Wave Functions and Form Factors for Relativistic Composite Particles. I

ARTHUR LEWIS LICHT* AND ANTONIO PAGNAMENTA†
Department of Physics, University of Illinois at Chicago Circle, Chicago, Illinois‡ 61801
(Received 28 January 1970)

We give a relativistic description of composite particles which behave as nonrelativistic clusters in their rest frames. We find that the relativistic form factor can be obtained from the nonrelativistic one by the substitution $S(t)=\alpha^{(1-n)/2}S_0(|t|/\alpha)$, where $\alpha=1-t/(4M_A^2)$, and where n is the number of subparticles in the cluster. An excellent one-parameter fit to the magnetic form factor of the proton is found.

I. INTRODUCTION

THERE are many composite systems which at low energies are adequately described by nonrelativistic wave functions. We think primarily of atoms and nuclei but we can also include the elementary particles if we accept a quark[1] or a parton model.[2]

Wave functions and nonrelativistic form factors are frequently used in the calculation of high-energy processes which involve one or more composites.[3] It has never been shown, however, how these wave functions should be boosted to relativistic velocities and how this affects the form factors which enter the differential cross sections.

We give a relativistic description of composite particles which behave as nonrelativistic clusters in their rest frames. In this first discussion, we treat spinless particles and clusters only. We find that the relativistic form factor $S(t)$ can be obtained from the nonrelativistic $S_0(|t|)$ by the substitution

$$S(t)=\alpha^{(1-n)/2}S_0(|t|/\alpha)\,,$$

where

$$\alpha=1-t/4M_A^2\,,$$

and n is the number of particles in cluster A, M_A is the rest mass of cluster A, and t (<0 for elastic collisions) is the square of the four-momentum transfer.

In Sec. II we establish our definitions by introducing the nonrelativistic quantities. Our main result is derived in Sec. III by boosting the nonrelativistic wave function. In Sec. IV we derive the simple substitution law for the elastic form factors. In Sec. V we apply our results to the magnetic form factor of the proton. We find an excellent one-parameter fit. In Sec. VI we compare our wave-function definition of the form factor with the dispersion approach and compare the analytic structures. Our approximations are emphasized in a critical discussion in Sec. VII, where we also indicate some generalizations of the present work.

II. NONRELATIVISTIC CLUSTERS

Consider a spinless cluster A formed from n identical spinless bosons. We describe A in its rest frame by a wave function $\psi_A(\mathbf{x},\ldots,\mathbf{x}_n)$. The $\mathbf{x}_i$ are the three-dimensional coordinates of the subparticles relative to the c.m. We fix the c.m. at the origin by

$$\sum_{i=1}^{n}\mathbf{x}_i=0. \tag{1}$$

Bose statistics require ψ to be symmetric under all permutations π of its arguments

$$\psi_A(\mathbf{x}_{\pi 1},\ldots,x_{\pi n})=\psi_A(x_1,\ldots,x_n). \tag{2}$$

In this first study we assume that A itself be spinless; therefore, ψ must be rotationally invariant,

$$\psi_A(R\mathbf{x}_1,\ldots,R\mathbf{x}_n)=\psi_A(\mathbf{x}_1,\ldots,\mathbf{x}_n) \tag{3}$$

for all rotations R.

To introduce the nonrelativistic form factor, we let A have the total four-momentum $p=(p_0,\mathbf{p})$ and denote the c.m. coordinates by $(t,\mathbf{X})$. We describe the c.m. motion by the plane wave

$$\phi_p{}^A(\mathbf{X},t)=(2\pi)^{-3/2}\exp[-i(p_0t-\mathbf{x}\cdot\mathbf{p})]. \tag{4}$$

In the nonrelativistic limit, the total wave function is simply the product

$$\psi_p{}^A(\mathbf{X},t;\mathbf{x}_1,\ldots,\mathbf{x}_n)=\phi_p{}^A(\mathbf{X},t)\psi_A(\mathbf{x}_1,\ldots,\mathbf{x}_n) \tag{5}$$

of the c.m. and internal wave functions.

Suppose A interacts with an external potential V_j which only acts on the jth subparticle. After this interaction the cluster is in a state B and carries four-momentum p'. The nonrelativistic matrix element for this process is

$$\langle p'B|V_j|pA\rangle=\int d^3Xdt\prod_{i=1}^{n-1}\int d^3x_i\phi_{p'}{}^{B*}(\mathbf{X},t)\psi_B{}^*(\{\mathbf{x}_i\})\times V_j(\mathbf{X}+\mathbf{x}_jt)\phi_p{}^A(\mathbf{X},t)\psi_A(\{\mathbf{x}_i\}). \tag{6}$$

We introduce the Fourier transform of V by

$$V_j(\mathbf{Z},t)=(2\pi)^{-4}\int d^3qdw\,e^{-iwt+i\mathbf{q}\cdot\mathbf{z}}\tilde{V}_j(w,\mathbf{q}). \tag{7}$$

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* On leave from U. S. Naval Ordnance Laboratory, White Oak, Silver Spring, Md. 20910. Visiting Max-Planck-Institut für Physik und Astrophysik, München, until 1 January 1970.

