High-Energy Deep-Inelastic Electron-Nucleon Scattering in a Renormalizable Theory*

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We have studied high-energy deep-inelastic electron-nucleon scattering in a neutral pseudoscalar-meson theory by summing an infinite set of diagrams. The diagrams analyzed are straight ladder unitary diagrams, with pions as the rungs. Explicit lower-order calculations indicate that this set of diagrams gives the leading $\ln|q^2|$ contribution, where q^2 is the momentum transfer squared, provided that nucleon-antinucleon pair creations and nucleon vertex corrections are ignored. The main results are: (1) The final hadrons fall naturally into two jets; (2) the Bjorken scaling law breaks down; (3) the number of pions increases as $\ln|q^2|$; and (4) a longitudinal impact-parameter space is realized. Some possible experimental consequences are deduced.

I. INTRODUCTION

THERE is great interest in the study of deep e^-p inelastic and e^-e^+ annihilation processes through various models.[1-7] The parton model, originally suggested by Feynman[1] and later developed by Bjorken,[2] gives a very appealing physical picture for these processes. It predicts many interesting features of e^-p inelastic scattering. One of the important predictions of this model is the validity of the Bjorken scaling law[8]: The ep inelastic form factors W_1 and νW_2 in the limit of large momentum transfer and energy transfer q^2, $m\nu$ are functions of their ratio $q^2/m\nu$ only. This scaling law is obeyed at least approximately by experiment.[9]

Although the original parton model is a physical picture of "bits" of the hadron scattering independently, Drell, Levy, and Yan[3] showed that a "parton-model" result can be derived for a large class of canonical field theories. Their results are based on the existence of certain infinite momentum limits; these conditions are satisfied in their model by introducing a transverse momentum cutoff so that there exists an asymptotic region in which q^2 and $m\nu$ can be made larger than the transverse momenta of all particles involved. In particular, they studied the cutoff neutral pseudoscalar-meson theory in detail.

By introducing a cutoff on the transverse momentum of the pions, the ps meson theory becomes a super-renormalizable theory rather than a renormalizable theory. In this paper we shall study the form factor W_2 in deep-inelastic ep scattering in a neutral ps meson theory *without* cutoff.

The main set of diagrams we considered are shown in Fig. 1. This is a set of ladder diagrams. The possible nucleon-antinucleon pair creations and pion vertex corrections (including nucleon self-energy corrections) are ignored, i.e., we are analyzing bremsstrahlung-type processes. It is clear that the vertex corrections at various vertices should be included. At present we do not know how this can be done efficiently. As we shall see later, the largest momentum transfer takes place at the photon vertex rather than at the individual pion vertices. We therefore conjecture that the over-all corrections to these vertices may be taken care of by including the nucleon electromagnetic form factors alone [Fig. 1(b)].

To each order in the pion-nucleon coupling constant g, we keep only the leading contribution in $\ln q^2$ in our calculation of $W_2(q^2,\nu)$. $W_{1,2}(q^2,\nu)$ are, of course, the form factors of ep inelastic scattering. In the forward, deep-inelastic regions,

$$m^2 \ll -q^2|, \quad 2m\nu \ll s,$$

where s is the square of the c.m. energy and m is the nucleon mass. Here only W_2 contributes.

In order to justify our choice of the diagrams of Fig. 1, we have looked explicitly in lower order at other diagrams. In particular, we found that diagrams with crossed rungs, with pions interacting between nucleons on different sides of the currents, or with pions joining over two or more vertices (see discussion in Sec. III), are at least order $\ln(|q^2|/m^2)$ smaller than the leading contribution of Fig. 1. This indicates that the straight ladder diagrams of Fig. 1 may well be the only leading diagrams (ignoring the nucleon pair creations and vertex corrections). Hence, it may not be a bad approximation to consider only the straight ladder diagrams. It is worth mentioning here that because we are summing and averaging over final and initial spins, the pseudo-

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[2] J. D. Bjorken, paper presented at the New York meeting of the American Physical Society, 1969 (unpublished); J. D. Bjorken and E. A. Paschos, Phys. Rev. **185**, 1975 (1969).
[3] S. D. Drell, D. J. Levy, and T. M. Yan, Phys. Rev. Letters **22**, 744 (1969); Phys. Rev. **187**, 2159 (1969); Phys. Rev. D **1**, 1035 (1970).
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[5] J. J. Sakurai, Phys. Rev. Letters **22**, 981 (1969); C. Cho, G. Gounaris, and J. Sakurai, Phys. Rev. **186**, 1734 (1969); W. K. Tung, Phys. Rev. Letters **23**, 1531 (1969); J. D. Sullivan (private communication).
[6] L. S. Brown, *Lectures in Theoretical Physics* (Gordon and Breach, New York, 1970).
[7] H. T. Nieh, State University of New York at Stony Brook report, 1969 (unpublished).
[8] J. D. Bjorken, Phys. Rev. **179**, 1547 (1969).
[9] E. D. Bloom *et al.*, Phys. Rev. Letters **23**, 930 (1969); M. Breidenbach *et al.*, *ibid.* **23**, 935 (1969).

scalar theory is effectively a scalar theory. We would also like to mention the work of Adler and Tung,[10] who have studied the leading logarithmic terms in the infrared region for all fourth-order diagrams in a related theory. Their conclusions agree with ours for the set of diagrams studied here.

The results of our calculation can be summarized as follows[11]:

(a) The final pions and proton fall naturally into two groups (jets) in which particles in a given group move close to each other. The first group contains all pions emitted *before* the proton interacts with the current (the outer "rainbow" of Fig. 1), while the second group contains the final proton and pions emitted *after* the proton interacts with the current (the inner rainbow of Fig. 1). The longitudinal momentum of the proton at the time of interaction, measured as a fraction x of the total longitudinal momentum, is still governed by the same $\delta(|q^2|/x-2m\nu)$ as in the parton model. This x measures the fraction of the longitudinal momentum left over by all the pions emitted *before* the proton interacts with the currents. The same conclusion was reached earlier by Drell, Levy, and Yan in the cutoff pseudoscalar-meson theory.

(b) The scaling law $\nu W_2=\nu W_2(q^2/2m\nu)$ is violated in an interesting manner. For a process with n final pions, the *partial* W_2 contains a q-dependent factor $[\ln(q^2/m^2)]^n$. Other than in $\delta(|q^2|/x-2m\nu)$, the partial W_2 for n pions emitted after the current insertion contains no x dependence; pions emitted before the current insertion introduce further x dependence.

(c) The total form factor W_2 is formed by summing over all pion ladders. As mentioned above, W_2 factors into two parts, each associated with one group of particles:

$$\lim_{\substack{q^2,\nu\to\infty\\ q^2/\nu \text{ fixed}}} W_2(q^2,\nu)=\int_0^1 dx\, 2mA_0(q^2,x) \times A_i(q^2)\delta\left(\frac{|q^2|}{x}-2m\nu\right),$$

where

$$\int_0^1 A_0(q^2,x)x^{\lambda-1}dx=\exp\left(\frac{g^2}{16\pi^2}\frac{\ln(|q^2|/m^2)}{\lambda(\lambda+1)}\right)-1\,,$$

$$A_i(q^2)=\exp\left(\frac{g^2}{32\pi^2}\ln\frac{|q^2|}{m^2}\right).$$

The explicit structure of this result is presumably quite model dependent. However, the facts that A_0 has a simple exponential structure in the Mellin transform space and that A_0 and A_i have explicit q^2 dependence may be the general properties of any renormalizable field theory. In analog to the eikonal form in the impact-parameter space, the Mellin transform space for the longitudinal momentum has a profound physical meaning of its own. It may be interpreted as a "longitudinal impact space."

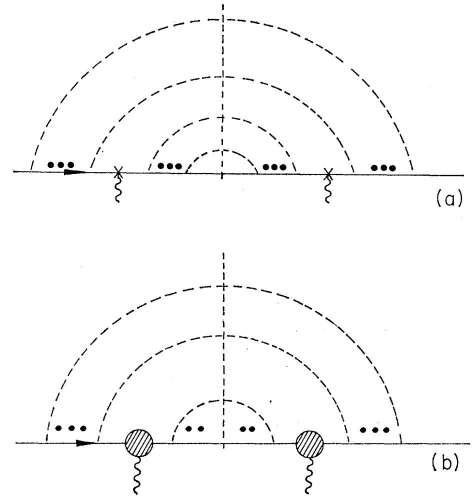

FIG. 1. Set of straight ladder unitary diagrams considered: (a) Without any vertex correction; (b) with form factor included.

(d) The number distribution in the pions emitted after the current insertion is a Poisson distribution. Because of the extra x dependence the distribution for the pions emitted before the current insertion is not Poisson; however, it is Poisson in the Mellin transform space. (The experimentally observed distribution is, of course, in the x space.) For the pions emitted after the current, the average number is easily calculated. For the pions emitted before the current, the average number is not simple. In both cases, however, $\bar{n}$ depends logarithmically on q^2.

(e) The longitudinal momentum distribution of the pions does *not* obey the simple dx/x rule, as suggested by phase space alone, because the integrand picks up extra x factors from the amplitude. Various experimental moments of this distribution are easily calculated. In more complicated field theories, it may be hard to predict the x dependence of the pion momenta.

(f) The largest momentum transfer takes place at the photon vertices rather than the individual pion vertices. This would indicate that inclusion of pure vertex corrections for the pions might have only small effect, but that corrections to the photon vertex alone may be important. In this sense it is simple to include such corrections; they are indicated in Fig. 1(b). Inclusion of these factors simply multiplies W_2 given above by squares of form factors. Physically it is clear that the possible momentum transfer in the process must be damped by the nucleon form factor.

(g) Inclusion of multiphoton exchange in the production process, rather than one-photon exchange, can be made by using the infinite-momentum technique.

[10] S. L. Adler and W. K. Tung, Phys. Rev. D 1, 2846 (1970).
[11] The main results of this paper have appeared as a Letter: S. J. Chang and P. M. Fishbane, Phys. Rev. Letters 24, 847 (1970).

Instead of the single-photon-exchange amplitude, one has an eikonal form whose driving term comes from one-photon exchange. The form of W_2 will be unchanged.

(h) Finally, we should note that accommodation of a factor like $\exp[(g^2/32\pi^2)\ln(|q^2|/m^2)]$ would not be a severe strain on the data. For $|q^2|$ running from 1 to 5, this factor varies from 1 to ~ 2.6, which is consistent with the present data.

The model studied here can easily be used to study the e^+e^- annihilation process. The results of this work will be published elsewhere.

The paper is organized as follows. In Sec. II, we review the kinematics for the e^-p inelastic scattering. In Sec. III, the contributions for various lower-order diagrams are analyzed, and the leading terms are identified. Only the leading diagrams (i.e., the rainbow diagrams of Fig. 1) are studied in Sec. IV, and the general result is obtained. In Sec. V, the physical meaning of the longitudinal impact space is examined, and possible experimental consequences are deduced. Finally, Sec. VI includes further discussions on various results. We also include two appendices, in the first of which we discuss the superrenormalizable $\lambda\phi^3$ theory.

II. STRUCTURE FUNCTIONS AND KINEMATICS

In this section we want to define more precisely the quantities we calculate and the framework in which we calculate them. Although the definition of the structure functions which describe inelastic scattering is not new, we give a brief description of them for completeness. We give a detailed description of the kinematics appropriate for so-called "infinite-momentum frame" calculations, as well as a brief review of the infinite-momentum techniques.

As usual, the process we are interested in is $e^\pm+\text{nucleon}\to e^\pm+(\text{hadrons})$, in which the energy and angle of the final electron are known, while no information about the hadron state is available. In particular, the process goes by one-photon exchange; we show this, with momentum labels, in Fig. 2(a). The two quantities upon which this effectively two-body$\to$two-body process depends are taken to be the invariant (spacelike) momentum transfer q^2 and the energy change of the electron in the lab frame ν.

