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# Nonlinear Hadron Couplings from Divergence Conditions. I. Pions and Nucleons

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It is shown that chiral-dynamical models for the pion-nucleon system can be obtained simply by requiring that the source function in the pion field equation be expressible as a complete divergence. Both nonderivative and derivative pion-nucleon couplings are considered. The nonderivative-coupling case yields models with one coupling constant, while the derivative-coupling case leads to models involving two coupling constants. Since our approach avoids the use of chiral symmetry, it does not raise the problems associated with a broken symmetry. It is also suggested that the nonlinear pion-nucleon coupling, arising from the divergence condition, might very well represent a fundamental coupling rather than merely an effective coupling.

## I. INTRODUCTION

When confronted with phenomena whose basic nature is unknown, it is often helpful to search for analogies with other phenomena that are better understood. The theories of electromagnetic and gravitational interactions have been developed to the extent where they are considered to be in the latter category, and are found to have significant similarities.[1] The fundamental basis of strong and weak interactions, on the other hand, is not well known. The purpose of this investigation is to explore the theory of strong interactions by emphasizing an analogy between the meson field equations and the electromagnetic and gravitational field equations. We shall confine ourselves to the interaction of pions and nucleons in this paper, while other mesons and baryons will be included in subsequent papers.

Since the source functions in the electromagnetic and gravitational field equations represent the current four-vector and the total energy-momentum tensor, they satisfy the condition of vanishing divergence, and it seems reasonable to look for appropriate divergence conditions for the source functions in the meson field equations. However, the source function in the pion field equation does not carry any tensor index, and we cannot demand that its divergence vanish. We shall, instead, impose the requirement that for the pion-nucleon system this source function be expressible as a complete divergence, and thus the pion field equation should be of the form

$$(\Box^2-m^2)\boldsymbol{\pi}=\partial_\mu \mathbf{J}_{\mu 5}. \tag{1.1}$$

It is also natural to assume that the Lagrangian density, which yields (1.1), should involve derivatives of the lowest possible order.

In recent years Weinberg,[2] Schwinger,[3] and others[4] have proposed Lagrangian schemes with broken chiral symmetry, which seem promising in the light of experimental results. We shall show that all the chiral-symmetry results together with the symmetry-breaking terms for the pion-nucleon system can be obtained simply as a consequence of requiring that the pion field equation be expressible in the form (1.1). Evidently, the field equation (1.1) can be written

$$\partial_\mu(\mathbf{J}_{\mu 5}-\partial_\mu\boldsymbol{\pi})=-m^2\boldsymbol{\pi}, \tag{1.2}$$

which can be interpreted as the well-known condition[5,6] for the partially conserved axial-vector current (PCAC), if we regard $\mathbf{J}_{\mu 5}-\partial_\mu\boldsymbol{\pi}$ with appropriate normalization as the PCAC current. *Thus, in the present treatment we regard the PCAC condition as fundamental but completely avoid the use of chiral symmetry.*[7]

Apart from its basic simplicity, our formulation has the following conceptual advantages.

(1) The analogy between the pion field equation and the electromagnetic and gravitational field equations suggests that the nonlinear pion-nucleon couplings resulting from our treatment might represent fundamental couplings and not merely effective couplings.

(2) Since we do not make use of the broken chiral symmetry, we are not faced with the question of why and how this symmetry is broken.

(3) There is no temptation here to introduce the scalar $\sigma$ meson.

We shall first consider the case of the nonderivative pion-nucleon coupling in Secs. II and III, then consider the case of the derivative pion-nucleon coupling in Secs. IV and V, and subsequently establish the relationship between them in Sec. VI, where the role of the chiral symmetry will also be briefly discussed.

We shall take $c=\hbar=1$, denote the space-time coordinates as $x_\mu=(x_1,x_2,x_3,ix_0)$. treat the $\gamma_\mu$ as Hermitian matrices, and use boldface letters to denote isovectors.

[1] S. N. Gupta, Phys. Rev. **96**, 1683 (1954).
[2] S. Weinberg, Phys. Rev. Letters **18**, 188 (1967).
[3] J. Schwinger, Phys. Letters **24B**, 473 (1967).
[4] For a review of Lagrangian schemes with chiral symmetry, see S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. **41**, 531 (1969).
[5] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).
[6] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).
[7] The fundamental role of the PCAC condition in the derivation of Lagrangian schemes for the pion-nucleon system has also been recognized by other authors. See, e.g., D. B. Fairlie and K. Yoshida, Phys. Rev. **174**, 1816 (1968); R. Dashen and M. Weinstein, *ibid.* **183**, 1261 (1969).

An asterisk will be used to denote the complex conjugate of a number or the Hermitian conjugate of an operator.

## II. LAGRANGIAN FORMALISM WITH NON-DERIVATIVE PION-NUCLEON COUPLING

In order to examine the restrictions imposed on the Lagrangian density for a pion-nucleon system by the divergence condition, we begin by considering the general nonlinear form[8]

$$L=L_{0\pi}+L_{0N}+L_{\pi\pi}+L_{\pi N}, \tag{2.1}$$

where

$$L_{0\pi}=-\tfrac{1}{2}(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}+m^2\boldsymbol{\pi}\cdot\boldsymbol{\pi}), \tag{2.2}$$

$$L_{0N}=-\bar{N}(\gamma\cdot\partial+M)N, \tag{2.3}$$

$$L_{\pi\pi}=a(\boldsymbol{\pi}^2)+b(\boldsymbol{\pi}^2)\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}+c(\boldsymbol{\pi}^2)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2, \tag{2.4}$$

$$L_{\pi N}=-M\bar{N}\alpha(i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})N. \tag{2.5}$$

Since $(\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})^2=\boldsymbol{\pi}^2$, the function $\alpha$ can be expressed as

$$\alpha(i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})=s(\boldsymbol{\pi}^2)+2it(\boldsymbol{\pi}^2)\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}, \tag{2.6}$$

and thus $L$ involves the five unknown Hermitian functions $a(\boldsymbol{\pi}^2)$, $b(\boldsymbol{\pi}^2)$, $c(\boldsymbol{\pi}^2)$, $s(\boldsymbol{\pi}^2)$, and $t(\boldsymbol{\pi}^2)$. It is the most general isospin-invariant Lagrangian density with nonlinearity in the pion field such that $L_{\pi\pi}$ is at most bilinear in the derivative of the pion field, and $L_{\pi N}$ contains no derivative coupling. Because $L$ must reduce to the free Lagrangian density $L_{0\pi}+L_{0N}$ in the absence of interaction, it is required that

$$a(0)=a'(0)=b(0)=s(0)=0, \tag{2.7}$$

where a prime denotes differentiation with respect to the argument, which is $\boldsymbol{\pi}^2$.