† Research sponsored by the U. S. Air Force Office of Scientific Research, Office of Aerospace Research, under AFSOR Grant No. AFOSR-69-1668.

‡ Part of this work was done at The Department of Physics, Rutgers University, New Brunswick, N. J.

[1] For a comprehensive, recent review we refer to J. J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969).

[2] J. D. Bjorken and E. A. Paschos, Phys. Rev. **185**, 1975 (1969).

[3] D. R. Harrington and A. Pagnamenta, Phys. Rev. **173**, 1599 (1968); E. Schrauner, L. Benofy, and D. W. Cho, *ibid.* **177**, 2590 (1969).

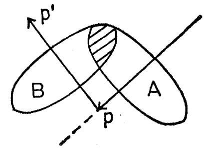


FIG. 1. The transition $(A,p) \rightarrow (B,p')$ as seen from an arbitrary Lorentz frame. The shaded region is the transition region.

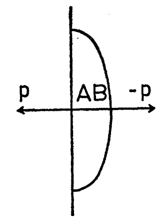


FIG. 2. The transition $(A,p) \rightarrow (B,p')$ as seen from the Breit frame.

We see that Eq. (6) may be written as

$$\langle p'B|V_j|pA\rangle = (2\pi)^{-3}S_{BA}{}^0(-(p-p')^2) \times V_j(p'^0-p^0, \mathbf{p}'-\mathbf{p}), \quad (8)$$

where we use the nonrelativistic form factor

$$S_{BA}{}^0(|\mathbf{q}|^2) = \prod_{i=1}^{n-1} \int d^3x_i \psi_B{}^*(\{\mathbf{x}_k\}) e^{i\mathbf{q}\cdot\mathbf{x}_j} \psi_A(\{\mathbf{x}_k\}). \quad (9)$$

This function is independent of j by Bose symmetry, Eq. (2), and it is a function of $\mathbf{q}^2$ alone by Eq. (5).

III. RELATIVISTIC CLUSTERS

In the relativistic case we expect the total wave function to be of the form

$$\psi_A{}^p(z,\mathbf{x}_1,\ldots,\mathbf{x}_n) = \phi_p{}^A(z)\psi_A{}^p(\mathbf{x}_1,\ldots,\mathbf{x}_n), \quad (10)$$

where $z=(t,\mathbf{X})$ is the c.m. four-position. Here $\psi_A{}^p(\{\mathbf{x}_i\})$ is some sort of single-time internal wave function. The $\mathbf{x}_i$ here are relative coordinates in a frame which will be determined below. We expect $\psi_A{}^p$ to depend on p since the internal spatial structure will be Lorentz contracted in the direction of motion.

We assume now that the matrix element of the potential (scalar interaction) can still be written as

$$\langle p'B|V_j|pA\rangle = \int d^4z \prod_{i=1}^{n-1} \int d^3x_i \psi_B{}^{*p'}(z,\{\mathbf{x}_k\}) \times V_j(\mathbf{X}+\mathbf{x}_j,t)\psi_A{}^p(z,\{\mathbf{x}_k\}). \quad (11)$$

This amounts to assuming that when the c.m. is at the point $(t,\mathbf{X})$, the jth subparticle is at $(t, \mathbf{X}+\mathbf{x}_j)$ where it interacts with the potential. The interaction is therefore assumed to be instantaneous. This is a physically reasonable approximation, and we show below that it is optional if we take the $\mathbf{x}_i$ in the Breit frame.

In Fig. 1 we show how the cluster A goes over into B in an arbitrary Lorentz frame. The clusters are moving in the directions of p and p', respectively, and are flattened in the direction of motion by Lorentz contraction. The interaction region AB (shaded in Fig. 1) varies rapidly with time as the two flat structures pass through each other. Therefore, we can not in general speak of an instantaneous interaction in such a frame.

In Fig. 2 we have drawn the same transition seen in the Breit frame. This is the center-of-momentum frame for the two four-vectors p, p'. In this frame $p=((m_A{}^2+\mathbf{p}^2)^{1/2},\mathbf{p})$ and $p'=((m_B{}^2+\mathbf{p}^2)^{1/2}, -\mathbf{p})$. The two clusters are moving in opposite directions, each being very flat. In this frame the interaction is clearly more nearly instantaneous than in any other.

By the above we can interpret the four-vectors $\xi_j=(0,\mathbf{x}_j)$ as the relative coordinates of the jth subparticle in the Breit frame. The c.m. of the cluster is located at $z=(t,X)$, so the jth subparticle is actually at $z_j=z+\xi_j$.