The covariant decomposition of the inelastic process is well known.[12] If the momentum of the final hadron state $|n\rangle$ is written as $p'=\sum p_i$, with p_i being the momentum for individual hadron, then the differential cross section in the final energy of the electron is

$$\frac{d^2\sigma^{(n)}}{dE'd\Omega_{l'}}=\frac{1}{(2\pi)^3}\frac{e^4}{(q^2)^2}\frac{l'}{l}M_{\alpha\beta}\,\mathrm{Im}T_{(n)}{}^{\alpha\beta}(\nu,q^2)\,, \quad (2.1)$$

[12] See, e.g., S. Drell and J. Walecka, Ann. Phys. (N. Y.) **28**, 18 (1964); see also Refs. 2 and 3.

where

$$M_{\alpha\beta}=\tfrac{1}{4}\,\mathrm{Tr}(\gamma_\alpha \boldsymbol{l}\gamma_\beta \boldsymbol{l}')=l_\alpha l_\beta'+l_\alpha' l_\beta+\tfrac{1}{2}q^2 g_{\alpha\beta}$$

comes from the electron-photon vertex and where

$$\frac{1}{\pi}\,\mathrm{Im}T_{(n)}{}^{\alpha\beta}(\nu,q^2)=(2\pi)^3\frac{E_p}{m}\int\prod_i d^3p_i\,\langle p|j^\alpha(0)|n\rangle \times\langle n|j^\beta(0)|p\rangle(2\pi)^3\delta^4(\textstyle\sum p_i-p-l)\,. \quad (2.2)$$

The l and l' are initial and final lepton momentum, m is the proton mass, and p is the momentum of the initial proton. A sum over all possible internal quantum numbers of the particles in $|n\rangle$ is understood in (2.2). Because of the symmetry of $M_{\alpha\beta}$, only symmetric terms contribute to $\mathrm{Im}T^{\alpha\beta}$, which is then real. A general tensor structure, together with current conservation $q_\alpha T^{\alpha\beta}=0$, gives

$$\frac{1}{\pi}\,\mathrm{Im}T_{(n)}{}^{\alpha\beta}(\nu,q^2)=\frac{1}{m^2}W_2{}^{(n)}(\nu,q^2)\left(p^\alpha-\frac{p\cdot q}{q^2}q^\alpha\right)\times\left(p^\beta-\frac{p\cdot q}{q^2}q^\beta\right)-W_1{}^{(n)}(\nu,q^2)\left(g^{\alpha\beta}-\frac{q^\alpha q^\beta}{q^2}\right). \quad (2.3)$$

[We shall denote all quantities without (n) as the corresponding quantities summed over (n). However, we sometimes leave off a subscript (n), if there is no possibility of confusion.] $T^{\alpha\beta}$ is the well-known M function for forward Compton scattering. In our case, for q^2 spacelike, $\mathrm{Im}T^{\alpha\beta}$ is the imaginary part of forward virtual spin-independent Compton scattering. $W_{1,2}\equiv\sum W_{1,2}{}^{(n)}$ are the structure functions for the inelastic process. W_1 contributes only to the Im part of the Compton amplitude for virtual *transverse* photons, while W_2 contributes for *both* transverse and longitudinal virtual photons. In terms of these quantities, the inelastic cross section is

$$\frac{d^2\sigma}{dE'd\Omega_{l'}}=\frac{1}{(2\pi)^2}\frac{e^4}{(q^2)^2}\times l'^2[W_2(\nu,q^2)\cos^2(\tfrac{1}{2}\theta)+2W_1(\nu,q^2)\sin^2(\tfrac{1}{2}\theta)]\,, \quad (2.4)$$

θ being the scattering angle. Diagrammatically, the structure functions are therefore calculated from the picture given as Fig. 2(b).

We shall be interested in the kinematic region $s\gg -q^2$, $\nu\gg m$, where $-q^2$ and $m\nu$ are of the same order. In this region we shall see that it is possible to find a frame in which the infinite-momentum techniques[13] are appropriate. Therefore we give a brief review of the technique. Instead of denoting a 4-vector a^μ as (a^0,a^1,a^2,a^3), we denote it as $(a_+,\mathbf{a},a_-)$, where $a_\pm=a^0\pm a^3$

[13] S. Weinberg, Phys. Rev. **150**, 1313 (1966); L. Susskind and G. Frye, *ibid.* **165**, 1535 (1968); **165**, 1547 (1968); **165**, 1553 (1968); K. Bardakci and M. B. Halpern, *ibid.* **176**, 1686 (1968); S. J. Chang and S. Ma, *ibid.* **180**, 1506 (1969); **188**, 2385 (1969).

and $\mathbf{a}=(a^1,a^2)$ is a 2-vector.[14] In terms of this decomposition, $a\cdot b=\frac{1}{2}(a_+b_-+a_-b_+)-\mathbf{a}\cdot\mathbf{b}$, so that the mass-shell condition for a^μ reads $a_+a_--\mathbf{a}^2=m^2$. Lorentz boosts along the 3-direction take a simple form: A boost with rapidity β leads to

$$a_\pm{}'=e^{\pm\beta}a_\pm\,,\quad \mathbf{a}'=\mathbf{a}\,.$$

Finally, a particle of momentum p moving rapidly along the positive 3-direction has a large p_+ and a small (of order $1/p_+$) p_-, and vice versa for a particle moving along the negative 3-direction.

We now define the kinematics of the inelastic process in terms of variables in the new decomposition. We define s as usual by $s=(p+l)^2$. Take the initial nucleon as traveling in the positive z direction. Then, in the center-of-mass system, up to terms of $O(m^2/s)$,

$$p^\mu=(\sqrt{s},0,m^2/\sqrt{s})\,,$$

while the exchanged photon has momentum

$$q^\mu=(O(1/\sqrt{s}),\mathbf{q},2m\nu/\sqrt{s})\,,$$

where $\mathbf{q}^2=-q^2$. We obtain $q_-=2m\nu/\sqrt{s}$ from

$$\begin{aligned} m\nu=p\cdot q&=\tfrac{1}{2}(p_+q_-+p_-q_+)-\mathbf{p}\cdot\mathbf{q}\\ &=\tfrac{1}{2}[(\sqrt{s})q_-+O(1/s)]\,. \end{aligned}$$

It is easy to see that q_+ is of order $O(1/\sqrt{s})$, since the photon must interact with the electron whose plus component is of $O(1/\sqrt{s})$. Finally, we take a Lorentz transformation (boost) along the z axis to a frame such that $p_+=1$. (Equivalently, we may view this as a scale transformation $p_+\rightarrow p_+/\sqrt{s}=1$, $p_-\rightarrow p_-\sqrt{s}$, $\mathbf{p}\rightarrow\mathbf{p}$.) This transformation leaves us with

$$\begin{aligned} p^\mu&=(1,0,m^2)\,,\\ q^\mu&=(O(1/s),\mathbf{q},2m\nu)\approx(0,\mathbf{q},2m\nu)\,. \end{aligned}$$

This is the standard frame in which we work. We wish to remark that in the large-s limit, the large contribution to the leptonic part $M_{\alpha\beta}$ comes from $\alpha=\beta=-$. Hence, the dominant contribution to the hadronic part $\mathrm{Im}T^{\alpha\beta}$ is $\alpha=\beta=+$. Since $g_{++}=q_+=0$, we shall actually calculate W_2 by means of diagrams like Fig. 2(c). We adopt a frame with initial $p_+=1$ rather than the lab frame because (1) this frame is related trivially to the c.m. frame by a simple boost and (2) the p_+'s of the intermediate particles all lie between 0 and 1 (actually, $\sum p_+=1$), and represent the fractions of the longitudinal momenta taken by these particles in the c.m. frame. The fraction of the longitudinal momentum and the transverse momentum $\mathbf{p}$ turn out to be the most convenient momentum variables to describe the high-energy scatterings.

III. LOW-ORDER CALCULATIONS

In this section we wish to illustrate some of our methods of calculation on low-order diagrams. We shall see how the infinite-momentum techniques simplify calculations, which types of diagrams are asymptotically large compared to others, and how the scaling law may break down when a transverse cutoff is not imposed on the particles produced.

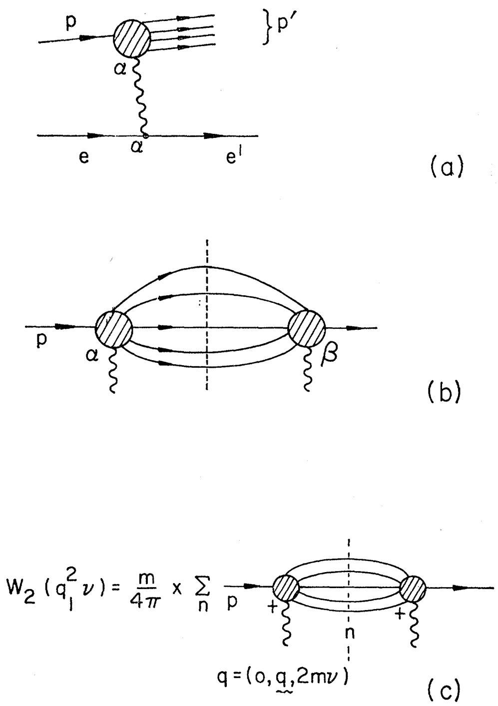

FIG. 2. (a) Picture of ep inelastic scattering; (b) general unitary diagram for $\mathrm{Im}T_{\alpha\beta}$; (c) graphical representation for W_2 as the imaginary part of a forward Compton scattering.

As remarked in the Introduction, we consider a neutral pseudoscalar-meson theory and ignore nucleon-antinucleon pair production. These limitations are not imposed by asymptotic considerations, but instead are necessary to make the calculations tractable. Thus the exchanged photon (actually the electromagnetic current) hooks only to the incoming nucleon line. We are essentially considering a bremmstrahlung model.

We have chosen four calculations which illustrate the salient features of our theory. These are illustrated, together with appropriate labelings, in Figs. 3(a)–3(d). We shall show in particular that 3(a) is large compared to 3(b)–3(d). The nucleons of mass m are represented by solid lines, the incoming photons by wavy lines, the currents by crosses, and the emitted pions of mass μ by dashed lines. We see in all four diagrams that the lepton current carries a large minus component but a vanishing plus component, so that only the plus component of the hadron part survives in the $\gamma^\mu j_\mu{}^{(\mathrm{lepton})}$ coupling.

We study W_2 as a function of q^2 and ν with $-q^2$, $m\nu\gg m^2$, but with a finite ratio $q^2/m\nu$. We then find that the N-pion contributions to $W_2(q,\nu)$ go like powers of $\ln|q^2|$. The meaning of "leading term" here deserves

[14] The set of variables $p_\pm$, $\mathbf{p}$ are also known as Sudakov variables. Further kinematics and transformation properties of these variables can be found in the articles of Chang and Ma in Ref. 13.

further explanation. As N, the number of $\bar{N}N\pi$ vertices, increases, there is no ceiling on the maximum power of $\ln|q^2|$. However, for a fixed N, i.e., a fixed power in g^2, the largest power in $\ln|q^2|$ is limited to $(\ln|q^2|)^N$. It is in this sense that we call a particular term "leading in $\ln|q^2|$." Mathematically, we are looking for terms which are leading in $g^2\ln|q^2|$ as $g^2\to 0$. It is in general unclear whether analysis of only the leading terms in $\ln|q^2|$ leads to a meaningful answer. Nevertheless, we keep only these leading terms (see also our discussion at the beginning of Sec. IV).

A. Diagram 3(a)

Since this diagram is symmetric, W_2 is actually obtained from an "amplitude" $\mathfrak{M}$ whose absolute value is squared and whose final-state momenta are integrated over. The integration contains δ functions expressing momentum conservation and the on-mass-shell condition of the intermediate states. For inelastic scattering from unpolarized protons, we must average the initial spin, and, of course, we must sum the "final" spin.