The above Lagrangian density yields the nucleon field equations

$$\begin{aligned}\partial_\mu\bar{N}\gamma_\mu&=M\bar{N}(1+s+2it\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}),\\ \gamma_\mu\partial_\mu N&=-M(1+s+2it\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})N,\end{aligned} \tag{2.8}$$

and the pion field equation

$$\begin{aligned}(\Box^2-m^2)\boldsymbol{\pi}=&\,2Ms'\bar{N}N\boldsymbol{\pi}+4Mt'(\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &+2Mt\bar{N}i\gamma_5\boldsymbol{\tau}N-2a'\boldsymbol{\pi}+2c'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}\\ &+4b'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}+2(c-b')\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}\\ &+2b'(\boldsymbol{\pi}\cdot\Box^2\boldsymbol{\pi})\boldsymbol{\pi}+2b\Box^2\boldsymbol{\pi}.\end{aligned} \tag{2.9}$$

In order that (2.9) be expressible in the form (1.1), $\mathbf{J}_{\mu5}$ must evidently be of the form

$$\mathbf{J}_{\mu5}=\beta\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N+d(\boldsymbol{\pi}^2)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}+e(\boldsymbol{\pi}^2)\partial_\mu\boldsymbol{\pi}, \tag{2.10}$$

with

$$\beta=\text{const},\quad e(0)=0, \tag{2.11}$$

where $\beta$ is required to be a constant rather than a function of $\boldsymbol{\pi}^2$ because the terms bilinear in the nucleon field in (2.9) do not involve any derivative of the pion field. Substitution of (2.10) into (1.1) gives

$$\begin{aligned}(\Box^2-m^2)\boldsymbol{\pi}=&\,\beta(\partial_\mu\bar{N})i\gamma_\mu\gamma_5\boldsymbol{\tau}N+\beta\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\partial_\mu N\\ &+2d'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}+d\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}\\ &+(d+2e')(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}+e\Box^2\boldsymbol{\pi},\end{aligned} \tag{2.12}$$

so that, on simplification with the help of (2.8),

$$\begin{aligned}(1-e)\Box^2\boldsymbol{\pi}=&\,2M\beta(1+s)\bar{N}i\gamma_5\boldsymbol{\tau}N-4M\beta t\bar{N}N\boldsymbol{\pi}\\ &+2d'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}+d\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}\\ &+(d+2e')(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}+m^2\boldsymbol{\pi}.\end{aligned} \tag{2.13}$$

By obtaining the value of $\Box^2\boldsymbol{\pi}$ from (2.13) and substituting it into (2.9), we obtain

$$\begin{aligned}&2[s'(1-e)+2(1-2b-2b'\boldsymbol{\pi}^2)t\beta]M\bar{N}N\boldsymbol{\pi}\\ &+2[t(1-e)+(2b-1)(1+s)\beta]M\bar{N}i\gamma_5\boldsymbol{\tau}N\\ &+4[t'(1-e)+b'(1+s)\beta]M(\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &+[2a'(e-1)+(2b-e)m^2+2b'm^2\boldsymbol{\pi}^2]\boldsymbol{\pi}\\ &+[4b'(1-e)+(2b-1)(d+2e')](\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}\\ &+[2(c-b')(1-e)+(2b-1)d+2b'd\boldsymbol{\pi}^2]\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}\\ &+2[c'(1-e)+(2b-1)d'+b'(d+2e')+2b'd'\boldsymbol{\pi}^2]\\ &\qquad\times(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}=0.\end{aligned} \tag{2.14}$$

The above equation can be satisfied only if the coefficients of the terms involving $\bar{N}N\boldsymbol{\pi}$, $\bar{N}i\gamma_5\boldsymbol{\tau}N$, $(\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}$, $\boldsymbol{\pi}$, $(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}$, $\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}$, and $(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}$ vanish separately. The solution of the resulting differential equations leads, as shown in Appendix A, to the following relationships.

Let us put

$$t(0)=f, \tag{2.15}$$

so that $f$ can be interpreted as a coupling constant that specifies the first-order pion-nucleon coupling in (2.5). The constant $\beta$ is then given by

$$\beta=f. \tag{2.16}$$

The functions $s$ and $t$ are not determined uniquely, but they satisfy the relation

$$(1+s)^2+4t^2\boldsymbol{\pi}^2=1, \tag{2.17}$$

while the functions $a$, $b$, $c$, $d$, and $e$ can be expressed in terms of $f$, $s$, and $t$ as

$$1-2a'/m^2=-s'/2ft, \tag{2.18}$$

$$1-2b=t^2/f^2, \tag{2.19}$$

$$c=(t^4-\tfrac{1}{4}s'^2)/2f^2t^2\boldsymbol{\pi}^2, \tag{2.20}$$

$$d=(2/f)[ts'-t'(1+s)], \tag{2.21}$$

$$e=1-(t/f)(1+s). \tag{2.22}$$

[8] It is understood that the Lagrangian density consists of ordered products of field operators, although for simplicity we have omitted the ordered-product notation. See S. N. Gupta, Phys. Rev. **107**, 1722 (1957).

## III. SPECIAL NONDERIVATIVE-COUPLING MODELS

The divergence condition provides the four relations (2.17), (2.18), (2.19), and (2.20), which are to be satisfied by the five functions $a$, $b$, $c$, $s$, and $t$ appearing in the Lagrangian density. Therefore, all these functions can be determined only by choosing one of them in some appropriate manner. We shall now discuss three special models such that in each model one of the functions $a$, $b$, and $c$ vanishes. This simple procedure, as we shall see, yields all the models considered by Chang and Gürsey.[9]

### Model A

We first consider the model obtained by putting

$$a\equiv 0. \tag{3.1}$$

Then, according to (2.18),

$$-s'/2ft=1$$

or, on using (2.17),

$$[1-(1+s)^2]^{-1/2}s'=-(1/\sqrt{\pi^2})f,$$

which gives, in view of the condition (2.7),

$$s=\cos(2f\sqrt{\pi^2})-1, \tag{3.2}$$

and, on using (2.17) again,

$$t=(1/2\sqrt{\pi^2})\sin(2f\sqrt{\pi^2}). \tag{3.3}$$

Substitution of (3.2) and (3.3) into (2.19)–(2.22) gives all the functions appearing in $L_{\pi N}$, $L_{\pi\pi}$, and $\mathbf{J}_{\mu 5}$, and thus

$$L_{\pi N}=-M[\cos(2f\sqrt{\pi^2})-1]\bar{N}N-M(1/\sqrt{\pi^2})\sin(2f\sqrt{\pi^2})\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N, \tag{3.4}$$

$$L_{\pi\pi}=\tfrac{1}{2}[1-(1/4f^2\pi^2)\sin^2(2f\sqrt{\pi^2})]\times[\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}-(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2/\pi^2], \tag{3.5}$$

$$\mathbf{J}_{\mu 5}=f\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N+(1/\pi^2)[1-(1/4f\sqrt{\pi^2})\times\sin(4f\sqrt{\pi^2})][\pi^2\partial_\mu\boldsymbol{\pi}-(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}]. \tag{3.6}$$