We assume in the following that the subparticles are all bound in the cluster A and that their relative motion within A can be neglected (impulse approximation). In this approximation each subparticle is moving on a world line W_j parallel to p as shown in Fig. 3. Let W_0 denote the world line of the c.m. The Breit-frame coordinate ξ_j at which the interaction occurs is therefore the point of intersection of W_j with the hyperplane $t=z^0$.

Each world line is specified by knowing z and the four-vector η_j which goes perpendicularly from W_0 to W_j. From Fig. 3 we see that

$$\eta_j = \Delta(p)\xi_j, \quad (12)$$

where

$$(\Delta(p))_\mu{}^\nu = g_\mu{}^\nu - p_\mu p^\nu/p^2 \quad (13)$$

is the matrix that projects out four-vectors orthogonal to p.

Let Σ_p be the hyperplane orthogonal to p which passes through z. The vectors η_j give, in the Breit frame, the positions of the subparticles in Σ_p relative to z.

Let $\hat{p}$ be the pure positive timelike vector $\hat{p}=((p^2)^{1/2},\mathbf{0})$ and let L_p denote the pure boost which takes $\hat{p}$ into p,

$$L_p\hat{p}=p.$$

The frame in which $\hat{p}=p$ is the rest frame of cluster A. There is a one-to-one correspondence between the

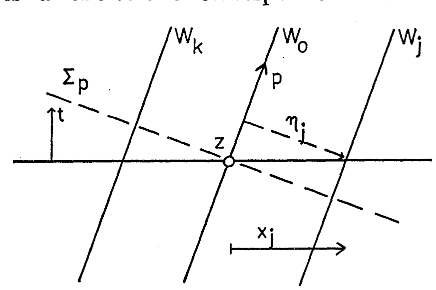


FIG. 3. The world lines of the cluster A as seen in the Breit frame. W_0 denotes the world line of the c.m. z. The jth subparticle moves along W_j.

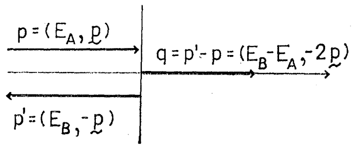


FIG. 4. The kinematics in the Breit frame.

four-vectors η_j in Σ_p and spatial vectors $\mathbf{y}_j$ in the rest frame given by

$$(0,\mathbf{y}_j)=L_p^{-1}\eta_j. \tag{14}$$

The configuration of the subparticles is given in the Breit frame by the four-vectors $\xi=(0,\mathbf{x}_j)$. Combining Eqs. (15) and (12), we see that the same configuration is determined in the rest frame of A by the vectors

$$(0,\mathbf{y}_j)=L_p^{-1}\Delta(p)(0,\mathbf{x}_j). \tag{15}$$

Summarizing, we repeat that $\mathbf{y}_i$ is the vector part of the rest frame coordinate of the ith subparticle. It is obtained from the vector part of the Breit-frame coordinate $\mathbf{x}_i$ by the transformation which connects the two frames (15). We assume now that both observers can describe a given matter distribution, each one in his own frame, by a probability amplitude which is a function of the three vectors only (single-time formalism). Both observers see the same distribution but each describes it in his own coordinates. Therefore, we identify

$$\psi_A{}^p(\{\mathbf{x}_i\})=\psi_A(\{\mathbf{y}_i\}), \tag{16}$$

where $\psi_A{}^p$ is the probability amplitude for the moving observer and ψ_A is the wave function in the rest frame. The coordinates $\mathbf{x}_i$ and $\mathbf{y}_i$ are connected by (15). Substituting (15) into (16), we see that the moving observer must use the wave function

$$\psi_A{}^p(\{\mathbf{x}_i\})=\psi_A(\mathrm{VP}\{L_p^{-1}\Delta(p)(0,\mathbf{x}_i)\}), \tag{17}$$

where VP denotes the vector part of the four-vectors in parentheses.

From Eqs. (7), 10), and (11) we find that we can write

$$\langle p'B|V_j|pA\rangle=(2\pi)^{-3}S_{BA}(p',p)\tilde{V}_j(p'-p), \tag{18}$$

where the relativistic form factor is, using (17),

$$S_{BA}(p',p)=\prod_{i=1}^{n-1}\int d^3x_i\,\psi_B{}^*(\mathrm{VP}\{L_p^{-1}\Delta(p)(0,\mathbf{x}_k)\})$$
$$\times e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}_j}\psi_A(\mathrm{VP}\{L_{p'}^{-1}\Delta(p')(0,\mathbf{x}_k)\}). \tag{19}$$