We label each of the final particles by $k_i=(x_i,\mathbf{k}_i,[\mathbf{k}_i^2+\mu^2(m^2)]/x_i)$, where x is the fraction of the longitudinal momentum taken by the particle in the ep c.m. frame. The "amplitude" for the process is

$$\begin{aligned}\mathfrak{M}&=g^2\bar{u}(k_3)\gamma_5(\not{k}_2+\not{k}_3+m)\gamma_+(\not{p}-\not{k}_1+m)\gamma_5 u(p)\\&\quad\times\{[(k_2+k_3)^2-m^2+i\epsilon]\\&\qquad\times[(p-k_1)^2-m^2+i\epsilon]\}^{-1}\\&=-g^2\bar{u}(k_3)\gamma_5\not{k}_2\gamma_+\not{k}_1\gamma_5 u(p)\\&\quad\times\{[(k_2+k_3)^2-m^2+i\epsilon]\\&\qquad\times[(p-k_1)^2-m^2+i\epsilon]\}^{-1}.\end{aligned}\tag{3.1}$$

Thus

$$\begin{aligned}\langle|\mathfrak{M}|^2\rangle&=\tfrac{1}{2}\sum_{\text{initial}}\sum_{\text{final}}|\mathfrak{M}|^2\\&=\tfrac{1}{2}g^4\,\mathrm{Tr}[(\not{k}_3+m)\gamma_5\not{k}_2\gamma_+\not{k}_1\gamma_5(\not{p}+m)\gamma_5\not{k}_1\gamma_+\not{k}_2\gamma_5]\\&\quad\times\{[(k_2+k_3)^2-m^2+i\epsilon][(p-k_1)^2-m^2+i\epsilon]\}^{-2}.\end{aligned}\tag{3.2}$$

The trace is simple. It is helpful to recognize that only the minus component of γ_μ survives when sandwiched between two γ_+'s:

$$\gamma_+{}^2=0=\gamma_+\gamma\gamma_+,\quad \gamma_+\gamma_-\gamma_+=4\gamma_+.$$

Hence,

$$\gamma_+\not{p}\gamma_+=2p_+\gamma_+.$$

Then we find for the trace in (3.2)

$$\begin{aligned}\tfrac{1}{4}\,\mathrm{Tr}[\cdots]&=2(2x_2k_2\cdot k_3-x_3\mu^2)(2x_1k_1\cdot p-\mu^2)\\&=8x_1x_2p\cdot k_1k_2\cdot k_3-4x_2\mu^2k_2\cdot k_3\\&\quad-4x_1x_3\mu^2p\cdot k_1+x_3\mu^4.\end{aligned}\tag{3.3}$$

The factorization of the numerator into two parts, one involving only the momentum variables before the current insertion and one involving only the momentum variables after the current insertion, is an important feature of our calculation. When we integrate over $d^4k_1d^4k_2d^4k_3$, the first term in (3.3) gives the largest contribution in $\ln|q^2|$. Thus we keep only this piece of the trace. This is the general philosophy of our field-theory calculations, i.e., we keep only the leading logarithmic terms for any given order of coupling constants. We sometimes refer to these leading terms as the "most divergent contribution" at large q^2. The last term of (3.3), which contains no momentum factors at all, represents the "most convergent" part of the integral. It is similar to what one would have in $\lambda\phi^3$ theory (see Appendix A). This term does not lead to any $\ln|q^2|$ term in the final expression. To complete the initial study of diagram 3(a), we include the phase-space factor (Appendix B)

$$\begin{aligned}P&=8\pi^2\frac{dx_1}{4\pi x_1}\frac{dx_2}{4\pi x_2}\frac{dx_3}{4\pi x_3}\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\\&\quad\times\delta(x_1+x_2+x_3-1)(2\pi)^2\delta^2(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{q})\\&\quad\times\delta\left(\frac{\mathbf{k}_1{}^2+\mu^2}{x_1}+\frac{\mathbf{k}_2{}^2+\mu^2}{x_2}+\frac{\mathbf{k}_3{}^2+m^2}{x_3}-2m\nu-m^2\right).\end{aligned}\tag{3.4}$$

The factor dx/x can be interpreted as the usual phase-space factor dk/E in the infinite-momentum frame. The phase-space factor carries over in obvious generalization when more pions are emitted.

The combination of (3.2), the first term of (3.3), and (3.4) gives us $W_2|_{\text{Fig. 3(a)}}$. Before we explicitly calculate W_2, we shall now show how a change of variables can separate the "inner rainbow" (the pion emitted after the first current insertion) from the "outer rainbow" (the pion emitted before the first current insertion). This is important because it will generalize when we study the production of more pions. We make a change of variables for particles in the inner rainbow, i.e., for k_2 and k_3 only (the new variables are temporarily denoted by a prime):

$$\begin{aligned}x_i&=x_i{}',\\\mathbf{k}_i&=\mathbf{k}_i{}'+\frac{x_i}{1-x_1}(\mathbf{q}-\mathbf{k}_1),\\k_{i-}&=k_{i-}{}'+2\mathbf{k}_i{}'\cdot\frac{\mathbf{q}-\mathbf{k}_1}{1-x_1}+x_i\frac{(\mathbf{q}-\mathbf{k}_1)^2}{(1-x_1)^2}\end{aligned}\tag{3.5}$$

for $i=2$ and 3. Note that the transformation we made is actually a Lorentz transformation. Under a Lorentz transformation, scalar quantities such as $k_{2,3}{}^2$ and $k_2\cdot k_3$ are invariant. This is important because it implies that the denominators are invariant under the change of variables. Another important fact is that this transformation also leaves the numerator function invariant. Hence, we can compute the numerator in either set of variables, and they both lead to the same result. The main properties of this Lorentz transformation will be discussed in detail in Sec. IV.

This transformation is designed so that $\mathbf{k}_1$ and $\mathbf{q}$ decouple from the transverse-momentum δ function: $\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{q})=\delta(\mathbf{k}_2'+\mathbf{k}_3')$. Using this δ function, we have for large $\mathbf{k}_3'^2$

$$k_2\cdot k_3=k_2'\cdot k_3'\approx[(x_2+x_3)^2/2x_2x_3]\mathbf{k}_3'^2.$$

Thus the leading part of $\langle|\mathfrak{M}|^2\rangle$ is

$$\begin{aligned}&2g^4 8x_1x_2 p\cdot k_1 k_2'\cdot k_3'\\&\times\{[(p-k_1)^2-m^2+i\epsilon][(k_2'+k_3')^2-m^2+i\epsilon]\}^{-2}\\&=4g^4\frac{x_1^2x_2^2x_3}{(x_2+x_3)^2}\frac{\mathbf{k}_1^2}{[\mathbf{k}_1^2+(1-x_1)\mu^2+x_1^2m^2]^2}\\&\times\frac{\mathbf{k}_3'^2}{\{\mathbf{k}_3'^2+[x_2/(x_2+x_3)]^2m^2+[x_3/(x_2+x_3)]\mu^2\}^2}.\end{aligned}\quad(3.6)$$

Finally, the minus-component δ function is

$$\delta_-=\delta\biggl(\frac{\mu^2}{x_1}+\frac{\mu^2}{x_2}+\frac{m^2}{x_3}-m^3+\frac{\mathbf{k}_1^2}{x_1}+\mathbf{k}_3'^2\Bigl(\frac{1}{x_2}+\frac{1}{x_3}\Bigr)+\frac{(\mathbf{q}-\mathbf{k}_1)^2}{1-x_1}-2m\nu\biggr).\quad(3.7)$$

The expression (3.6) tells us that the integrand damps for large $\mathbf{k}_1^2$ and $\mathbf{k}_3'^2$ and also for small x_1, x_2, or x_3. Thus for sufficiently large $\mathbf{q}^2$ and ν, as we shall see, the δ function will finally simplify to

$$\delta\Bigl(\frac{\mathbf{q}^2}{1-x_1}-2m\nu\Bigr).\quad(3.8)$$

To calculate the leading piece of W_2, it will be necessary to leave this δ function intact:

$$\begin{aligned}W_2|_{\text{Fig. 3(a)}}&=\frac{m}{4\pi}\langle|\mathfrak{M}|^2\rangle P\\&=2m\frac{g^4}{16\pi^2}\int\frac{dx_1dx_2dx_3}{(1-x_1)^2}x_1x_2\delta(x_1+x_2+x_3-1)\\&\times\int\frac{d^2k_1}{(2\pi)^2}\frac{\mathbf{k}_1^2}{[\mathbf{k}_1^2+(1-x_1)\mu^2+x_1^2m^2]^2}\int\frac{d^2k_3'}{(2\pi)^2}\\&\times\frac{\mathbf{k}_3'^2}{\{\mathbf{k}_3'^2+[x_3/(x_2+x_3)]\mu^2+[x_2/(x_2+x_3)]^2m^2\}^2}\delta_-.\end{aligned}\quad(3.9)$$

Except for the last δ function, we see that W_2 factors into two pieces, one dependent on the inner rainbow and one dependent on the outer rainbow.

We can now proceed with the calculation of W_2 by actually performing the integrals over $\mathbf{k}_1$ and $\mathbf{k}_3'$. It is sufficient for us to calculate only, say, the $\mathbf{k}_1$ integral. To begin, we consider

$$I(q,\nu)=\int\frac{d^2k_1}{(2\pi)^2}\frac{\mathbf{k}_1^2}{(\mathbf{k}_1^2+a)^2}\delta\Bigl(\frac{\mathbf{q}^2}{1-x_1}-2m\nu+b\Bigr),\quad(3.10)$$

where $a=(1-x_1)\mu^2+x_1^2m^2$ and b is the extra term as given in Eq. (3.7). b involves both $\mathbf{k}_1^2$ and $\mathbf{q}\cdot\mathbf{k}_1$ terms. To keep only the most divergent piece of (3.10), write

$$\frac{\mathbf{k}_1^2}{(\mathbf{k}_1^2+a)^2}=\frac{1}{\mathbf{k}_1^2+a}-\frac{a}{(\mathbf{k}_1^2+a)^2}.\quad(3.11)$$

The second term in this expression is more convergent than the first (again, it is a term like one finds in $\lambda\phi^3$), and can be ignored. Thus

$$I(q,\nu)\approx\int\frac{d^2k_1}{(2\pi)^2}\frac{1}{\mathbf{k}_1^2+a}\delta\Bigl(\frac{\mathbf{q}^2}{1-x_1}-2m\nu+b\Bigr).\quad(3.12)$$

To find the most divergent behavior in $\mathbf{q}^2$, we divide $I(q,\nu)$ into two regions of integration, only one of which gives the most divergent piece. This division calls for a split in the integral over the *magnitude* of $\mathbf{k}_1$ [15]:

$$\begin{aligned}I(q,\nu)&=\Bigl(\int_0^{\epsilon|q|}+\int_{\epsilon|q|}^{\infty}\Bigr)\frac{k_1dk_1d\theta}{(2\pi)^2}\frac{1}{\mathbf{k}_1^2+a}\\&\times\delta\Bigl(\frac{\mathbf{q}^2}{1-x_1}-2m\nu+b\Bigr)\quad(3.13)\\&=I_0^{\epsilon}(q,\nu)+I_\epsilon^{\infty}(q,\nu),\quad(3.14)\end{aligned}$$

where $\epsilon\ll1$, but $q\epsilon\gg m,\mu$. In $I_0^\epsilon(q,\nu)$, one can ignore b completely together with its $\mathbf{k}_1^2$ and $\mathbf{q}\cdot\mathbf{k}_1$ dependence, compared to $\mathbf{q}^2/(1-x_1)$. It is then straightforward to calculate

$$I_0^\epsilon(q,\nu)\approx\frac{1}{4\pi}\ln\Bigl(\frac{\mathbf{q}^2}{m^2}\Bigr)\delta\Bigl(\frac{\mathbf{q}^2}{1-x_1}-2m\nu\Bigr),\quad(3.15)$$

where the m^2 in the logarithm is an arbitrary mass scale, and is chosen as the proton mass for convenience. Equation (3.15) is the leading part of $I_0^\epsilon(q,\nu)$. In $I_\epsilon^\infty(q,\nu)$, we cannot ignore b compared to $\mathbf{q}^2$. It is simple to estimate this integral and to see that it gives an $I_\epsilon^\infty(q,\nu)$ which is at most of $O(\ln\epsilon)$ rather than $\ln\mathbf{q}^2$. Thus we find I_0^ϵ dominates:

$$I(q,\nu)\approx\frac{1}{4\pi}\ln\frac{\mathbf{q}^2}{m^2}\delta\Bigl(\frac{\mathbf{q}^2}{1-x_1}-2m\nu\Bigr).$$

The integral over d^2k_3' of course gives the same result. Note that there is no additional $(\ln\mathbf{q}^2)^2$ contributions

[15] For a thorough discussion on computing the leading logarithmic term, see R. J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge U. P., London, 1966).

from the end points in the x integrals.[16] This can be verified by examining the x dependence in the original integrand.

