using presently available optical lasers (frequency $\omega_0\sim3\times10^{15}\ \text{sec}^{-1}$) correspond to peak-fields of the order of 7×10^{12} V/cm. These extremely intense fields can only be obtained in pulses of order 10^{-12} sec and are concentrated in volumes $\sim10^{-9}\ \text{cm}^3$. Our two small parameters are, under these circumstances,

$$\hbar\omega_0/mc^2\sim4\times10^{-6},\quad eE\hbar/m^2c^3\sim3\times10^{-4},$$

and, not too surprisingly, the parameter γ is also small:

$$\gamma=m\omega_0c/eE\sim10^{-2},$$

indicating that the frequency plays no significant role. With these numerical values the probability of pair creation is zero to a fantastic accuracy [$\sim\exp(-10^4)$]. Clearly the observation of the effect would require either an increase in the field, or in the frequency, or both.

The condition for observing pairs in vacuum can thus be summarized as

$$eE \gtrsim \pi m^2 g(\gamma)\,,$$

or, using the very good approximate form for $g(\gamma) \simeq (4/\pi\gamma)\sinh^{-1}\gamma$,

$$eE \gtrsim \frac{m\omega_0 c}{\sinh(\hbar\omega_0/4mc^2)}. \tag{51}$$

Hence even if x-ray lasers would become feasible, $\hbar\omega_0/mc^2$ would still remain very small and the effect could only be observed through a huge increase of the intensity of four orders of magnitude.

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Neutron Polarization in $\pi^- p$ Charge-Exchange Scattering at 310 MeV*

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We report the measurement of the polarization in the reaction $\pi^- + p \rightarrow \pi^0 + n$ at an incident-pion kinetic energy (lab) of 310 MeV and at an angle of 30° in the c.m. system. The polarization was obtained from measurements of the left-right asymmetry in the scattering of the neutrons from liquid helium at lab-scattering angles of 75° and 125°. The measured polarization is 0.24 ± 0.07.

I. INTRODUCTION

The experiment reported here was performed some time ago as part of a program to obtain sufficient experimental data on pion-nucleon scattering at $T_\pi = 310$ MeV so that a unique set of π-N phase shifts could be determined. Although this goal was only partly realized, subsequent experiments and detailed phase-shift analyses have established the π-N phase shifts rather uniquely up to 1 GeV, and possibly up to 2 GeV.

Although the result reported here has been used in some of the detailed phase-shift analyses performed over the last few years,[1–4] it has apparently been omitted in some of the others.[5–7] Because of this, and because our result has been omitted from a recent compilation of pion-nucleon scattering data,[8] we feel that it should be properly published rather than only be available in its present obscure form.[9,10]

Apart from the result of our polarization measurement, the experimental technique of using liquid helium as a polarization analyzer continues to be of interest.[11–13]

II. MOTIVATION

In 1959 an extensive set of measurements was begun on pion-proton scattering at an incident lab kinetic energy of 310 MeV. Measurements were first made of

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