This is independent of j because of Eq. (2). We show now that it can be written in the manifestly covariant form

$$S_{BA}=S_{BA}((p'-p)^2). \tag{20}$$

Let ξ_k be an arbitrary four-vector

$$\xi_k=(\xi_k{}^0,\boldsymbol{\xi}_k). \tag{21}$$

It can easily be shown that in the Breit frame

$$[(p'+p)^2]^{1/2}d^4\xi_k\,\delta(\xi_k\cdot(p'+p))=d^3x_k. \tag{22}$$

In the Breit frame, the δ function constrains ξ_k to be $(0,\mathbf{x}_k)$. Let us also introduce $\mathbf{H}_p$ by

$$\mathbf{H}_p\xi_k=\mathrm{VP}(L_p^{-1}\Delta(p)\xi_k). \tag{23}$$

Now we can write Eq. (19) as

$$S_{BA}(p',p)=[(p'+p)^2]^{(n-1)/2}\prod_{i=1}^{n-1}\int d^4\xi_i\delta(\xi_i\cdot(p'+p))$$
$$\times\psi_B{}^*(\{\mathbf{H}_{p'}\xi_k\})e^{-i(p'-p)\cdot\xi_n}\psi_A(\{\mathbf{H}_p\xi_k\}), \tag{24}$$

where

$$\xi_n=-\sum_{i=1}^{n-1}\xi_i. \tag{25}$$

The rotational invariance expressed in Eq. (7) implies that $\psi_A(\{\mathbf{y}_k\})$ is a function of the scalar products y_iy_j alone,

$$\psi_A(\{\mathbf{y}_k\})=\tilde{\psi}_A(\{\mathbf{y}_i\cdot\mathbf{y}_j\}). \tag{26}$$

For an arbitrary vector ξ_k, we can compute the Lorentz scalar product

$$\begin{aligned}\hat{p}\cdot L_p^{-1}\Delta(p)\xi_k&=L_p\hat{p}\cdot\Delta(p)\xi_k\\&=p\cdot\Delta(p)\xi_k=0.\end{aligned} \tag{27}$$

Thus $L_p^{-1}\Delta(p)\xi_k$ has a zero time component and can be written

$$L_p^{-1}\Delta(p)\xi_k=(0,\mathbf{H}_p\xi_k). \tag{28}$$

Therefore,

$$\begin{aligned}\mathbf{H}_p\xi_k\cdot\mathbf{H}_p\xi_l&=-(L_p^{-1}\Delta(p)\xi_k,L_p^{-1}\Delta(p)\xi_l)\\&=-\xi_k\cdot\Delta(p)\xi_l,\end{aligned} \tag{29}$$

which is a manifestly Lorentz-invariant term.

Introducing Eqs. (26) and (29) in (24), we see that

$$S_{BA}(p',p)=[(p+p')^2]^{(n-1)/2}\prod_{i=1}^{n-1}\int d^4\xi_i\,\delta(\xi_i\cdot(p'+p))$$
$$\times\tilde{\psi}_B{}^*(\{-\xi_k\Delta(p')\xi_l\})e^{-i(p'-p)\cdot\xi_n}$$
$$\times\tilde{\psi}_A(\{-\xi_k\Delta(p)\xi_l\}). \tag{30}$$

This is manifestly Lorentz invariant, and we see that

$$S_{BA}(p',p)=S_{BA}((p'-p)^2) \tag{31}$$

is a function of the four-momentum transfer only.

IV. ELASTIC FORM FACTORS

Of special interest is the case $B=A$ which applies in elastic collisions.

The kinematics of the Breit frame are shown in Fig. 4. We take the z axis along p. The x and y directions are then not affected by the boost operators L_p and $L_{p'}$. We can therefore represent these boosts as 2×2

matrices acting on the t, z coordinates alone. Thus,

$$L_p=\frac{1}{M_A}\begin{pmatrix} E_A & p \\ p & E_A \end{pmatrix}, \tag{32}$$

where M_A is the rest mass of the cluster A and E_A is its energy in the Breit frame. Also,

$$L_p^{-1}=L_{-p}.$$

The boost $L_{p'}$ is obtained from L_p by exchanging $A\leftrightarrow B$ and $p\leftrightarrow -p$. Also, $L_{p'}^{-1}=L_{-p'}=L_p$. Now from (13),

$$\begin{aligned}\mathbf{H}_p\xi_k&=\mathrm{VP}\{L_p^{-1}(\xi_k-p(p\cdot\xi_k)/p^2)\}\\&=\mathrm{VP}\{(L_p^{-1}\xi_k)-\hat{p}(p\cdot\xi_k)/p^2\}\\&=\mathrm{VP}\{L_p^{-1}\xi_k\},\end{aligned}$$

which becomes

$$\mathbf{H}_p\xi_k=(0;x_k,y_k,(E_A/M_A)z_k). \tag{33}$$

Similarly,

$$\mathbf{H}_{p'}\xi_k=(0;x_k,y_k,(E_B/M_B)z_k). \tag{34}$$

If now $B\equiv A$, the form (30) can be simplified considerably. We get

$$S_{AA}(q^2)=\prod_{i=1}^{n-1}\int d^3\xi_i\left|\psi_A\left(\{x_k,y_k,\frac{E_A}{M_A}z_k\}\right)\right|^2\times\exp(2ip\sum_{j=1}^{n-1}z_j). \tag{35}$$