We therefore have for W_2

$$W_2|_{\text{Fig. 3(a)}}=2m\int_0^1 dx_1x_1\frac{\ln(\mathbf{q}^2/m^2)}{16\pi^2}\frac{\ln(\mathbf{q}^2/m^2)}{32\pi^2}\times\delta\left(\frac{\mathbf{q}^2}{1-x_1}-2m\nu\right). \quad (3.16)$$

Several features of this expression will generalize for multipion rainbows. Factorization occurs between the inner and outer rainbows, and there is different x dependence for the inner and outer rainbows (we shall delve more deeply into this point in Sec. IV), and the "parton" δ function $\delta(\mathbf{q}^2/x-2m\nu)$ occurs. The particular factors of $\ln\mathbf{q}^2$ appearing for one rainbow or the other are characteristic of the number of pions in the rainbow. We shall see in Sec. IV that for N pions in a given rainbow, the factor is $(\ln\mathbf{q}^2)^N/N!$.

B. Diagram 3(b)

This is, of course, not a symmetric diagram. It represents a type of diagram which, as we shall now show, can be ignored in comparison to the rainbow diagrams we have just studied. We represent W_2 as

$$W_2|_{\text{Fig. 3(b)}}=\frac{mg^2}{32\pi^3}\int\frac{dx_1dx_2}{x_1x_2}\delta(x_1+x_2-1)\times\int d^2k_1d^2k_2\ \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{q})\times\delta\left(\frac{\mathbf{k}_1^2+\mu^2}{x_1}+\frac{\mathbf{k}_2^2+m^2}{x_2}-2m\nu-m^2\right)\frac{N}{D}, \quad (3.17\text{a})$$

where

$$\begin{aligned}D^{-1}&=[(p-k_1)^2-m^2+i\epsilon][(k_1+k_2)^2-m^2+i\epsilon]\\&=x_2^{-1}(-\mathbf{k}_1^2+a+i\epsilon)(\mathbf{k}_2'^2+b+i\epsilon),\end{aligned} \quad (3.17\text{b})$$

$$\begin{aligned}N&=\tfrac{1}{2}\operatorname{Tr}[(\not p+m)\gamma_5(\not p-\not k_1+m)\gamma_+\\&\qquad\times(\not k_2+m)\gamma_5(\not k_1+\not k_2+m)\gamma_+]\\&=-4(x_2\mathbf{k}_1^2-x_1\mathbf{k}_1\cdot\mathbf{k}_2+x_1^2m^2),\end{aligned} \quad (3.17\text{c})$$

and $\mathbf{k}_2'=\mathbf{k}_2-(x_2/x_1)\mathbf{k}_1=\mathbf{q}-(1/x_1)\mathbf{k}_1$; a and b are $\mathbf{k}$-independent factors. Throw the nonleading term $x_1^2m^2$ out of N and use $\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{q})$ to perform d^2k_2. Then

$$N=-4(\mathbf{k}_1^2-x_1\mathbf{k}_1\cdot\mathbf{q})=4\mathbf{k}_1\cdot\mathbf{k}_2'. \quad (3.18)$$

We are left with a $\mathbf{k}_1$ integral. Examination of this integral in the manner prescribed in Sec. III A shows that it does not contribute in $O(\ln\mathbf{q}^2)$. This is essentially because of the fact that the two terms in the denominators cannot become small simultaneously.

Diagram 3(b) is the simplest example for a pion joining from an inner to an outer rainbow. Its contribution is one order of $\ln|q^2|$ smaller than the corresponding "pure" rainbow. Physically, this can be understood as follows: Since a pion emitted from a nucleon tends to be soft with respect to the nucleon, it can hardly be reabsorbed by the nucleon after the electromagnetic interaction. We have already seen in case (a) that the dominant contribution comes from the region where the pion has a relatively small transverse momentum with respect to the nucleon. Since this result seems to be physically reasonable and general, we shall ignore in the next section all diagrams with at least one pion connecting the inner and outer rainbows.

C. Diagram 3(c)

As a third low-order example, we consider a "crossed" outer rainbow, as shown in Fig. 3(c). Following our previous examples, it is not difficult to see that for the dominant contribution the minus-component δ function again reduces to the "parton" δ function $\delta(\mathbf{q}^2/x_3-2m\nu)$. Then

$$W_2|_{\text{Fig. 3(c)}}=\frac{8m}{(4\pi)^6}\int_0^1\frac{dx_1dx_2dx_3}{x_1x_2x_3}\delta(x_1+x_2+x_3-1)\times\int d^2k_1d^2k_2d^2k_3\times\delta^2(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{q})\delta(\mathbf{q}^2/x_3-2m\nu)N/D, \quad (3.19\text{a})$$

where

$$\begin{aligned}D^{-1}=&[(p-k_1)^2-m^2+i\epsilon][(p-k_1-k_2)^2-m^2+i\epsilon]^2\\&\times[(p-k_2)^2-m^2+i\epsilon]\end{aligned} \quad (3.19\text{b})$$

and

$$\begin{aligned}N=\tfrac{1}{2}\operatorname{Tr}[&(\not p+m)\gamma_5\not k_1\gamma_5(\not p-\not k_1-\not k_2+m)\gamma_+\\&\times(\not k_3+m)\gamma_+(\not p-\not k_1-\not k_2+m)\gamma_5\not k_2\gamma_5],\end{aligned} \quad (3.19\text{c})$$

with the understanding that the integrals are evaluated for $0<|\mathbf{k}_1|,|\mathbf{k}_2|<\epsilon|\mathbf{q}|$. Taking only the *leading* term of the trace, we have

$$\begin{aligned}N=&\,8x_3(p\cdot k_1)(p\cdot k_2)\\&+2(p-k_1-k_2)^2(k_1\cdot k_2-xp\cdot k_1-xp\cdot k_2)\\\approx&\,8x_3(p\cdot k_1)(p\cdot k_2)-2(p-k_1-k_2)^2\mathbf{k}_1\cdot\mathbf{k}_2.\end{aligned} \quad (3.20)$$

[16] The naïve way of extracting the $\ln q^2$ term will fail if the remaining x integral diverges. When any term, even if it is an order or so smaller in $\ln q^2$, becomes divergent *after* the $\ln q^2$ terms have been taken out, we must keep this term in its original form and perform the x integral first. In general, the latter integral converges, and there appears a natural cutoff of order $\ln q^2$. The over-all $\ln q^2$ dependence can then be different from its naïve dependence. We refer to this type of additional leading $\ln q^2$ contribution as "divergent contribution from x integral." However, this type of contribution never appears in the leading diagrams we studied in this paper.

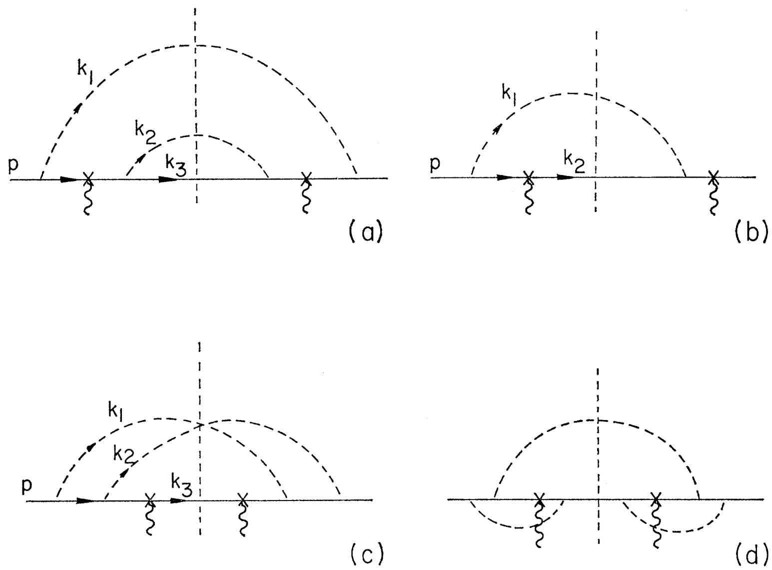

FIG. 3. (a) Typical straight ladder diagram with two rungs; (b) diagram with a pion joining from an inner to an outer rainbow; (c) lower-order ladder diagram with crossed rungs; (d) diagram with pions interacting on different sides of the current. Diagrams (b)–(d) do not contribute to the leading $\ln|q^2|$ terms.

Using (3.20), we write the leading part of N/D as

$$\sim x_3 \frac{1}{[(p-k_1-k_2)^2-m^2+i\epsilon]^2} - \frac{\mathbf{k}_1\cdot\mathbf{k}_2}{(p\cdot k_1-m^2+i\epsilon)[(p-k_1-k_2)^2-m^2+i\epsilon](p\cdot k_2-m^2+i\epsilon)}. \tag{3.21}$$

We have here used the same trick as we did in Eq. (3.11) to pick out the leading piece.

For the first term of (3.21), a change of variable eliminates the cross term in the denominator. Then the $\mathbf{k}_1,\mathbf{k}_2$ integrals can be put into the form

$$\int dZ_1 dZ_2 \frac{1}{(Z_1+Z_2+C)^2}, \quad Z_i \sim \mathbf{k}_i{}^2.$$

Using the method described in detail in Sec. III A, this integral contributes to order $\ln\mathbf{q}^2$. Similarly, the second term of (3.21) has $\mathbf{k}_1\cdot\mathbf{k}_2$ in the numerator, and examination shows that $\int d^2k_1 d^2k_2$ cannot contribute terms higher than $\ln\mathbf{q}^2$ either.

Thus diagram 3(c) contributes to $O(\ln\mathbf{q}^2)$. This is to be compared to the $O((\ln\mathbf{q}^2)^2)$ which the pure two-pion outer rainbow contributes. We can therefore ignore this "crossed" rainbow compared to the "pure" rainbow of the same order. This example prompts us in the next section to ignore all such diagrams.

D. Diagram 3(d)

This diagram is again symmetric. The methods of studying this diagram are no different from those used in Secs. III A–III C, but the expressions are far more complicated. Because only the result is of interest, we shall not give the details here.

We find that this diagram cannot contribute $O((\ln|q^2|)^3)$, which is what a leading diagram of order g^6 should contribute. We shall therefore ignore such diagrams in the future. This also tells us that a symmetric diagram (i.e., a diagram which contributes an absolute value squared of an amplitude) does not necessarily contribute to a leading order in $\ln|q^2|$. However, the smallness of this diagram *does* suggest that diagrams with pionic corrections for more than two vertices at a time [Fig. 4(a)] tend to be at least an order of $\ln|q^2|$ smaller. This is because of the fact that the extra pionic corrections do not lead to extra divergence, and hence the diagram has the same degree of divergence as the original diagram. But the pionic corrections *do* contribute to extra g^2 factors. Therefore, we can no longer have a leading diagram.

However, the argument given above does not apply to the pionic corrections to a single vertex, nor to a self-energy diagram [Fig. 4(b)]. These diagrams have extra vertex and self-energy divergences, and may pick up extra $\ln|q^2|$ factors. Physically, we need the vertex corrections to supply us the damping factor appearing in the form factors. According to the established rule, one should first compute the vertex functions from the field theory, and then put them into each of the πNN vertices. At present, however, this is too difficult. We therefore try to bypass this point by evaluating all amplitudes without considering any vertex (and self-energy) corrections and later including these corrections in the final expression (see discussion in Sec. VI).