### Model B

To obtain another model we require that

$$b\equiv 0, \tag{3.7}$$

so that (2.19), together with (2.15), yields

$$t=f, \tag{3.8}$$

and, on using (2.17),

$$s=(1-4f^2\pi^2)^{1/2}-1. \tag{3.9}$$

Again, the above values of $s$ and $t$ enable us to determine all the required functions, and lead to the results

$$L_{\pi N}=-M[(1-4f^2\pi^2)^{1/2}-1]\bar{N}N-2fM\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N, \tag{3.10}$$

$$L_{\pi\pi}=(m^2/4f^2)[(1-4f^2\pi^2)^{1/2}-1+2f^2\pi^2]-2f^2(1-4f^2\pi^2)^{-1}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2, \tag{3.11}$$

$$\mathbf{J}_{\mu 5}=f\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N-4f^2(1-4f^2\pi^2)^{-1/2}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}+[1-(1-4f^2\pi^2)^{1/2}]\partial_\mu\boldsymbol{\pi}. \tag{3.12}$$

This model was suggested by Gell-Mann and Lévy,[5] and further developed by Weinberg[2] and by Brown.[10]

### Model C

If we put

$$c\equiv 0, \tag{3.13}$$

it follows from (2.20) that

$$\tfrac{1}{4}s'^2=t^4 \quad\text{or}\quad -\tfrac{1}{2}s'=t^2,$$

where, in addition to $t(0)=f$ we have used the condition $s'(0)=-2f^2$, which is obtained by differentiating (2.17) with respect to $\pi^2$ and then putting $\pi^2=0$. The above relation becomes, on using (2.17),

$$[1-(1+s)^2]^{-1}s'=-(1/2\pi^2),$$

which gives, when integrated with the condition (2.7),

$$s=-2f^2\pi^2(1+f^2\pi^2)^{-1}, \tag{3.14}$$

and, on using (2.17) again,

$$t=f(1+f^2\pi^2)^{-1}. \tag{3.15}$$

After determining the required functions with the help of (3.14) and (3.15), we obtain

$$L_{\pi N}=2Mf^2\pi^2(1+f^2\pi^2)^{-1}\bar{N}N-2Mf(1+f^2\pi^2)^{-1}\times\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N, \tag{3.16}$$

$$L_{\pi\pi}=\tfrac{1}{2}m^2[\pi^2-(1/f^2)\ln(1+f^2\pi^2)]+\tfrac{1}{2}[1-(1+f^2\pi^2)^{-2}](\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}), \tag{3.17}$$

$$\mathbf{J}_{\mu 5}=f\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N-2f^2(1+f^2\pi^2)^{-2}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}+[1-(1-f^2\pi^2)(1+f^2\pi^2)^{-2}]\partial_\mu\boldsymbol{\pi}. \tag{3.18}$$

This model is related to the work of Schwinger discussed in Sec. V.

Besides the above three models, we have been able to find two other reasonably simple models corresponding to $d=0$ and $s'=\text{const}$, respectively. The values of $s$ and $t$ for these models are

$$s=(1+4f^2\pi^2)^{-1/2}-1,\quad t=f(1+4f^2\pi^2)^{-1/2}$$

for $d=0$, and

$$s=-2f^2\pi^2,\quad t=f(1-f^2\pi^2)^{1/2}$$

for $s'=\text{const}$. The derivation of $L_{\pi N}$, $L_{\pi\pi}$, and $\mathbf{J}_{\mu 5}$ from the above values of $s$ and $t$ is quite straightforward.

[9] P. Chang and F. Gürsey, Phys. Rev. **164**, 1752 (1967).

[10] L. S. Brown, Phys. Rev. **163**, 1802 (1967).

## IV. LAGRANGIAN FORMALISM WITH DERIVATIVE PION-NUCLEON COUPLING

We shall now reinvestigate the Lagrangian formalism of Sec. II by giving up the condition that $L_{\pi N}$ contains no derivative coupling and assuming instead that $L_{\pi N}$ contains a single derivative of the pion field. Thus, we retain the Lagrangian density of Sec. II except that $L_{\pi N}$, given by (2.5), is replaced by

$$L_{\pi N}=\alpha_1(\boldsymbol{\pi}^2)\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N+\alpha_2(\boldsymbol{\pi}^2)\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N +\alpha_3(\boldsymbol{\pi}^2)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N\,, \quad (4.1)$$

with $\alpha_1$, $\alpha_2$, and $\alpha_3$ unknown Hermitian functions of $\boldsymbol{\pi}^2$. In addition, we must now include analogous terms in the general form of $\mathbf{J}_{\mu5}$, which becomes

$$\mathbf{J}_{\mu5}=\beta_1(\boldsymbol{\pi}^2)\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N+\beta_2(\boldsymbol{\pi}^2)\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\boldsymbol{\pi}N +\beta_3(\boldsymbol{\pi}^2)(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi} +d(\boldsymbol{\pi}^2)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}+e(\boldsymbol{\pi}^2)\partial_\mu\boldsymbol{\pi}. \quad (4.2)$$

The Lagrangian formalism yields the nucleon field equations

$$\begin{aligned}\partial_\mu\bar{N}\gamma_\mu&=\bar{N}[M-\alpha_1 i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}-\alpha_2 i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi} -\alpha_3(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}]\,,\\ \gamma_\mu\partial_\mu N&=-[M-\alpha_1 i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}-\alpha_2 i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi} -\alpha_3(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}]N\,,\end{aligned} \quad (4.3)$$

and the pion field equation

$$\begin{aligned}(\Box^2-m^2)\boldsymbol{\pi} &=\partial_\mu[\alpha_1\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N+\alpha_2\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\boldsymbol{\pi}N +\alpha_3\boldsymbol{\pi}(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)]-2\boldsymbol{\pi}[a_1'\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N\\ &\quad+\alpha_2'\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N+\alpha_3'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N] +\alpha_2\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\partial_\mu\boldsymbol{\pi}N-\alpha_3\bar{N}i\gamma_\mu\gamma_5(\boldsymbol{\tau}\cdot\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}N\\ &\quad-\alpha_3\bar{N}i\gamma_\mu\gamma_5(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\tau}N -2a'\boldsymbol{\pi}+2c'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}+4b'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}\\ &\quad+2(c-b')\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}+2b'(\boldsymbol{\pi}\cdot\Box^2\boldsymbol{\pi})\boldsymbol{\pi} +2b\Box^2\boldsymbol{\pi}.\end{aligned} \quad (4.4)$$