This can be related to the nonrelativistic form factor by the substitutions

$$z_k=(M_A/E_A)z_k',\quad x_k=x_k',\quad y_k=y_k'.$$

Then

$$S_{AA}(q^2)=\left(\frac{M_A}{E_A}\right)^{n-1}\prod_{i=1}^{n-1}\int d^3\xi_i|\psi_A(\{\xi_i\})|^2\times\exp\left(i\mathbf{q}\cdot\frac{M_A}{E_A}\sum_{j=1}^{n-1}z_j\hat{k}\right). \tag{36}$$

The energy in the Breit frame is related to the momentum transfer by

$$E_A=(M_A{}^2+\tfrac{1}{4}\mathbf{q}^2)^{1/2}\quad\text{and}\quad q^2=-\mathbf{q}^2=t. \tag{37}$$

Therefore, we find the simple substitution law

$$S_{AA}(t)=\alpha^{(1-n)/2}S_{AA}{}^0(|t|/\alpha), \tag{38}$$

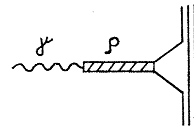
with

$$a=1-t/4M_A{}^2. \tag{39}$$

Note that for elastic scattering, $t\leq 0$.

This result has a simple interpretation. In the Breit frame the wave packet, which describes the internal configuration of the cluster, is contracted in the z direction by a factor $(1-\beta^2)^{1/2}=M_A/E_A$. The overlap integral should therefore be decreased by $(M_A/E_A)^{n-1}$, a factor M/E coming from each z integration.

Fig. 5. The photon-proton vertex under the assumption of ρ dominance is represented by the product of the ρ propagator $(1-t/m_\rho{}^2)^{-1}$ and the strong proton form factor $S(t)$.



The dispersion along the z axis in the Breit frame, $\Delta z'$, is therefore

$$\Delta z'=(M_A/E_A)\Delta z, \tag{40}$$

where Δz denotes the dispersion in the rest frame. By the uncertainty principle the momentum scale factor $\Delta p'$ must be changed by

$$\Delta p'=\Delta p(E_A/M_A). \tag{41}$$

The nonrelativistic form factor $S^0(\mathbf{q}^2)$ should be a function of $\mathbf{q}^2/(\Delta p)^2$. Therefore, we expect the relativistic form factor to be a function of

$$\frac{q^2}{(\Delta p')^2}=\frac{q^2}{(\Delta p)^2}\frac{M_A}{E_A}. \tag{42}$$

V. PROTON ELECTROMAGNETIC FORM FACTOR

As a simple application of our results, we show how the proton electromagnetic form factor as computed from Gaussian wave functions is modified in a way which greatly improves agreement with experimental data.

In the three-quark model, the wave function of the proton is often written as

$$\psi(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=N\exp[-\tfrac{1}{2}a^2(\mathbf{x}_1{}^2+\mathbf{x}_2{}^2+\mathbf{x}_3{}^2)], \tag{43}$$

with the constraint $\mathbf{x}_3=-\mathbf{x}_1-\mathbf{x}_2$. This Gaussian form is preferred because the quarks are assumed to be bound in very deep potential wells. For such a deep well, the ground-state wave function is well approximated for small distances by a Gaussian. This would be the exact solution for a harmonic-oscillator potential.

The nonrelativistic form factor corresponding to this wave function is

$$S^0(\mathbf{q}^2)=\exp[-\mathbf{q}^2/(6a^2)], \tag{44}$$

a Gaussian in $|\mathbf{q}|$. The photon-proton vertex is shown in Fig. 5. Neglecting spin and relativistic corrections, we expect the proton magnetic form factor to be given by

$$G_m(t)/\mu=(1-t/m_\rho{}^2)^{-1}S^0(t). \tag{45}$$

Here μ is the proton magnetic moment and we have assumed that the photon couples to each quark only through the ρ meson.[4] Equations (44) and (45) predict an exponential decrease of $G_m(t)$ with $|t|$. This contradicts the experimental information which is closely

[4] N. M. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

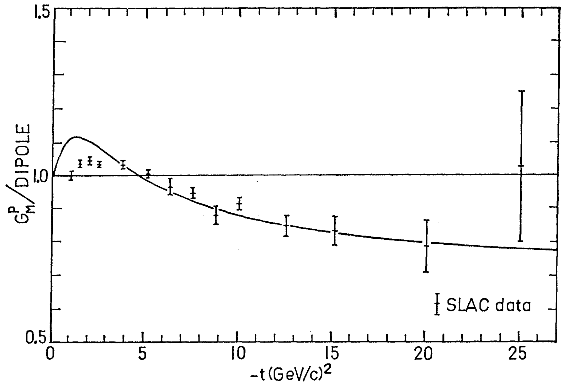


FIG. 6. Comparison of Eq. (47) with the SLAC data (Ref. 5) and the dipole fit.