We conclude this section by comparing our results with the important work of Adler and Tung.[10] They have worked on a similar γ_5 coupling theory, but with a massless nucleon. Instead of letting q^2 be large, they studied the infrared properties of the ep inelastic form

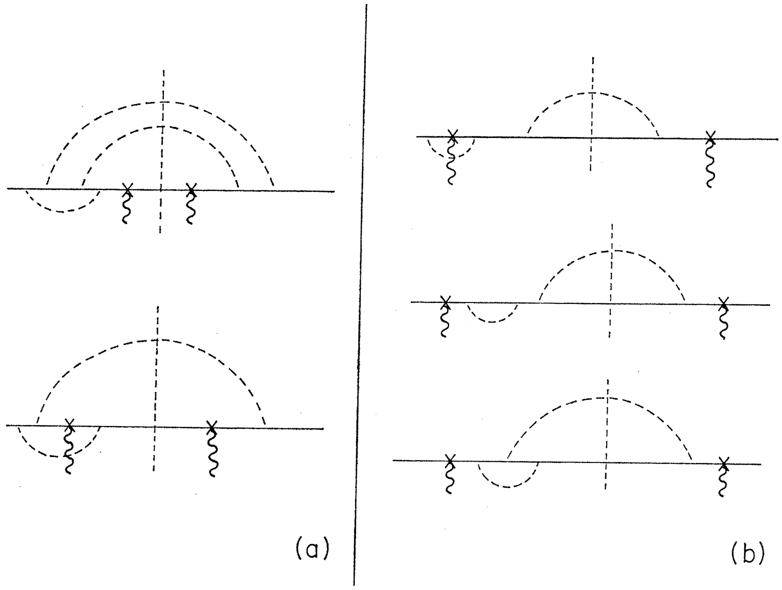

FIG. 4. (a) Diagrams with pionic corrections for two (or more than two) vertices at a time; (b) diagrams with self-energy and vertex corrections.

factors as $\mu \to 0$. The algebraic structure of their expressions are practically the same as the large q^2 structure studied here. They have studied the leading logarithmic terms for all fourth-order diagrams, and their conclusions agree with ours for the set of diagrams studied earlier.

IV. GENERAL RESULT

In this section we shall calculate W_2 for many-pion production. The spirit of this calculation will be that for any given order in W_2 we take only the leading piece in $\ln|q^2|$. The calculations of Sec. III have already given us a clue as to where to find the leading piece. Following those results, we ignore diagrams like those of Figs. 3(b)–3(d), and compute only diagrams analogous to Fig. 3(a). We emphasize that we have actually calculated diagrams 3(b)–3(d) only in the order shown. We have not shown in general that such diagrams are smaller than those analogous to Fig. 3(a), although the manner of our low-order calculations strongly suggest that they are indeed smaller.

It is worth discussing this point in more detail.[17] While the rainbow diagrams for n pions will behave like $(\ln q^2)^n$, the lower-order diagrams behave as $(\ln q^2)^{n-1}$, $(\ln q^2)^{n-2}$, etc. For example, rainbow diagrams with one pion pair crossed, as in Fig. 3(c), behave as $(\ln q^2)^{n-1}$, as do some other types of diagrams. Now simple counting arguments indicate that the diagrams of $O(\ln q^2)^{n-1}$ may increase as n. Since n in turn will turn out to increase as $\ln q^2$, it is *conceivable* that these diagrams may sum up to be as big as the leading rainbow diagrams. We are assuming that this does not happen. This assumption is equivalent to a kind of random-phase approximation in the following manner: In Appendix B we point out that W_2 is given by the integral over phase space of an amplitude $\mathfrak{M}$ squared. In our model, $\mathfrak{M}$ is represented by a sum of terms, each of which represents the emission of pions in a given order. When we take $|\mathfrak{M}|^2$, the diagonal terms in the sum squared correspond to the rainbow diagrams. These terms are of course positive. The off-diagonal terms, which are not necessarily positive, are the lower-order diagrams. The assumption that the off-diagonal terms tend to cancel one another is the random-phase approximation. Although we have not explicitly checked this point, it is of course interesting and important to do so.

We have also—somewhat arbitrarily—excluded diagrams with nucleon-antinucleon pair creation and annihilation. The fact that we do not consider such effects implies that pion production is a bremsstrahlung-like process. This is indeed what diagrams analogous to Fig. 3(a) tell us. The nucleon receives an impulse from the electromagnetic current, shaking pions off in the process. Indeed, we shall see many bremsstrahlung features as we develop the general result, both in this section and in Sec. V.

A. *N*-Pion Emission

The general diagram we consider is shown in Fig. 5. According to our previous analysis, this should be the only leading diagram if pair creations and vertex corrections are ignored. Note that this is also similar to a set of diagrams studied earlier by Drell, Levy, and Yan[3] in a related model. There are M pions in the outer rainbow (M pions produced before the current insertion) and $N-M$ pions in the inner rainbow ($N-M$ pions produced after the current insertion). As for Fig. 3(a), W_2 is an integral over an amplitude $\mathfrak{M}$ squared. In addition, we have denoted by Δ the inverse propagator function: If an internal line carries momentum p, the

[17] One of us (P. M. F.) is grateful to Professor S. Gasiorowicz for a helpful conversation on this point.

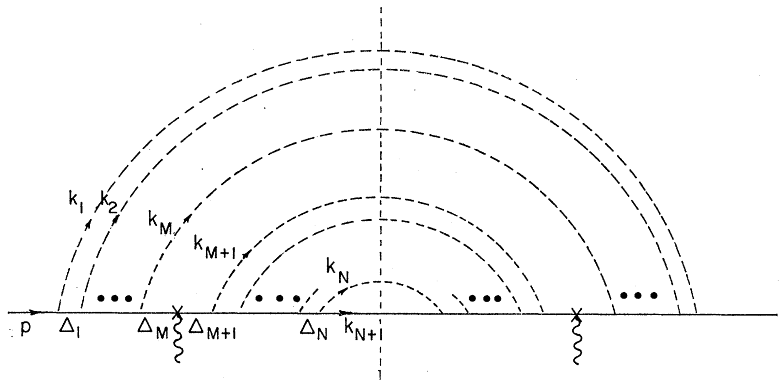

FIG. 5. Typical straight ladder unitary diagram considered. This is Fig. 1(a) in more detail. The notations used are $p=(1,0,m^2)$; $k_i=(x_i,\mathbf{k}_i,(\mathbf{k}_i^2+\mu^2)/x_i)$, $i=1,\ldots,N$; $k_{N+1}=(x_{N+1},\mathbf{k}_{N+1},(\mathbf{k}_{N+1}^2+m^2)/x_{N+1})$, and for an internal line carrying momentum k, $\Delta=k^2-m^2+i\epsilon$.

corresponding Δ for that line is $\Delta=p^2-m^2+i\epsilon$. The labeling of the Δ's is as shown in Fig. 5. We call the contribution of W_2 for Fig. 5 $W_2^{M,N-M}$. We have

$$W_2^{M,N-M}=2\pi mg^{2N}\int\frac{dx_1\cdots dx_{N+1}}{4\pi x_1\cdots 4\pi x_{N+1}}$$
$$\times\delta(x_1+\cdots+x_{N+1}-1)\int\frac{d^2k_1}{(2\pi)^2}\cdots\frac{d^2k_{N+1}}{(2\pi)^2}$$
$$\times(2\pi)^2\delta(\mathbf{k}_1+\cdots+\mathbf{k}_{N+1}-\mathbf{q})$$
$$\times\delta\Big(\sum\frac{k_i^2+\mu^2(m^2)}{x_i}-2m\nu-m^2\Big)\frac{N}{D},\quad(4.1\text{a})$$

where

$$D^{-1}=(\Delta_1\Delta_2\cdots\Delta_M\Delta_{M+1}\cdots\Delta_N)^2\qquad(4.1\text{b})$$

and for the numerator (not to be confused with the index N),

$$N=\tfrac{1}{2}\operatorname{Tr}[(\not p+m)\gamma_5(\not p-\not k_1+m)\gamma_5\cdots\gamma_5(\not p-\not k_1-\cdots-\not k_M+m)\gamma_+$$
$$\times(\not k_{M+1}+\cdots+\not k_N+\not k_{N+1}+m)\gamma_5\cdots\gamma_5(\not k_N+\not k_{N+1}+m)\gamma_5(\not k_{N+1}+m)$$
$$\times\gamma_5(\not k_N+\not k_{N+1}+m)\gamma_5\cdots\gamma_5(\not k_{M+1}+\cdots+\not k_{N+1}+m)\gamma_+$$
$$\times(\not p-\not k_1-\cdots-\not k_M+m)\gamma_5\cdots\gamma_5(\not p-\not k_1+m)\gamma_5].\qquad(4.1\text{c})$$

As our first general result, we can show that the dominant term in $W_2^{M,N-M}$ factors into two parts, one involving outer rainbow quantities and one involving inner rainbow quantities. This is the generalization of what we showed in Sec. III A. The propagator factors in D^{-1} break into two such pieces, $\Delta_1\cdots\Delta_M$ being outer quantities, while $\Delta_{M+1}\cdots\Delta_N$ are inner quantities. Furthermore, the trace in N is of the form Tr[(outer quantities)$\times\gamma_+$ (inner quantities)$\times\gamma_+$]. When we pick the leading piece from the trace, it consists of dot products of outer quantities times dot products of inner quantities. Then we can make a change of variables in the inner quantities. The new variables, denoted by primes, are given by

$$x_i=x_i',$$
$$\mathbf{k}_i=\mathbf{k}_i'+(x_i/x)\mathbf{k},\qquad(4.2)$$
$$k_{i-}=k_{i-}'+(2/x)\mathbf{k}_i'\cdot\mathbf{k}+(x_i/x^2)\mathbf{k}^2,\quad i=M+1,\ldots,N+1,$$

where

$$x=1-\sum_{j=1}^{M}x_j,\quad \mathbf{k}=\mathbf{q}-\sum_{j=1}^{M}\mathbf{k}_j.$$

If we put $\mathbf{a}\equiv\mathbf{k}/x$, then this transformation on a 4-vector p is

$$p\to p'=e^{-i\mathbf{a}\cdot\mathbf{E}}pe^{i\mathbf{a}\cdot\mathbf{E}},\qquad(4.3)$$

where $\mathbf{E}=(K_1+L_2,K_2-L_1)$ are the commuting generators of the infinite-momentum $E(2)$ subgroup.[18] L and K are, of course, the conventional rotation and boost generators. Thus the transformation (4.2) is a Lorentz transformation. The (invariant) N/D in W_2 is unchanged under this transformation. The transverse-component δ function is affected as follows:

$$\delta\Big(\sum_{j=1}^{N+1}\mathbf{k}_j-\mathbf{q}\Big)\to\delta\Big(\sum_{j=M+1}^{N+1}\mathbf{k}_j'\Big).\qquad(4.4)$$

In the minus-component δ function, the terms involving $\mathbf{q}^2$ and ν become $\mathbf{q}^2/x-2m\nu$. For the leading behavior, this is the only part of this δ function that survives. Thus the inner and outer rainbows completely decouple. If we introduce an identity

$$\int_0^1 dx\;\delta\Big(\sum_{j=1}^{M}x_j-1+x\Big)=1,$$

then the decoupling takes the form

$$W_2^{M,N-M}=2m\int_0^1 dx\;\delta(\mathbf{q}^2/x-2m\nu)\times A^{M,\text{outer}}A^{N-M,\text{inner}},\qquad(4.5)$$

where the A's are related to the W_2 of the "pure" outer and inner rainbows through

$$W_2^{N,\text{outer/inner}}=2m\int_0^1 dx\;\delta(\mathbf{q}^2/x-2m\nu)A^{N,\text{outer/inner}}.$$

We have thus far said nothing about the x and q^2 dependence of A^{inner} and A^{outer}. In order to find this

[18] See, e.g., S. J. Chang, J. G. Kuriyan, and L. O'Raifeartaigh, Phys. Rev. **169**, 1275 (1968); S. J. Chang and L. O'Raifeartaigh, J. Math. Phys. **10**, 21 (1969).

dependence, it is necessary to look at these quantities in detail. In order to examine one of these two functions, it is sufficient to assume there are no pions emitted in the other.