On the other hand, substitution of (4.2) into the divergence condition (1.1) gives

$$\begin{aligned}(\Box^2-m^2)\boldsymbol{\pi}&=\partial_\mu[\beta_1\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N+\beta_2\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\boldsymbol{\pi}N +\beta_3\boldsymbol{\pi}(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)]+2d'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}\\ &\quad+d\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}+(d+2e')(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}+e\Box^2\boldsymbol{\pi}.\end{aligned} \quad (4.5)$$

The total divergence terms appearing in (4.4) and (4.5) have the same structure, and they can be simplified by using the nucleon field equations (4.3) and the commutation property of the isospin matrices

$$[\boldsymbol{\tau}\cdot\mathbf{A},\boldsymbol{\tau}]=2i\boldsymbol{\tau}\times\mathbf{A}\,, \quad (4.6)$$

along with the standard vector identities such as

$$\begin{aligned}\boldsymbol{\tau}\times(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})&=(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}-(\boldsymbol{\tau}\cdot\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}\,,\\ (\boldsymbol{\tau}\cdot\boldsymbol{\pi})\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}&=(\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi} -(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\tau}\times\boldsymbol{\pi}+\boldsymbol{\pi}^2\boldsymbol{\tau}\times\partial_\mu\boldsymbol{\pi}.\end{aligned} \quad (4.7)$$

Thus, it is possible to express (4.4) as

$$\begin{aligned}(\Box^2-m^2)\boldsymbol{\pi}&=2M\alpha_1\bar{N}i\gamma_5\boldsymbol{\tau}N+2M\alpha_3(\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &\quad+(2\alpha_1'-\alpha_3-2\alpha_1\alpha_2-2\alpha_2\alpha_3\boldsymbol{\pi}^2)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N\\ &\quad-(2\alpha_1'-\alpha_3-2\alpha_1\alpha_2-2\alpha_2\alpha_3\boldsymbol{\pi}^2)(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &\quad-2(\alpha_2'-\alpha_2^2+\alpha_1\alpha_3)(\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &\quad+2(\alpha_2'-\alpha_2^2+\alpha_1\alpha_3)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\boldsymbol{\pi}N\\ &\quad+2(\alpha_2+\alpha_1^2+\alpha_2^2\boldsymbol{\pi}^2)\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\partial_\mu\boldsymbol{\pi}N\\ &\quad-2\alpha'\boldsymbol{\pi}+2c'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}+4b'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}\\ &\quad+2(c-b')\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}+2b'(\boldsymbol{\pi}\cdot\Box^2\boldsymbol{\pi})\boldsymbol{\pi}+2b\Box^2\boldsymbol{\pi}\,,\end{aligned} \quad (4.8)$$

and (4.5) as

$$\begin{aligned}(1-e)\Box^2\boldsymbol{\pi}&=2M\beta_1\bar{N}i\gamma_5\boldsymbol{\tau}N+2M\beta_3(\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &\quad+2(\beta_1'-\alpha_1\beta_2-\alpha_3\beta_2\boldsymbol{\pi}^2)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N\\ &\quad+2(\beta_3'-\alpha_2\beta_3+\alpha_3\beta_2)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &\quad+(\beta_3+2\alpha_2\beta_1+2\alpha_2\beta_3\boldsymbol{\pi}^2)(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &\quad+2(\alpha_2\beta_2-\alpha_1\beta_3)(\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &\quad+(\beta_3+2\alpha_1\beta_2-2\alpha_2\beta_1)(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\partial_\mu\boldsymbol{\pi}\\ &\quad+2(\beta_2'-\alpha_2\beta_2+\alpha_3\beta_1)(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\boldsymbol{\pi}N\\ &\quad+(\beta_2+2\alpha_1\beta_1+2\alpha_2\beta_2\boldsymbol{\pi}^2)\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\partial_\mu\boldsymbol{\pi}N\\ &\quad+2d'(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}+d\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}\\ &\quad+(d+2e')(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}+m^2\boldsymbol{\pi}.\end{aligned} \quad (4.9)$$

By obtaining the value of $\Box^2\boldsymbol{\pi}$ from (4.9) and substituting it into (4.8), we find

$$\begin{aligned}&[2M\beta_1(1-2b)-2M\alpha_1(1-e)](\bar{N}i\gamma_5\boldsymbol{\tau}N)+[2M\beta_3(1-2b)-4Mb'(\beta_1+\beta_3\boldsymbol{\pi}^2)-2M\alpha_3(1-e)](\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &\quad+[2(\beta_1'-\alpha_1\beta_2-\alpha_3\beta_2\boldsymbol{\pi}^2)(1-2b)-(2\alpha_1'-\alpha_3-2\alpha_1\alpha_2-2\alpha_2\alpha_3\boldsymbol{\pi}^2)(1-e)](\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N)\\ &\quad+[2(\beta_3'-\alpha_2\beta_3+\alpha_3\beta_2)(1-2b)-2b'(2\beta_1'+\beta_3-2\alpha_2\beta_1+2\beta_3'\boldsymbol{\pi}^2-2\alpha_2\beta_3\boldsymbol{\pi}^2)](\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &\quad+[(\beta_3+2\alpha_2\beta_1+2\alpha_2\beta_3\boldsymbol{\pi}^2)(1-2b)-2b'\boldsymbol{\pi}^2(\beta_3+2\alpha_2\beta_1+2\alpha_2\beta_3\boldsymbol{\pi}^2)+(2\alpha_1'-\alpha_3-2\alpha_1\alpha_2-2\alpha_2\alpha_3\boldsymbol{\pi}^2)(1-e)]\\ &\quad\times(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N)\boldsymbol{\pi}+[2(\alpha_2\beta_2-\alpha_1\beta_3)(1-2b)+2b'(\beta_2+2\alpha_1\beta_1+2\alpha_1\beta_3\boldsymbol{\pi}^2)-2(\alpha_2^2-\alpha_2'-\alpha_1\alpha_3)(1-e)]\\ &\quad\times(\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N)\boldsymbol{\pi}+[(\beta_3+2\alpha_1\beta_2-2\alpha_2\beta_1)(1-2b)](\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\partial_\mu\boldsymbol{\pi}\\ &\quad+[2(\beta_2'-\alpha_2\beta_2+\alpha_3\beta_1)(1-2b)+2(\alpha_2^2-\alpha_2'-\alpha_1\alpha_3)(1-e)](\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\boldsymbol{\pi}N)\\ &\quad+[(\beta_2+2\alpha_1\beta_1+2\alpha_2\beta_2\boldsymbol{\pi}^2)(1-2b)-2(\alpha_1^2+\alpha_2+\alpha_2^2\boldsymbol{\pi}^2)(1-e)](\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\partial_\mu\boldsymbol{\pi}N)-[2a'(e-1)+(2b-e)m^2+2b'm^2\boldsymbol{\pi}^2]\boldsymbol{\pi}\\ &\quad-[4b'(1-e)+(2b-1)(d+2e')](\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}-[2(c-b')(1-e)+(2b-1)d+2b'd\boldsymbol{\pi}^2]\partial_\mu(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}\\ &\quad-2[c'(1-e)+(2b-1)d'+b'(d+2e')+2b'd'\boldsymbol{\pi}^2](\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\boldsymbol{\pi}=0.\end{aligned} \quad (4.10)$$