represented by the empirical form[5]

$$G_m(t)/\mu \sim (1-t/0.71)^{-2}. \tag{46}$$

If instead we apply our relativistic corrections to the strong-interaction form factor in (44), we find

$$\frac{G_m(t)}{\mu} = \left(1-\frac{t}{m_\rho^2}\right)^{-1}\left(1-\frac{t}{4M_p^2}\right)^{-1} \times \exp\left[\frac{t}{6a^2}\left(1-\frac{t}{4M_p^2}\right)^{-1}\right]. \tag{47}$$

We note that the relativistic corrections have the effect of removing the objectionable exponential falloff. Moreover, we see that the observed $|t|^{-2}$ behavior of the proton form factor can be accounted for by one term $|t|^{-1}$ coming from an intermediate meson, and another one coming from the relativistic contraction factor.

The result (47) is very close to experiment.[5] In fact, for large q^2, it is much better than (46). We can consider it as a one-parameter fit and adjust a^2. This has been done in Fig. 6, which shows the fit obtained for

$$2M_p^2/3a^2 = 2.001 \quad \text{or} \quad a^2 = \tfrac{1}{3}M_p^2. \tag{48}$$

We cannot expect detailed agreement yet since we have neglected spin and we have oversimplified the photon quark coupling.

It is amusing to note that the value (48) for a^2 can be obtained from a heuristic argument. This would make Fig. 6 a parameter-free prediction. The mean square radius of the quark wave function is given by (9):

$$\langle r^2\rangle = -6\frac{d}{dq^2}S_0(q^2)\,|_{q^2=0}. \tag{49}$$

On the other hand, $\langle r^2\rangle = 3\langle x^2\rangle$ for a spherically symmetric distribution. Using (44), we find

$$\langle r^2\rangle = a^{-2} \quad \text{or} \quad \langle x^2\rangle = 1/3a^2. \tag{50}$$

It is suggestive to take

$$\langle x^2\rangle = M_p^{-2},$$

the Compton wave length of the nucleon. This gives in (50), $a^2=\frac{1}{3}M_p^2$, which is the value found from the data (48). The interpretation is straightforward. The quark form factor gives the nucleon a dimension M_p^{-1}, while the δ propagator, representing in a simplified manner the effect of the meson cloud, increases the nucleon's effective diameter.

VI. ANALYTICITY

According to dispersion theory,[6] the form factor $S(t)$ should be analytic in t, with a cut on the positive real axis from a lowest threshold $t=t_0$ to $+\infty$, and with possibly some poles between $t=0$ and $t=t_0$.

We claim that this behavior can be obtained by a proper choice of wave functions. We prove this for the subparticle numbers $n=2$ and $n=3$. The generalization to higher n is straightforward.

For two subparticles, we have

$$S_0(\mathbf{q}^2) = \int d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}}|\psi(\mathbf{x})|^2. \tag{51}$$

Consider, for some weight function $\varphi(\nu)$, the wave function

$$\psi(x) = \int_{\nu_0}^{\infty} d\nu\, \varphi(\nu) e^{-\nu r} r^{-1/2}. \tag{52}$$

Then

$$|\psi(x)|^2 = \int_{\mu_0}^{\infty} d\mu \frac{\sigma(\mu)e^{-\mu r}}{r}, \tag{53}$$

where $\mu_0 = 2\nu_0$, and

$$\sigma(\mu) = \int_{\nu_0}^{\infty} d\nu_1 \int_{\nu_0}^{\infty} d\nu_2\, \delta(\mu-\nu_1-\nu_2)\,\varphi(\nu_1)\,\varphi(\nu_2). \tag{54}$$

This gives

$$S_0(\mathbf{q}^2) = 4\pi \int_{\mu_0}^{\infty} d\mu \frac{\sigma(\mu)}{\mu^2+\mathbf{q}^2}. \tag{55}$$

For $n=2$ we find (38) the relativistic form factor

$$S(t) = \left(1-\frac{t}{4M^2}\right)^{-1/2} S_0\left(\frac{-t}{1-t/4M^2}\right). \tag{56}$$

After some algebra this can be written as

$$S(t) = \left(1-\frac{t}{4M^2}\right)^{+1/2} \int_{t_0}^{4M^2} \frac{V(t')dt'}{(1-t'/4M^2)(t'-t)}, \tag{57}$$

[5] D. H. Coward *et al.*, Phys. Rev. Letters **20**, 292 (1968); P. N. Kirk *et al.*, Stanford Linear Accelerator Center Report No. SLAC-PUB-656, 1969, (TH) and (EXP) (unpublished).