First look at $A^{M,\text{outer}}$:

$$A^{M,\text{outer}}=\pi g^{2M}\int\frac{dx_1\cdots dx_M}{4\pi x_1\cdots 4\pi x_M 4\pi x}\delta\Big(\sum_{j=1}^{M}x_j-1+x\Big)\times\int\frac{d^2k_1}{(2\pi)^2}\cdots\frac{d^2k_M}{(2\pi)^2}\frac{N^{\text{outer}}}{D^{\text{outer}}},\quad(4.6\text{a})$$

where

$$\mathbf{k}_{M+1}=\mathbf{q}-\sum_{j=1}^{M}\mathbf{k}_j$$

and

$$(D^{\text{outer}})^{-1}=(\Delta_1\Delta_2\cdots\Delta_M)^2.\quad(4.6\text{b})$$

Now we must examine the leading behavior in the $\mathbf{k}_i$. For this purpose, we need the $\mathbf{k}_n{}^2$ behavior in Δ_n:

$$\begin{aligned}\Delta_n&=(p-k_1-\cdots-k_n)^2-m^2\\&=-\mathbf{k}_n{}^2[(1-x_1-\cdots-x_n)/x_n+1]+\cdots\\&=-[(1-x_1-\cdots-x_{n-1})/x_n]\mathbf{k}_n{}^2+\cdots.\end{aligned}\quad(4.7)$$

The $+\cdots$ in (4.7) is $\sim\mathbf{k}_j{}^2$ and $\mathbf{k}_i\mathbf{k}_j$, $i\leq n, j<n$. The cross terms in $\mathbf{k}_n\cdot\mathbf{k}_j$ can be eliminated by a change of variable in $\mathbf{k}_n$. Therefore, the denominator function behaves like

$$\begin{aligned}(D^{\text{outer}})^{-1}&=\frac{(1-x_1)^2(1-x_1-x_2)^2\cdots(1-x_1-\cdots-x_{M-1})^2}{x_1{}^2x_2{}^2\cdots x_M{}^2}\\&\quad\times(\mathbf{k}_1{}^2+a_1)^2\cdots(\mathbf{k}_M{}^2+c\mathbf{k}_{M-1}{}^2+\cdots+a_M)^2,\end{aligned}\quad(4.8)$$

where the a's are $\mathbf{k}$ independent. The maximum power of $\ln\mathbf{q}^2$ is obtained from a numerator function with the behavior $N\sim\mathbf{k}_1{}^2\cdots\mathbf{k}_M{}^2$. Keeping this in mind, it is quite straightforward to pick out the leading term from the trace

$$\begin{aligned}N^{\text{outer}}=\tfrac{1}{2}\,\mathrm{Tr}[&(p+m)\gamma_5(p-k_1+m)\gamma_5\cdots\\&\gamma_5(p-k_1-\cdots-k_M+m)\gamma_+\\&\times(k_{M+1}+m)\gamma_+(p-k_1-\cdots-k_M+m)\gamma_5\cdots\\&\gamma_5(p-k_1+m)\gamma_5].\end{aligned}$$

The leading term from the trace is

$$N^{\text{outer}}\approx(2p\cdot k_1)(2k_1\cdot k_2)\cdots(2k_{M-1}\cdot k_M)2x_M2x_{M+1},$$

which is just

$$4x\mathbf{k}_1{}^2\mathbf{k}_2{}^2\cdots\mathbf{k}_M{}^2.\quad(4.9)$$

Putting Eqs. (4.8) and (4.9) into (4.6), we find

$$\begin{aligned}A^{M,\text{outer}}=\Big(\frac{g^2}{16\pi^2}\Big)^M\int&\frac{x_1dx_1}{(1-x_1)^2}\frac{x_2dx_2}{(1-x_1-x_2)^2}\cdots\frac{x_{M-1}dx_{M-1}}{(1-x_1-\cdots-x_{M-1})^2}x_Mdx_M\\&\times\delta\Big(\sum_{j=1}^{M}x_j-1+x\Big)\int d\mathbf{k}_1{}^2\cdots d\mathbf{k}_M{}^2\frac{\mathbf{k}_1{}^2\mathbf{k}_2{}^2\cdots\mathbf{k}_M{}^2}{(\mathbf{k}_1{}^2+a_1)^2(\mathbf{k}_2{}^2+b\mathbf{k}_1{}^2+a_2)^2\cdots(\mathbf{k}_M{}^2+c\mathbf{k}_{M-1}{}^2+\cdots+a_M)^2}.\end{aligned}\quad(4.10)$$

We denote the multiple integral over the x_i's as $F_M(x)$; the leading term in the multiple integral over $\mathbf{k}^2$'s is I_M.

First look at I_M. We have shown in Sec. III A that $I_1=\ln(\mathbf{q}^2/m^2)$. We shall show now that $I_M=(1/M!)\times[\ln(\mathbf{q}^2/m^2)]^M$. The proof is by induction. I_M is of the form

$$I_M=\int_0^{\epsilon q^2}\cdots\int_0^{\epsilon q^2}dz_1\cdots dz_M\times\frac{z_1z_2\cdots z_M}{(z_1+a_1)^2(z_2+bz_1+a_2)^2\cdots(z_M+cz_{M-1}+\cdots)^2}.\quad(4.11)$$

Let $I_M=c_M[\ln(\mathbf{q}^2/m^2)]^M$. We have

$$c_M=\frac{1}{1!}c_{M-1}-\frac{1}{2!}c_{M-2}+\frac{1}{3!}c_{M-2}-\cdots.$$

We know $c_1=1/1!$. Assume now that $c_n=1/n!$ up to $n=M-1$. Then

$$\begin{aligned}c_M&=\frac{1}{(M-1)!}-\frac{1}{2!}\frac{1}{(M-2)!}+\frac{1}{3!}\frac{1}{(M-3)!}-\cdots\\&=\frac{1}{M!}\Big(M-\frac{1}{2!}M(M-1)\\&\qquad+\frac{1}{3!}M(M-1)(M-2)-\cdots\Big)\\&=\frac{1}{M!}A.\end{aligned}$$

Note that $1-A$ is just the binomial expansion for $(1-1)^M$, which is zero. Hence $A=1$. This completes the proof by induction:

$$I_M=\frac{1}{M!}\Big(\ln\frac{\mathbf{q}^2}{m^2}\Big)^M.\quad(4.12)$$

We would like to remind ourselves that m^2 appearing in the logarithm is an arbitrary scale mass. This arbitrariness will be significant after we sum up the series.

Next look at $F_M(x)$. If the integrals in x_i are done in the order shown in (4.10), then x_1 goes from 0 to $1-x$, x_2 goes from 0 to $1-x-x_1,\ldots$, and x_M goes from 0 to $1-x-x_1-\cdots-x_{M-1}$. A change of variables can map all the limits from 0 to 1. The new variables y_i are defined by

$$x_j=(1-x_1-\cdots-x_{j-1})y_j =(1-y_1)(1-y_2)\cdots(1-y_{j-1})y_j\,,\quad j=1,2,\ldots,M. \tag{4.13}$$

In terms of these variables,

$$F_M(x)=\int_0^1 y_1dy_1\int_0^1 y_2dy_2\cdots\int_0^1 y_Mdy_M \times\delta[x-(1-y_1)(1-y_2)\cdots(1-y_M)]. \tag{4.14}$$

This form of $F_M(x)$ is not very enlightening. There is, however, a form in which F_M becomes simple. This form is the Mellin transform of $F_M(x)$. Denote the Mellin transform of $F_M(x)$ by

$$\tilde{F}_M(\tau)=\int_0^1 dx\, x^{\tau-1}F_M(x). \tag{4.15}$$

The Mellin transform of the δ function in (4.1) is

$$\int_0^1 dx\, x^{\tau-1}\delta(x-(1-y_1)\cdots(1-y_M)) =(1-y_1)^{\tau-1}\cdots(1-y_M)^{\tau-1}.$$

Then

$$\tilde{F}_M(\tau)=\prod_{j=1}^{M}\int_0^1 dy_j\, y_j(1-y_j)^{\tau-1}=[\tau(\tau+1)]^{-M}. \tag{4.16}$$

This gives for the Mellin transform of $A^{\text{outer}}(x,q^2)$

$$\tilde{A}^{M,\text{outer}}(\tau,q^2)=\frac{1}{M!}\left[\frac{\ln(\mathbf{q}^2/m^2)}{16\pi^2\tau(\tau+1)}\right]^M. \tag{4.17}$$

Next we look at $A^{N,\text{inner}}$ (we set $M=0$, i.e., no pions in the outer rainbow, and $p_+=x$ to look at A^{inner}). We have

$$A^{N,\text{inner}}=\pi g^{2N}\int\frac{dx_1\cdots dx_{N+1}}{4\pi x_1\cdots 4\pi x_{N+1}}\delta\Big(\sum_{j=1}^{N+1}x_j-x\Big) \times\int\frac{d^2k_1}{(2\pi)^2}\cdots\frac{d^2k_{N+1}}{(2\pi)^2}(2\pi)^2 \times\delta\Big(\sum_{j=1}^{N+1}\mathbf{k}_j\Big)\frac{N^{\text{inner}}}{D^{\text{inner}}}, \tag{4.18a}$$

$$(D^{\text{inner}})^{-1}=(\Delta_1\Delta_2\cdots\Delta_N)^2 =(k^2-m^2+i\epsilon)^2[(k-k_1)^2-m^2+i\epsilon]\cdots [(k_N+k_{N+1})^2-m^2+i\epsilon]^2, \tag{4.18b}$$

and

$$N^{\text{inner}}=(1/2x) \times\mathrm{Tr}[(\not p+m)\gamma_+(\not k+m)\gamma_5(\not k-\not k_1+m)\gamma_5\cdots \gamma_5(\not k_{N+1}+m)\gamma_5\cdots\gamma_5(\not k+m)\gamma_+]. \tag{4.18c}$$

We have here put $k\equiv k_1+k_2+\cdots+k_{N+1}$. The factor $1/2x$ is associated with the factorization of the trace into the products of two traces (i.e., the N^{inner} and the N^{outer}). The inverse propagators behave as

$$\begin{aligned}
\Delta_N&=(k_N+k_{N+1})^2-m^2=(x_{N+1}/x_N)(k_N'^2+a_N)\,,\\
\Delta_{N-1}&=(k_{N-1}+k_N+k_{N+1})^2-m^2\\
&=[(x_N+x_{N+1})/x_{N-1}]\times(k_{N-1}'^2+bk_N'^2+a_{N-1})\,,\\
&\cdots\\
\Delta_1&=(k_1+\cdots+k_{N+1})^2-m^2\\
&=[(x-x_1)/x_1](k_1'^2+ck_2'^2+\cdots+a_1)\,,
\end{aligned} \tag{4.19}$$

where

$$k_N'=k_N-(x_N/x_{N+1})k_{N+1}\,,$$
$$k_{N-1}'=k_{N-1}-[x_{N-1}/(x_N+x_{N+1})](k_N+k_{N+1})\,,$$
$$\cdots.$$

The leading term in N is

$$N^{\text{inner}}=4k_{1+}(2k_1\cdot k_2)(2k_2\cdot k_3)\cdots(2k_N\cdot k_{N+1}).$$

In terms of the new transverse momenta, the leading term in N^{inner} reduces to

$$4x_{N+1}\mathbf{k}_1'^2\mathbf{k}_2'^2\cdots\mathbf{k}_N'^2. \tag{4.20}$$

Finally, the phase-space factor becomes

$$\frac{d^2k_1}{(2\pi)^2}\cdots\frac{d^2k_{N+1}}{(2\pi)^2}(2\pi)^2\delta\Big(\sum\mathbf{k}_j\Big) =\left(\frac{x_{N+1}}{x}\right)^2\frac{d^2k_1'}{(2\pi)^2}\cdots\frac{d^2k_N'}{(2\pi)^2}. \tag{4.21}$$

Combining Eqs. (4.18)–(4.21), just as (4.10), we have

$$A^{N,\text{inner}}=\left(\frac{g^2}{16\pi^2}\right)^N\int\frac{x_1dx_1}{(x-x_1)^2} \times\frac{x_2dx_2}{(x-x_1-x_2)^2}\cdots\frac{x_Ndx_N}{(x-x_1-\cdots-x_N)^2} \times\frac{x_{N+1}^2dx_{N+1}}{x^2}\delta\Big(\sum_{j=1}^{N+1}x_j-x\Big)I_N. \tag{4.22}$$

In this case, the sequence of x_i integrals can be done easily because of the appearance of x_{N+1}^2 rather than x_{N+1} as in A^{outer}. The result of the x_i integrals is just $(\frac{1}{2})^N$, and hence

$$A^{N,\text{inner}}=\frac{1}{N!}\left(\frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)^N. \tag{4.23}$$

Contrary to A^{outer}, there is *no* x dependence in A^{inner}. Equations (4.23) and (4.17) are the final results of Sec. IV A.