Again, as in Sec. II, the coefficients of all the terms on the left-hand side of (4.10) must vanish separately. Then, after solving the differential equations, as shown in Appendix B, we arrive at the following results.

By introducing two functions $s(\boldsymbol{\pi}^2)$ and $t(\boldsymbol{\pi}^2)$ such that

$$(1+s)^2+4t^2\boldsymbol{\pi}^2=1\,, \tag{4.11}$$

with

$$s(0)=0\,,\quad t(0)=f\,, \tag{4.12}$$

it is possible to express the functions $a$, $b$, $c$, $d$, and $e$ again in terms of $f$, $s$, and $t$ by means of the relations (2.18) to (2.22). We also put

$$\alpha_1(0)=g\,, \tag{4.13}$$

so that $g$ can be interpreted as a coupling constant that specifies the first-order pion-nucleon coupling in (4.1). We are then able to express the functions $\alpha_1$, $\alpha_2$, and $\alpha_3$ in terms of $g$, $f$, $s$, and $t$ as

$$\begin{aligned}\alpha_1&=gt/f\,,\\ \alpha_2&=s/2\boldsymbol{\pi}^2\,,\\ \alpha_3&=(2g/f)[(1+s)t'-s't+st/2\boldsymbol{\pi}^2]\,,\end{aligned} \tag{4.14}$$

and the functions $\beta_1$, $\beta_2$, and $\beta_3$ as

$$\begin{aligned}\beta_1&=g(1+s)\,,\\ \beta_2&=-2ft\,,\\ \beta_3&=-gs/\boldsymbol{\pi}^2.\end{aligned} \tag{4.15}$$

Note the appearance of two independent coupling constants $f$ and $g$ in the above relations.

## V. SPECIAL DERIVATIVE-COUPLING MODELS

It is possible to obtain derivative-coupling models corresponding to those described in Sec. III. However, $L_{\pi N}$ and $\mathbf{J}_{\mu 5}$ now acquire a more complicated form, although $L_{\pi\pi}$ remains the same. The first two models described below have not been derived before with two coupling constants, while the third model has been given by Schwinger.

### Model A′

When $a=0$, the functions $b$, $c$, $d$, $e$, $s$, and $t$ have the same values as in Model A of Sec. III. Moreover, the functions $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$, and $\beta_3$ can be determined by substituting the values of $s$ and $t$, given by (3.2) and (3.3), into (4.14) and (4.15). We thus obtain

$$\begin{aligned}L_{\pi N}=&(g/2f\sqrt{\boldsymbol{\pi}^2})\sin(2f\sqrt{\boldsymbol{\pi}^2})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N\\ &-(1/\boldsymbol{\pi}^2)\sin^2(f\sqrt{\boldsymbol{\pi}^2})\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N\\ &+(g/\boldsymbol{\pi}^2)[1-(1/2f\sqrt{\boldsymbol{\pi}^2})\sin(2f\sqrt{\boldsymbol{\pi}^2})]\\ &\times(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N\,,\end{aligned} \tag{5.1}$$

$$\begin{aligned}\mathbf{J}_{\mu 5}=&g\cos(2f\sqrt{\boldsymbol{\pi}^2})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N-(f/\sqrt{\boldsymbol{\pi}^2})\sin(2f\sqrt{\boldsymbol{\pi}^2})\\ &\times\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\boldsymbol{\pi}N+(g/\boldsymbol{\pi}^2)[1-\cos(2f\sqrt{\boldsymbol{\pi}^2})]\\ &\times(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}+(1/\boldsymbol{\pi}^2)[1-(1/4f\sqrt{\boldsymbol{\pi}^2})\\ &\times\sin(4f\sqrt{\boldsymbol{\pi}^2})][\boldsymbol{\pi}^2\partial_\mu\boldsymbol{\pi}-(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}]\,.\end{aligned} \tag{5.2}$$

### Model B′

This model, for which $b=0$, corresponds to Model B of Sec. III. By following the same procedure as described above for Model A′, we find

$$\begin{aligned}L_{\pi N}=&g\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N-(1/2\boldsymbol{\pi}^2)[1-(1-4f^2\boldsymbol{\pi}^2)^{1/2}]\\ &\times\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N-(g/\boldsymbol{\pi}^2)[1-(1-4f^2\boldsymbol{\pi}^2)^{-1/2}]\\ &\times(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N\,,\end{aligned} \tag{5.3}$$

$$\begin{aligned}\mathbf{J}_{\mu 5}=&g(1-4f^2\boldsymbol{\pi}^2)^{1/2}\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N-2f^2\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\boldsymbol{\pi}N\\ &+(g/\boldsymbol{\pi}^2)[1-(1-4f^2\boldsymbol{\pi}^2)^{1/2}](\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &-4f^2(1-4f^2\boldsymbol{\pi}^2)^{-1/2}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}\\ &+[1-(1-4f^2\boldsymbol{\pi}^2)^{1/2}]\partial_\mu\boldsymbol{\pi}\,.\end{aligned} \tag{5.4}$$

### Model C′

The model, obtained by taking $c=0$, corresponds to Model C of Sec. III, and gives

$$\begin{aligned}L_{\pi N}=&g(1+f^2\boldsymbol{\pi}^2)^{-1}\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N\\ &-f^2(1+f^2\boldsymbol{\pi}^2)^{-1}\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N\,,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\mathbf{J}_{\mu 5}=&g(1-f^2\boldsymbol{\pi}^2)(1+f^2\boldsymbol{\pi}^2)^{-1}\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N\\ &-2f^2(1+f^2\boldsymbol{\pi}^2)^{-1}\bar{N}i\gamma_\mu\boldsymbol{\tau}\times\boldsymbol{\pi}N\\ &+2gf^2(1+f^2\boldsymbol{\pi}^2)^{-1}(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N)\boldsymbol{\pi}\\ &-2f^2(1+f^2\boldsymbol{\pi}^2)^{-2}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\boldsymbol{\pi}\\ &+[1-(1-f^2\boldsymbol{\pi}^2)(1+f^2\boldsymbol{\pi}^2)^{-2}]\partial_\mu\boldsymbol{\pi}\,,\end{aligned} \tag{5.6}$$

in agreement with Schwinger's results.[3]

It is, of course, possible to obtain additional derivative-coupling models as indicated in Sec. III.