[6] S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. **115**, 731 (1959).

where

$$t_0=\frac{\mu_0}{1+\mu_0/4M^2}$$

and

$$V(t)=4\pi\sigma\left(\frac{t}{1-t/4M^2}\right). \tag{58}$$

This $S(t)$ is cut from $t=t_0$ to $+\infty$. The cut from t_0 to $4M^2$ is arbitrary. Its discontinuity depends directly on the function $\sigma(\mu)$. The cut from $t=4M^2$ to ∞ comes from the Lorentz factor $(1-t/4M^2)^{+1/2}$.

For three subparticles, the form factor is

$$S_0(\mathbf{q}^2)=\int d^3x_1 d^3x_2\, e^{i\mathbf{q}\cdot\mathbf{x}_1}|\psi(\mathbf{x}_1,\mathbf{x}_2)|^2. \tag{59}$$

We consider the wave function

$$\psi(\mathbf{x}_1,\mathbf{x}_2)=\int_{\nu_0}^{\infty} d\nu\, \varphi(\nu)e^{-\nu R}R^{-5/2}, \tag{60}$$

where $R=(\mathbf{x}_1^2+\mathbf{x}_2^2+|\mathbf{x}_1+\mathbf{x}_2|^2)^{1/2}$. Then

$$|\psi|^2=\int_{\mu_0}^{\infty} d\mu\, \sigma(\mu)e^{-\mu R}R^{-5},$$

with $\sigma(\mu)$ as given above (54). After doing some integrations, we find

$$S_0(\mathbf{q}^2)=\int_{s_0}^{\infty}\frac{ds\, W(s)}{s+\mathbf{q}^2}, \tag{61}$$

where $s_0=3\mu_0^2$ and

$$W(s)=\int_{\mu_0}^{(s/3)^{1/2}} d\mu\, \sigma(\mu)\left(1-\frac{3\mu^2}{s}\right)^{3/2}. \tag{62}$$

From this we get

$$S(t)=\int_{t_0}^{4M^2}\frac{W(s(t'))dt'}{(1-t'/4M^2)(t'-t)}, \tag{63}$$

where

$$s(t)=\frac{t}{1-t/4M^2},\qquad t_0=\frac{s_0}{1+s_0/4M^2}, \tag{64}$$

which has the correct analytic structure.

We observe from (63) that, for $n=3$, there is no cut in $S(t)$ above $4M^2$. When $n=2$ there is a purely kinematic cut from the Lorentz factor $(1-t/4M^2)^{+1/2}$. This result is actually more general. It depends only on the parity of n, because, in general, we have

$$S(t)=(1-t/4M^2)^{(1-n)/2}\prod_{i=1}^{n-1}\int d^3x_i|\psi(\{\mathbf{x}_1\})|^2 \times\exp(i[-t/(1-t/4M^2)]^{1/2}\hat{n}\cdot\mathbf{x}_1), \tag{65}$$

where $\hat{n}$ is some unit vector. As t approaches the positive real axis above $4M^2$ from above, this goes to

$$S(t+i\epsilon)=e^{-i\pi(1-n)/2}(t/4M^2-1)^{(1-n)/2} \times\prod_{i=1}^{n-1}\int d^3x_i|\psi(\{\mathbf{x}_1\})|^2 e^{i[t/(t/4M^2-1)]^{1/2}\hat{n}\cdot\mathbf{x}}. \tag{66}$$

The integral gives the same limit when the real axis is approached from below. The Lorentz factor, however, gives a factor of $\exp[i\pi(1-n)/2]$. Thus, when n is odd, $n=2m+1$, we get, for $t>4M^2$,

$$S(t+i\epsilon)-S(t-i\epsilon)=0. \tag{67}$$

When n is even, $n=2m$, we find

$$S(t+i\epsilon)-S(t-i\epsilon)=-2i(-1)^m|t/4M^2-1|^{(1-n)/2}\times\text{integral}. \tag{68}$$

We interpret the cut below $4M^2$ as due to the internal structure of the cluster. Our wave-function approach does give this cut. The part of the cut above $4M^2$ is mostly due to cluster-anticluster scattering. Our form factor could be corrected for this by using it as the input term in a Bethe-Salpeter equation for the form factor.[7] The singularity of $4M^2$ is, however, quite distant from the region $t<0$ which is the one of interest in our study of the nucleon form factor.

The t^{-2} falloff of the electromagnetic form factor comes from the combination of meson propagator and wave function, each of which satisfies a dispersion relation and goes as t^{-1}. If the product is rewritten as a single dispersion relation, this implies trivially a superconvergence relation.