B. Summation over Pions

In an actual inelastic experiment in the kinematic regime we are discussing, only the energy and angle of the final electron are measured. Thus the W_2 we want is the sum of $W_2{}^{M,N-M}$ over N and M. Due to our factorization property we can separately sum M and $N-M$. If we denote

$$A_0=\sum_N A^{N,\mathrm{outer}},\quad A_i=\sum_N A^{N,\mathrm{inner}},$$

then

$$W_2=2m\int_0^1 dx\ \delta(\mathbf{q}^2/x-2m\nu)A_0(x,q^2)A_i(q^2). \quad (4.5')$$

We have seen that it is quite easy to sum $A^{N,\mathrm{inner}}$:

$$A_i=\sum_N A^{N,\mathrm{inner}}=\exp\left(\frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right). \quad (4.24)$$

Note that the arbitrariness in the scale mass m^2 is now important. A change of m^2 will induce an over-all constant in A_i. Hence, by analyzing the leading $\ln|q^2|$ term, we can only determine W_2 up to a constant multiplicative factor.

$A^{N,\mathrm{outer}}(x,q^2)$ is, however, difficult to sum in the x space. We have seen that the Mellin transform $\tilde{A}^{N,\mathrm{outer}}(\tau,q^2)$ of $A^{N,\mathrm{outer}}(x,q^2)$ is quite simple, and we can indeed take its sum:

$$\tilde{A}_0(\tau,q^2)=\exp\left(\frac{g^2}{16\pi^2\tau(\tau+1)}\ln\frac{\mathbf{q}^2}{m^2}\right)-1. \quad (4.25)$$

When we discuss our results further in Sec. V, Eq. (4.25) and the significance of τ will play a central role. Without diminishing the importance of Eq. (4.25) we would like to point out that the inverse Mellin transform of $\tilde{A}_0$ can actually be taken. We regard neither this fact nor the actual result to be of special significance, and we urge all but the most dedicated reader to move on to Sec. V.

Instead of taking an inverse Mellin transform, it is convenient to regard $\tilde{A}_0(\tau,q^2)$ as a Laplace transform over the variable $z=-\ln x$, $0<z<\infty$. Then, we find that $A_0(z,q^2)$ is given by

$$\int_0^\infty A_0(z,q^2)e^{-\tau z}dz=\exp\left(\frac{a}{\tau(\tau+1)}\right)-1, \quad (4.26)$$

$$a=\frac{g^2}{16\pi^2}\ln\frac{\mathbf{q}^2}{m^2}.$$

To find $A_0(z,q^2)$, write

$$e^{a/\tau(\tau+1)}-1=(e^{a/\tau}-1)(e^{-a/(\tau+1)}-1)+(e^{-a/(\tau+1)}-1)+(e^{a/\tau}-1). \quad (4.27)$$

The last two terms in (4.27) have known inverse transforms, and the first term can be performed as a convolution. The result is

$$A_0(z,q^2)=(a/z)^{1/2}[I_1(2a^{1/2}z^{1/2})-e^{-z}J_1(2a^{1/2}z^{1/2})] -a\int_0^z dz'\,[z'(z-z')]^{-1/2}e^{-z'}I_1[2a^{1/2}(z-z')^{1/2}] \times J_1(2a^{1/2}z'^{1/2}). \quad (4.28)$$

This rather formidable looking result does not correspond to any familiar distribution in x. It does simplify a bit in limiting cases. [This can be seen either by taking the limit in (4.26) and doing further inverse transforms or by taking the limits directly in (4.28).] In the limit $x\rightarrow 1$ (z small),

$$A_0(x,q^2)\rightarrow(q^2/16\pi^2)(1-x)\ln(\mathbf{q}^2/m^2). \quad (4.29)$$

In the limit $x\rightarrow 0$ (z large),

$$A_0(x,q^2)\rightarrow(4\pi)^{-1/2}(a/z^3)^{1/4}\exp(2a^{1/2}z^{1/2}). \quad (4.30)$$

V. LONGITUDINAL IMPACT PARAMETER

We have seen that in our neutral pseudoscalar-meson theory the structure function has a particularly striking form in a new space. This space is the Laplace space of the logarithm of a longitudinal momentum—or, equivalently, the Mellin space of that momentum. In this section we would like to examine some possible reasons why this space might be of fundamental importance. Our arguments rely heavily on analogy and generalization, but we think that they add up to a rather convincing set of facts.

The analogy we shall try to develop is with the eikonal form for the transverse part of a scattering process. Consider the two sets of variables[19]

$$b_1=E_1/p_+,\quad b_2=E_2/p_+,\quad \tau=K_3 \quad (5.1)$$

and

$$p_1,\quad p_2,\quad \ln(p_+/m), \quad (5.2)$$

where $(E_1,E_2)=(K_1+L_1,K_2-L_2)$ are the $E(2)$ generators introduced earlier, and K_3 is a boost along the z direction. The first set of variables is a set of Lorentz transformations. The second set of variables refers to momenta of our problem.

The commutator relations among these variables are well known:

$$[E_1,E_2]=[\mathbf{E},p_+]=0,$$
$$i[K_3,\mathbf{E}]=\mathbf{E},\quad i[K_3,p_+]=p_+,\quad i[K_3,\mathbf{p}]=0, \quad (5.3)$$
$$[p_\mu,p_\nu]=0,\quad i[E_i,p_j]=\delta_{ij}p_+,\quad i,j=1,2,\quad \mu=+,1,2.$$

It is easy to see that (b_1,b_2,τ) and $(p_1,p_2,\ln(p_+/m))$ form two commuting sets. For conjugate pairs, we have

$$[b_1,p_1]=[b_2,p_2]=-i. \quad (5.4)$$

The commutator between the last pair of variables is slightly more difficult to compute. Using

$$e^{i\lambda K_3}p_+e^{-i\lambda K_3}=e^\lambda p_+,$$

we have

$$e^{i\lambda K_3}\ln(p_+/m)e^{-i\lambda K_3}=\ln(e^\lambda p_+/m)=\ln(p_+/m)+\lambda.$$

Hence, to first order in λ, we have

$$[K_3,\ln(p_+/m)]=-i. \quad (5.5)$$

[19] We are indebted to Professor R. Dashen for pointing out this analogy to us.

One can verify that all remaining commutators between these two sets of variables vanish. Equations (5.3)–(5.5) show that the two sets of commuting variables (5.1) and (5.2) are canonical sets. Now it is well known that the usual impact-parameter representation comes from a two-dimensional Fourier transform over the transverse momenta $\mathbf{p}$, and is realized in the E_2 space.[20] We have found in Sec. IV that the W_2 function is simple when we take a Laplace transform over the longitudinal quantity $\ln x = \ln(p_+/m)$. The variable of this new space, τ, is therefore the parameter of a K_3 space. Just as the parameter of the E_2 space is known as the "impact parameter," so we term our new parameter a "longitudinal impact parameter."

The impact parameter has a very simple classical basis which we would like to supply for the longitudinal impact parameter. It turns out that the longitudinal impact representation indeed has a simple physical meaning. Its existence can be established quite generally for a large class of bremsstrahlung processes, satisfying the following assumptions: (1) The process is invariant under the acceleration along the longitudinal direction. This is a kind of scale invariance for the process under $p_\pm \to e^{\pm\lambda}p_\pm$, i.e., under a translation in $z=-\ln p_+$, z running from 0 to ∞. (2) The emission of one pion is independent of the emission of all the others. These assumptions are reasonable if one integrates over all transverse momenta of the emission pions, as we did earlier in our computation. Let $F_N(z)$ be the probability of emission of N pions from a nucleon of initial momentum p_+ to a final nucleon of $p_+'=e^{-z}p_+$. Under this assumption, given $F_1(z)$, we have

$$\begin{aligned} F_2(z) &= \frac{1}{2!}\int_0^z dz_1 F_1(z_1)F_1(z-z_1)\,, \\ F_3(z) &= \frac{1}{3!}\int_0^z dz_1 dz_2 F_1(z_2)F_1(z_1-z_2)F_1(z-z_1)\,. \end{aligned} \tag{5.6}$$

The $1/N!$ comes from the $\mathbf{k}$ phase-space integrals in our model. Consider now the representation of $F_1(z)$ in the longitudinal impact space:

$$\tilde{F}_1(\tau) = \int_0^\infty dz\, e^{-\tau z}F_1(z)\,. \tag{5.7}$$

The transforms of F_2, F_3, etc., are

$$\begin{aligned} \tilde{F}_2(\tau) &= (1/2!)[\tilde{F}_1(\tau)]^2\,, \\ \tilde{F}_3(\tau) &= (1/3!)[\tilde{F}_1(\tau)]^3\,, \\ &\cdots \\ \tilde{F}_M(\tau) &= (1/M!)[\tilde{F}_1(\tau)]^M\,. \end{aligned} \tag{5.8}$$

We see that the independence of pion emission implies a Poisson distribution in the τ space, in which the amplitude for n-pion emission is expressed in terms of the one-pion emission amplitude. Summation over all pions in this τ space then gives an exponential whose argument is the one-pion emission amplitude. This is perfectly analogous to the standard eikonal result: The amplitude for a complete transverse process, summed over all elementary processes, is, in the impact-parameter $\mathbf{b}$ space, of the form $\exp[i\chi(b)]-1$; the amplitude for the elementary transverse process in the $\mathbf{b}$ space is just $\chi(b)$. In order for this representation to hold, the amplitude for an elementary transverse process must not depend on previous occurrence of that transverse process.

In our own particular case, we have seen that

$$\tilde{A}_0 = \exp[a/\tau(\tau+1)]-1\,.$$

The one-pion emission diagram has an amplitude in the τ space of

$$\tilde{A}^{1,\mathrm{outer}} = a/\tau(\tau+1)\,.$$

Thus our result tells us how to calculate n-pion emission in terms of single-pion emission. As we stated above, this is the typical property of a bremsstrahlunglike process. This single-pion emission amplitude has the property that for large z (small τ),

$$\tilde{A}^{1,\mathrm{outer}} \sim a/\tau\,,$$

corresponding to $A^{1,\mathrm{outer}}(z)=\text{const}$, independent of z. For small z (large τ),

$$\tilde{A}^{1,\mathrm{outer}} \sim a/\tau^2\,,$$

corresponding to $A^{1,\mathrm{outer}}(z)\sim z$, i.e., just the phase space.

It seems to us likely that the particular form of the longitudinal impact-parameter eikonal function $a/\tau(\tau+1)$ is very model dependent. However, the fact that the amplitude becomes simple in the longitudinal impact space is possibly a fundamental one. Study of other models is necessary to verify this potentially important statement.

Experimental Consequences

We conclude this section by suggesting some experimental tests of the idea of the longitudinal impact-parameter space. First, let us look into the ep inelastic scattering. According to our discussion and to the results of Ref. 3, the hadrons should fall into two groups. The first group consists of all pions emitted before the nucleon interacts with the current, while the second group consists of all particles created after the nucleon interacts with the current. If one measures, in addition to the energy loss ν and momentum transfer $\mathbf{q}$ from the final electron, the partial cross section for N nonresonant pions in the first group, then one can very easily check the validity of the longitudinal impact space. In particular, the partial $A^{N,\mathrm{outer}}(q^2,\nu)$ should exponentiate in the Mellin transform space when summed over N. It is important to note that for a multipion final state the pions come from nonresonant states. Thus, two pions coming from a decaying ρ emitted from the

[20] N. P. Chang, Phys. Rev. **172**, 1796 (1968).

nucleon must not be counted in the same way as the bremsstrahlung pions, since their emission is correlated. This is undoubtedly the most difficult experimental problem to be faced.