The constants $g$ and $f$ appearing in the above models remain unrelated unless additional considerations are introduced into the formalism.[11] Moreover, since $L_{\pi N}$ can be expanded in powers of $f^2$, it is sufficient to require that $g$ and $f^2$ be real to ensure that $L_{\pi N}$ is Hermitian. We shall, however, for simplicity take $f$ itself to be real.

## VI. CHIRAL SYMMETRY AND TRANSFORMATION OF PION-NUCLEON COUPLING

Let us define a function $U(i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})$ as

$$U=1+s+2it\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi} \tag{6.1}$$

in terms of the functions $s(\boldsymbol{\pi}^2)$ and $t(\boldsymbol{\pi}^2)$ of Secs. II or IV which satisfy the relation

$$(1+s)^2+4t^2\boldsymbol{\pi}^2=1\,. \tag{6.2}$$

Then,

$$U^*U=(1+s)^2+4t^2\boldsymbol{\pi}^2=1\,,$$

so that $U$ is unitary and

$$U^{-1}=U^*=1+s-2it\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}. \tag{6.3}$$

[11] Such considerations have been advanced, e.g., by J. Schwinger, Phys. Rev. Letters 18, 923 (1967).

It can also be established, with the help of (6.2), that

$$U^{\pm 1/2}=(1+\tfrac{1}{2}s)^{1/2}\pm i(1+\tfrac{1}{2}s)^{-1/2}t\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\,, \quad (6.4)$$

which implies that

$$(U^{\pm 1/2})^*=U^{\mp 1/2}\,,\quad \gamma_\mu U^{\pm 1/2}=U^{\mp 1/2}\gamma_\mu\,. \quad (6.5)$$

Evidently, in the nonderivative-coupling case the sum of (2.3) and (2.5) takes the form

$$L_{0N}+L_{\pi N}=-\bar{N}(\gamma\cdot\partial+MU)N\,. \quad (6.6)$$

We also find that

$$\mathrm{Tr}(\partial_\mu U\partial_\mu U^{-1})=8[t^2\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}+(s'^2+4t'^2\boldsymbol{\pi}^2+4tt')(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2]\,,$$

where Tr denotes the trace of the products of isospin matrices, and consequently, by virtue of (2.19), (2.20), and (6.2),

$$-(1/16f^2)\,\mathrm{Tr}(\partial_\mu U\partial_\mu U^{-1})=-\tfrac{1}{2}(1-2b)\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}+c(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\,. \quad (6.7)$$

Thus, the sum of (2.2) and (2.4) can be expressed as

$$L_{0\pi}+L_{\pi\pi}=-(1/16f^2)\,\mathrm{Tr}(\partial_\mu U\partial_\mu U^{-1})+L'\,, \quad (6.8)$$

where

$$L'=-\tfrac{1}{2}m^2\boldsymbol{\pi}^2+a=-\tfrac{1}{2}m^2\int_0^{\boldsymbol{\pi}^2}\left(1-\frac{2a'}{m^2}\right)d(\boldsymbol{\pi}^2)$$

or, in view of (2.18),

$$L'=\frac{m^2}{4f}\int_0^{\boldsymbol{\pi}^2}\frac{s'}{t}\,d(\boldsymbol{\pi}^2)\,. \quad (6.9)$$

In the derivative-coupling case, the sum of $L_{0N}$ and $L_{\pi N}$ becomes, on the substitution of (4.14) into (4.1),

$$L_{0N}+L_{\pi N}=-\bar{N}(\gamma\cdot\partial+M)N+(g/f)t\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N+(s/2\boldsymbol{\pi}^2)\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N+(g/f)[2(1+s)t'-2s't+st/\boldsymbol{\pi}^2](\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N\,, \quad (6.10)$$

and it can be verified by direct calculation that (6.10) can be put in the form

$$L_{0N}+L_{\pi N}=-\bar{N}(\gamma\cdot\partial+M)N-[(f+g)/2f]\bar{N}\gamma_\mu(U^{1/2}\partial_\mu U^{-1/2})N-[(f-g)/2f]\bar{N}\gamma_\mu(U^{-1/2}\partial_\mu U^{1/2})N\,, \quad (6.11)$$

while $L_{0\pi}+L_{\pi\pi}$ is again given by (6.8).

*The relations (6.6), (6.11), and (6.8) exhibit chiral symmetry except for the symmetry-breaking term $L'$, which disappears only if the pion mass is unrealistically assumed to vanish.*[12]

By putting[13]

$$N=U^{-1/2}N'\,,\quad \bar{N}=\bar{N}'U^{-1/2}\,, \quad (6.12)$$

where the second relation follows from the first one by virtue of (6.5), it is possible to transform (6.6) into the form

$$L_{0N}+L_{\pi N}=-\bar{N}'(\gamma\cdot\partial+M)N'-\bar{N}'\gamma_\mu(U^{1/2}\partial_\mu U^{-1/2})N'\,, \quad (6.13)$$

which shows that the nonderivative pion-nucleon coupling is equivalent to a special case of the derivative coupling (6.11) corresponding to $g=f$. We have also investigated the Lagrangian formalism by taking a mixture of the nonderivative and derivative pion-nucleon couplings, and verified that the nonderivative coupling can again be eliminated by a unitary transformation so that the total pion-nucleon coupling is of the form (6.11).

We conclude that (6.11) together with (6.8) provides the most general form of the Lagrangian density for the pion-nucleon system that satisfies the divergence condition (1.1). However, the choice among the various models discussed in Sec. V must await further theoretical and experimental developments.