VII. DISCUSSION

We have shown how the nonrelativistic definition of the form factor can be extended into the relativistic domain by Lorentz-transforming the variables of the rest-frame wave function into the Breit frame and rewriting the results in covariant form. This led us to the substitution law (38). In Sec. V we have used our results to compute the magnetic form factor of the proton. Our fit which for large $\mathbf{q}^2$ is a double pole agrees quite well with the data. If our t^{-2} falloff is confirmed, this would prove[7] finite compositeness of the proton. A steeper falloff would imply a higher degree of compositeness (38).

In Sec. VI we have shown how to reconcile the wave-function definition of the form factor with the dispersion approach. We have shown that it is possible to find wave functions such that the form factor has the correct analytic structure below $t=4M_p^2$. To get the right analyticity above $4M_p^2$, one would probably have to give up the single time formulation and use

[7] D. Amati, R. Jengo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Letters **27B**, 38 (1968); M. Ciafaloni and P. Menotti, Phys. Rev. **173**, 1575 (1968).

Bethe-Salpeter wave functions. Our discussion has also shown how a superconvergent dispersion relation could come about without the need to introduce spurious mesons.

Clearly we cannot expect perfect agreement with the data yet because we have oversimplified. There is a spurious essential singularity at $t=4M_p^2$, due to our Gaussian approximation, which is not valid for large x. This point in the timelike region is, of course, in the physical region for $\bar{p}p\to$ leptons. This and the small discrepancies with the data for small q^2 can be remedied by the selection of a more realistic wave function, perhaps with an exponential tail. In the following paper we give a formal treatment of the spin problem and present the definition of the electromagnetic form factor of the nucleons in a quark model. In a future paper we hope to present detailed calculations which include all the known vector mesons ρ, ω, and ϕ.

PHYSICAL REVIEW D VOLUME 2, NUMBER 6 15 SEPTEMBER 1970

Wave Functions and Form Factors for Relativistic Composite Particles. II

ARTHUR LEWIS LICHT* AND ANTONIO PAGNAMENTA†
Department of Physics, University of Illinois at Chicago Circle, Chicago, Illinois‡ 61801
(Received 6 March 1970)

We extend our relativistic description of composite particles to the case of clusters of spin-$\frac{1}{2}$ particles. From this, explicit expressions for the nucleon vector current form factors are derived. We find that the nucleon's electric form factor has the form

$$G^E(t)=\alpha^{-1}S_0(-t/\alpha)\sum_{i=1}^{3}q^iG_{ib}{}^1(t),$$

where S_0 is the nonrelativistic form factor, $\alpha=1-t/4M^2$, and q^i is the charge and $G_{ib}{}^1(t)$ the electric form factor for the ith quark. A similar expression is obtained for the magnetic form factor.

I. INTRODUCTION

In the preceding paper,[1] we discussed spinless composite particles which behave as nonrelativistic clusters in their rest frames. Here we generalize these results to include spin.[2] In particular, we derive explicit expressions for the nucleon form factors.

We derive a substitution law for the relativistic form factors which permits us to obtain them from the nonrelativistic form factors. We apply these results to the quark model of the nucleons and find for the proton electric form factor,

$$G_p{}^E(t)=\alpha^{-1}S_p\left(\frac{-t}{\alpha}\right)\sum_{i=1}^{3}q^iG_{ib}{}^1(t)\,,$$

where S_p is the nonrelativistic strong-interaction form factor of the proton, $G_{ib}{}^1$ is the electric form factor of the ith quark, q^i is the charge of the ith quark, and

$$\alpha=1-t/4M_p{}^2.$$

A similar expression holds for the magnetic form factor.

In Sec. II, to define our notation, we review briefly the description of elementary relativistic particles with spin. This is generalized in Sec. III to the case of relativistic particles with an arbitrary complex internal structure. In Sec. IV we consider particles whose rest-frame structure is described by a wave function. In Sec. V we relate the matrix elements of transition operators to the form factors. We derive our formulas for the form factors in Sec. VI. The proof of two mathematical relations is delegated to Appendices A and B.

II. RELATIVISTIC SPINNING PARTICLES

Let the ket $|pjma\rangle$ denote an elementary-particle state with four-momentum p, spin j with z component m in the rest frame, and other discrete quantum numbers a.[3,4] These states have the inner product

$$\langle p'j'm'a'|pjma\rangle=2p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{jj'}\delta_{mm'}\delta_{aa'}.\qquad(1)$$

We denote an inhomogeneous Lorentz transformation by (Λ,h). Let L_p denote a pure Lorentz transformation that takes the rest-frame vector $\hat{p}=((p^2)^{1/2},\mathbf{0})$ into p.

* On leave from the U. S. Naval Ordnance Laboratory, White Oak Silver Spring, Md. 20910. Visiting the Max-Planck-Institut für Physik und Astrophysik, München, until 1 January 1970.

† Research sponsored by the U. S. Air Force Office of Scientific Research, Office of Aerospace Research, under AFOSR Grant No. AFOSR-69-1668.

‡ Part of this work was done at the Department of Physics, Rutgers University, New Brunswick, N. J.

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