## APPENDIX A: SOLUTION OF EQUATIONS FOR NONDERIVATIVE-COUPLING CASE

The vanishing of the coefficient of each of the seven terms in (2.14) yields the differential equations

$$s'(1-e)=-2(1-2b-2b'\boldsymbol{\pi}^2)t\beta\,, \quad (A1)$$

$$t(1-e)=(1-2b)(1+s)\beta\,, \quad (A2)$$

$$t'(1-e)=-b'(1+s)\beta\,, \quad (A3)$$

$$1-2a'/m^2=(1-2b-2b'\boldsymbol{\pi}^2)/(1-e)\,, \quad (A4)$$

$$d=4b'(1-e)/(1-2b)-2e'\,, \quad (A5)$$

$$c=b'+\tfrac{1}{2}d(1-2b-2b'\boldsymbol{\pi}^2)/(1-e)\,, \quad (A6)$$

$$c'(1-e)=(1-2b-2b'\boldsymbol{\pi}^2)d'-b'(d+2e')\,. \quad (A7)$$

By putting $\boldsymbol{\pi}^2=0$ in (A2), we find, in view of (2.7) and (2.11),

$$\beta=t(0)\equiv f\,. \quad (A8)$$

It also follows from (A1)–(A3) that

$$2(1+s)s'=-4(t^2+2tt'\boldsymbol{\pi}^2)\,, \quad (A9)$$

which gives

$$(1+s)^2+4t^2\boldsymbol{\pi}^2=1\,, \quad (A10)$$

where the constant of integration is determined by the condition that $s=0$ for $\boldsymbol{\pi}^2=0$.

According to (A2) and (A3), we have

$$t'/t=-b'/(1-2b)$$

[12] Properties of the Lagrangian-density terms of the form (6.6), (6.11), and (6.8) under chiral transformations have been discussed in Ref. 9. Also note that these authors use chiral symmetry for the determination of the symmetry-preserving terms and the PCAC condition for the determination of the symmetry-breaking terms.

[13] According to Ref. 2, the above transformation can be interpreted as a redefinition of the nucleon field operator.

or

$$\ln t = \tfrac{1}{2}\ln(1-2b) + \ln f\,,$$

where we have used the condition that $b=0$ and $t=f$ for $\boldsymbol{\pi}^2=0$, and thus

$$1-2b = t^2/f^2. \qquad \text{(A11)}$$

On using (A8) and (A11), we also obtain from (A2)

$$e = 1-(t/f)(1+s). \qquad \text{(A12)}$$

The relation (A4) becomes, by virtue of (A1) and (A8),

$$1-2a'/m^2 = -s'/2ft\,, \qquad \text{(A13)}$$

while (A5) gives, with the help of (A11) and (A12),

$$d = (2/f)[ts' - t'(1+s)]. \qquad \text{(A14)}$$

By using (A11), (A12), and (A14), and simplifying by means of (A9) and (A10), it is possible to express (A6) as

$$c = -(2/f^2)(\tfrac{1}{4}s'^2 + tt' + t'^2\boldsymbol{\pi}^2) \qquad \text{(A15)}$$

or

$$c = (t^4 - \tfrac{1}{4}s'^2)/2f^2t^2\boldsymbol{\pi}^2. \qquad \text{(A16)}$$

It can also be verified that (A7) is indeed satisfied by $b$, $e$, $d$, and $c$, given by (A11), (A12), (A14), and (A15).

## APPENDIX B: SOLUTION OF EQUATIONS FOR DERIVATIVE-COUPLING CASE

The differential equations resulting from the vanishing of the coefficients of all the terms in (4.10) are given by

$$\beta_1(1-2b) = \alpha_1(1-e)\,, \qquad \text{(B1)}$$

$$\beta_3(1-2b) - 2b'(\beta_1+\beta_3\boldsymbol{\pi}^2) = \alpha_3(1-e)\,, \qquad \text{(B2)}$$

$$2[\beta_1' - \beta_2(\alpha_1+\alpha_3\boldsymbol{\pi}^2)](1-2b) = [2\alpha_1' - \alpha_3 - 2\alpha_2(\alpha_1+\alpha_3\boldsymbol{\pi}^2)](1-e)\,, \qquad \text{(B3)}$$

$$(\beta_3' - \alpha_2\beta_3 + \alpha_3\beta_2)(1-2b) - b'[\beta_3 + 2\beta_1' + 2\beta_3'\boldsymbol{\pi}^2 - 2\alpha_2(\beta_1+\beta_3\boldsymbol{\pi}^2)] = 0\,, \qquad \text{(B4)}$$

$$[\beta_3 + 2\alpha_2(\beta_1+\beta_3\boldsymbol{\pi}^2)](1-2b) - 2b'\boldsymbol{\pi}^2[\beta_3 + 2\alpha_2(\beta_1+\beta_3\boldsymbol{\pi}^2)] = -[2\alpha_1' - \alpha_3 - 2\alpha_2(\alpha_1+\alpha_3\boldsymbol{\pi}^2)](1-e)\,, \qquad \text{(B5)}$$

$$(\alpha_2\beta_2 - \alpha_1\beta_3)(1-2b) + b'[\beta_2 + 2\alpha_1(\beta_1+\beta_3\boldsymbol{\pi}^2)] = (\alpha_2^2 - \alpha_2' - \alpha_1\alpha_3)(1-e)\,, \qquad \text{(B6)}$$

$$(\beta_3 + 2\alpha_1\beta_2 - 2\alpha_2\beta_1)(1-2b) = 0\,, \qquad \text{(B7)}$$

$$(\beta_2' - \alpha_2\beta_2 + \alpha_3\beta_1)(1-2b) = -(\alpha_2^2 - \alpha_2' - \alpha_1\alpha_3)(1-e)\,, \qquad \text{(B8)}$$

$$(\beta_2 + 2\alpha_1\beta_1 + 2\alpha_2\beta_2\boldsymbol{\pi}^2)(1-2b) = 2(\alpha_1^2 + \alpha_2 + \alpha_2^2\boldsymbol{\pi}^2)(1-e)\,, \qquad \text{(B9)}$$

together with the relations (A4)–(A7).

By subtracting $2\alpha_2$ times (B1) and $(2\alpha_2\boldsymbol{\pi}^2+1)$ times (B2) from (B5), we obtain the simple relation

$$\beta_1 b' = -\alpha_1'(1-e)$$

or, on using (B1),

$$\alpha_1'/\alpha_1 = -b'/(1-2b)\,,$$

which has the solution

$$\alpha_1 = g(1-2b)^{1/2}\,, \quad \alpha_1(0) \equiv g. \qquad \text{(B10)}$$

It then also follows from (B1) that

$$\beta_1 = g(1-e)(1-2b)^{-1/2}\,, \quad \beta_1(0) = g. \qquad \text{(B11)}$$

By adding (B3), (B5), (B7), and $2\boldsymbol{\pi}^2$ times (B4), we find

$$(\beta_1' + \beta_3 + \beta_3'\boldsymbol{\pi}^2)(1-2b-2b'\boldsymbol{\pi}^2) = 0$$

or, $(1-2b-2b'\boldsymbol{\pi}^2)$ being nonzero because it obviously cannot vanish for $\boldsymbol{\pi}^2=0$,

$$\beta_1' + \beta_3 + \beta_3'\boldsymbol{\pi}^2 = 0\,,$$

which can be integrated immediately to yield

$$\beta_1 + \beta_3\boldsymbol{\pi}^2 = g\,, \qquad \text{(B12)}$$

where the constant of integration is determined by the condition $\beta_1(0)=g$, obtained above. Substitution of (B11) into (B12) gives

$$\beta_3 = (g/\boldsymbol{\pi}^2)[1-(1-e)(1-2b)^{-1/2}]\,, \qquad \text{(B13)}$$

and a further substitution of (B12) and (B13) into (B2) gives

$$\alpha_3 = (g/\boldsymbol{\pi}^2)[(1-2b-2b'\boldsymbol{\pi}^2)/(1-e) - (1-2b)^{1/2}]. \qquad \text{(B14)}$$

According to (B7),

$$\beta_3 + 2\alpha_1\beta_2 - 2\alpha_2\beta_1 = 0\,, \qquad \text{(B15)}$$

while (B9) can be reduced, with the help of (B1), to the form

$$\alpha_1\beta_2 - 2\alpha_2\beta_1 + 2\alpha_2\boldsymbol{\pi}^2(\alpha_1\beta_2 - \alpha_2\beta_1) = 0$$

or, in view of (B15),

$$\beta_3 + 2\alpha_2(\beta_1+\beta_3\boldsymbol{\pi}^2) = 0. \qquad \text{(B16)}$$

By substituting the earlier results into (B15) and (B16), we can express $\alpha_2$ and $\beta_2$ in terms of $b$ and $e$ as

$$\alpha_2 = (1/2\boldsymbol{\pi}^2)[(1-e)(1-2b)^{-1/2} - 1]\,, \qquad \text{(B17)}$$

$$\beta_2 = (1/2\boldsymbol{\pi}^2)[(1-e)^2/(1-2b) - 1](1-2b)^{-1/2}. \qquad \text{(B18)}$$

We have now found all of the nucleon functions in terms of $b$ and $e$. However, we have used only six of our nine equations, because only a linear combination of (B3) and (B4) has been used, while (B6) and (B8) have not been used at all. When the results for the nucleon functions are substituted into (B3), (B4), (B6), and (B8), we obtain in each case the same relationship between $b$ and $e$, given by

$$\left[\frac{1-(1-e)^2/(1-2b)}{(1-2b)\boldsymbol{\pi}^2}\right]' = 0\,,$$

which yields

$$(1-e)^2/(1-2b)+\lambda(1-2b)\boldsymbol{\pi}^2=1\,, \qquad \text{(B19)}$$

where $\lambda$ is the constant of integration. It is convenient to introduce $f$, $s$, and $t$, defined as

$$4f^2=\lambda\,,\quad 1+s=(1-e)/(1-2b)^{1/2}\,,\quad t=f(1-2b)^{1/2}\,, \qquad \text{(B20)}$$

so that (B19) can be expressed in the form

$$(1+s)^2+4t^2\boldsymbol{\pi}^2=1\,. \qquad \text{(B21)}$$

By using (B20) and (B21) and remembering that the relations (A4) to (A7) also hold in the present case, the functions $a$, $b$, $c$, $d$, and $e$ can be expressed in terms of $s$ and $t$ in the same form as in Appendix A. Moreover, the functions $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$, and $\beta_3$, when expressed in terms of $s$ and $t$, give the relations (4.14) and (4.15).

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# Nonlinear Lagrangian Transformations

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In nonlinear Lagrangian schemes it is useful to transform the nonderivative pion-nucleon coupling into the derivative coupling. The effect of such a transformation on renormalization constants and closed-loop diagrams is investigated by direct calculations of the second-order nucleon and pion self-energies and the fourth-order nucleon-nucleon interaction with the inclusion of the renormalization diagrams. By carefully combining the contributions of appropriate diagrams, the complete equivalence of the nonderivative and derivative couplings is demonstrated.

## I. INTRODUCTION

TRANSFORMATION of the pion-nucleon couplings[1] has recently been carried out for nonlinear Lagrangian schemes by various authors.[2,3] It is thus found that the nonderivative coupling, given by

$$L_{\pi N}=-2fM\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N+2f^2M\boldsymbol{\pi}^2\bar{N}N-2nf^3M\boldsymbol{\pi}^2\bar{N}i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N+O(f^4)\,, \qquad (1)$$

can be transformed into the derivative form

$$L_{\pi N}'=f\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N-f^2\bar{N}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}N+2(1+n)f^3(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}N+nf^3\boldsymbol{\pi}^2\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}N+O(f^4)\,, \qquad (2)$$

where $M$ is the nucleon mass, $f$ is the coupling constant, and the value of the dimensionless parameter $n$ depends on the choice of the Lagrangian scheme.[4] These schemes also involve nonlinear pion-pion couplings which, however, are not affected by the transformation and will not be considered here.

The conversion of (1) into (2) essentially involves a redefinition of the nucleon field by means of a unitary transformation. The physical interpretation of the resulting nucleon field is somewhat obscure, since it has associated with it any number of pion fields, which appear in the series expansion of the unitary transformation function. Therefore, while there is general agreement that the tree-diagram contributions remain unchanged under the above transformation of couplings, the situation with regard to diagrams with closed loops is not entirely clear. It is also doubtful whether the renormalization constants remain unaltered, because the derivative coupling at least superficially appears to be more divergent than the nonderivative coupling.

In order to clarify and reinforce the general theoretical arguments, we shall investigate the equivalence of the nonderivative and derivative couplings by direct calculations of the second-order nucleon and pion self-energies and the fourth-order nucleon-nucleon interaction. As we shall see, the demonstration of equivalence by direct calculations requires extensive manipulations, and it brings out several interesting features that afford a deeper understanding of the relationship between the two couplings.

Besides using the standard notation $\boldsymbol{\pi}$, $N$, and $\bar{N}$ for the pion and nucleon field operators, we shall denote the pion mass as $m$ to distinguish it from the nucleon mass $M$. We shall also take $c=\hbar=1$.

## II. NUCLEON AND PION SELF-ENERGIES

The second-order self-energy diagrams due to the nonlinear pion-nucleon coupling are shown in Fig. 1, where the "leaf" diagrams [Figs. 1(b) and 1(d)] arise

[1] For the older work on this subject, see S. S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961), p. 301.
[2] S. Weinberg, Phys. Rev. Letters **18**, 188 (1967).
[3] For a general discussion, see S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969), and earlier papers quoted there.
[4] For instance, $n$ takes the values $-\frac{2}{3}$, 0, and $-1$, respectively, in the models referred to as A, B, and C by S. N. Gupta and W. H. Weihofen, preceding paper, Phys. Rev. D **2**, 1123 (1970).