Since the basic assumptions for deriving the longitudinal impact-parameter representation are quite general, one may tend to believe that the longitudinal impact space might also be realized in the pure hadron processes, such as pp and πp scatterings. Experimentally, our ideas should be easier to verify in these processes than in ep scattering. Let us consider high-energy pp scattering: $p+p\to A+B$, where A and B. are two jets of particles moving along, each containing at least one nucleon. We now concentrate on the first jet—say, A. We select these events in A (but include all final states in B) such that A contains a single nucleon of longitudinal momentum $p_{\parallel}'=xp_{\parallel}$, where $p_{\parallel}$ is the incident longitudinal momentum, and N nonresonant pions. This should give us a measure of the probability function $F_N(x)$. By transforming this probability function $F_N(x)$ into Mellin transform space, one can check whether the longitudinal impact space is compatible with the present hadron data.

VI. DISCUSSION

The purpose of this section is to review and elucidate particular features of our model which are of interest. We shall look at distribution properties of the emitted pions, at scaling properties, factorization properties, inclusion of vertex corrections, etc.

A. Distribution Properties

A most striking feature of our model is, in common with superrenormalizable theories, that the pions produced in the inelastic process group into two "jets," one associated with pions emitted before the current insertion—the outer rainbow pions—and the other associated with pions emitted after the current insertion—the inner rainbow pions. The properties of these two jets, as expressed by $A_0(x,q^2)$ and $A_i(q^2)$, are different. The full amplitude has a structure function W_2 which is the product of these quantities times a δ function of argument $\mathbf{q}^2/x-2m\nu$. By measuring $\mathbf{q}^2$ and $2m\nu$, we therefore measure x. The x referred to here is the fraction of the original nucleon p_+ that the nucleon has just after the current insertion.

With this much clear, we can now distinguish two general types of distribution associated with W_2. The first type is the distribution in x, as expressed by $A_0(x,q^2)=\sum_M A^{M,\text{outer}}(x,q^2)$. The second type is the distribution in the number of outer pions, expressed by the M dependence of $A^{M,\text{outer}}(x,q^2)$, and of inner pions, expressed by the N dependence of $A^{N,\text{inner}}(q^2)$.[21]

We shall first examine the distribution in x. We have seen that $A^{\text{outer}}(x,q^2)$ is not very simple in the x space, but is quite simple in the Mellin transform space:

$$\tilde{A}_0(\tau,q^2)=\int_0^1 A_0(x,q^2)x^{\tau-1}dx$$
$$=\exp\left(\frac{g^2}{16\pi^2\tau(\tau+1)}\ln\frac{\mathbf{q}^2}{m^2}\right)-1. \qquad (6.1)$$

If we do not measure x (i.e., if we integrate over all x), then we end up with

$$\int_0^1 A_0(x,q^2)\,dx=\tilde{A}_0(1,q^2)=\exp\left(\frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)-1. \quad (6.2)$$

But this is just $A_i(q^2)-1$:

$$A_i(q^2)=\exp\left(\frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right). \qquad (6.3)$$

Thus, the contribution for the inner pions, in our model, is equivalent to the x-integrated contribution for the outer pions. Physically, this can be understood as follows: In our theory, the contribution for the inner pions is invariant under a boost along the z direction. Hence, $A^{\text{inner}}(q^2)$ is x independent. Therefore, we can evaluate $A^{\text{inner}}(q^2)$ in the same initial frame ($p_+=1$) as we compute A^{outer}. The only distinction is, of course, that in A^{outer} the final p_+' of the nucleon is fixed by $p_+'=xp_+$, while in A^{inner} there is no restriction on the final p_+' of the nucleon at all. It is now not surprising to see that $A_i(q^2)$ is the x integral of $A_0(x,q^2)$.

The above interpretation actually suggests that, if the longitudinal momentum of the final proton is measured to be $p_+'=y\mathbf{q}^2/2m\nu$, then its probability distribution in y is also governed by the same distribution function $A_0(y,q^2)$. That this is true can be verified explicitly. Hence, the partial contribution to W_2 for p_+' (or y) is

$$dW_2(q^2,\nu,p_+')=2m\int_0^1 dx\;\delta(\mathbf{q}^2/x-2m\nu)$$
$$\times A_0(x,q^2)A_0(y,q^2)dy, \quad (6.4)$$

with

$$p_+'=xy=y\mathbf{q}^2/2m\nu.$$

The above relation reveals that the distribution function $A_0(x,q^2)$ is of fundamental importance and has a universal meaning in our model. It simply describes the correlation in the longitudinal momentum of the proton between two interactions, by viewing the pion emissions as a background. Therefore, we would like to learn more about this distribution.

We can use the distribution properties in x to determine various moments in the longitudinal momentum x.

[21] We can, of course, consider simultaneous distributions in M and x, but this does not seem very interesting to us.

Thus the average value of x in our process is

$$\begin{aligned}\langle x\rangle &= \int_0^1 xA_0(x,q^2)\,dx \Big/ \int_0^1 A_0(x,q^2)\,dx \\ &= \tilde{A}_0(2,q^2)/\tilde{A}_0(1,q^2) \\ &= \exp\left(-\frac{g^2}{48\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right). \end{aligned} \tag{6.5}$$

Similarly,

$$\begin{aligned}\langle x^2\rangle &= \tilde{A}_0(3,q^2)/\tilde{A}_0(1,q^2) \\ &= \exp\left(-\frac{5}{12}\frac{g^2}{16\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right). \end{aligned} \tag{6.6}$$

It is quite clear that one could in this way calculate any moment of x. Numerically (6.5) implies for the average x

$$\langle x\rangle \approx (\mathbf{q}^2/m^2)^{-0.4},$$

and for the spread in x,

$$\langle x^2\rangle - \langle x\rangle^2 \approx (\mathbf{q}^2/m^2)^{-0.5}.$$

We see that both $\langle x\rangle$ and $\langle x^2\rangle - \langle x\rangle^2$ decrease moderately with increasing q^2 in our model.

We now address ourselves to the problem of number distribution of pions. We have seen that for the outer rainbow pions,

$$\tilde{A}^{M,\mathrm{outer}}(\tau,q^2) = \frac{1}{M!}\left(\frac{g^2}{16\pi^2\tau(\tau+1)}\ln\frac{\mathbf{q}^2}{m^2}\right)^M, \tag{6.7}$$

and for the inner rainbow pions,

$$A^{N,\mathrm{inner}}(q^2) = \frac{1}{N!}\left(\frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)^N. \tag{6.8}$$

Both (6.7) and (6.8) are Poisson distributions in the number of pions. It is important to recognize, however, that (6.7) for $A^{M,\mathrm{outer}}$ is a Poisson distribution only in the Mellin transform space. It is emphatically not Poisson in the x space, which is the physically measured distribution. This can be seen in the discussion centering around Eq. (4.28), where we looked at the sum over M of $A^{M,\mathrm{outer}}(x,q^2)$.

Because the distribution of inner rainbow pions is Poisson, we can immediately see that

$$\begin{aligned}\langle N\rangle_{\mathrm{inner}} &= \sum_{N=0}^{\infty} NA^{N,\mathrm{inner}}(q^2) \Big/ \sum_{N=0}^{\infty} A^{N,\mathrm{inner}}(q^2) \\ &= \frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2} \end{aligned} \tag{6.9}$$

and

$$\begin{aligned}\langle N\rangle_{\mathrm{inner}}{}^2 &= \sum N^2 A^{N,\mathrm{inner}}(q^2) \Big/ \sum A^{N,\mathrm{inner}}(q^2) \\ &= \frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\left(1+\frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right). \end{aligned} \tag{6.10}$$

This is, of course, a well-known result for Poisson distributions. Naturally, Eq. (6.9) gives for the average number of inner pions

$$\langle N\rangle_{\mathrm{inner}} = 0.6\ln(\mathbf{q}^2/m^2)$$

and Eqs. (6.9) and (6.10) give the same result for the spread $\langle N^2\rangle - \langle N\rangle^2$.

For the outer rainbow pions we shall only give the result in certain limiting cases. In the limit τ small, which corresponds to x small,

$$\tilde{A}^{M,\mathrm{outer}}(\tau,q^2) \approx \frac{1}{M!}\left(\frac{g^2}{16\pi^2\tau}\ln\frac{\mathbf{q}^2}{m^2}\right)^M. \tag{6.11}$$

We find for the inverse Mellin transform

$$A^{M,\mathrm{outer}}(x,q^2) \approx \frac{1}{M!(M-1)!}\left(\frac{g^2}{16\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)^M \times\left(\ln\frac{1}{x}\right)^{M-1}. \tag{6.12}$$

Then, with $a=(g^2/16\pi^2)\ln(\mathbf{q}^2/m^2)$,

$$\begin{aligned}\langle M\rangle &\approx \sum\left[\frac{M}{M!(M-1)!}a^M\left(\ln\frac{1}{x}\right)^{M-1}\right] \Big/ \sum\left[\frac{1}{M!(M-1)!}a^M\left(\ln\frac{1}{x}\right)^{M-1}\right] \\ &\approx \left(\frac{g^2}{16\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\ln\frac{1}{x}\right)^{1/2}. \end{aligned} \tag{6.13}$$

For large τ, corresponding to $x\approx 1$, we have

$$\tilde{A}^{M,\mathrm{outer}}(\tau,q^2) \approx \frac{1}{M!}\left(\frac{g^2}{16\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)^M \tau^{-2M},$$

which gives

$$A^{M,\mathrm{outer}}(x,q^2) = \frac{1}{M!(2M-1)!}\left(\frac{g^2}{16\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)^M \times\left(\ln\frac{1}{x}\right)^{2M-1}. \tag{6.14}$$

The average M is very small ($\sim 1-x$) at $x=1$, i.e., we only see the inner rainbow here.

There are two more distribution properties which are of interest. One is the transverse momentum distribution of the emitted pions, and the other is the average charge properties of the inelastic final states.

In the previously studied field-theory model of inelastic scattering,[3] the transverse momentum of the emitted pions has a cutoff externally imposed. While this cutoff was necessary to ensure the scaling law, it does make the question of the transverse pion momentum a trivial one. In our model the transverse momentum of pions is not imposed by outside and its distribution is of interest. In order to study this, we must return to the original amplitude which determines A^{inner} or A^{outer}. In particular, the difference between the inner and outer pions lies in the longitudinal properties, and we expect the transverse properties of the two jets to be similar. The transverse momenta, by understanding, are defined relative to the longitudinal directions of their respective jets. To calculate these properties, we return to the original expression (4.1) for W_2. We found in Sec. IV that the $\mathbf{q}^2$ dependence comes from integration over transverse momenta of the emitted pions. The factor which gives this is given by (4.11) and (3.13):

$$I_N=\int_0^{\epsilon\mathbf{q}^2}\cdots\int_0^{\epsilon\mathbf{q}^2}dz_1dz_2\cdots dz_N \times\frac{1}{(z_1+a_1)(z_2+z_1+a_2)\cdots(z_N+z_{N-1}+\cdots+a_N)}, \quad (6.15)$$

with $z_i\propto\mathbf{k}^2$ for the ith pion. The limit of integration comes from the minus-component δ function [see Eq. (3.12) *et seq.*]; it provides a cutoff on the z_i integrals at $\epsilon\mathbf{q}^2$. The integrand of Eq. (6.15) may be regarded as a distribution function for the transverse momenta. Thus we have for the average transverse momentum squared of an emitted pion,

$$\langle\mathbf{k}^2\rangle\approx\frac{1}{\sum_N I_N}\sum_N\int(\prod_i dz_i)\times\frac{N^{-1}\sum_i z_i}{(z_1+a_1)(z_1+z_2+a_2)\cdots(z_N+z_{N-1}+\cdots+a_N)} \sim\mathbf{q}^2\frac{\sum_N N^{-1}I_{N-1}}{\sum_N I_N}. \quad (6.16)$$

But we already know that [see (4.12)]

$$I_N=\frac{1}{N!}\left(\ln\frac{\mathbf{q}^2}{m^2}\right)^N, \quad (6.17)$$

so that Eq. (6.16) becomes

$$\langle\mathbf{k}^2\rangle\sim\mathbf{q}^2/\ln(\mathbf{q}^2/m^2). \quad (6.18)$$

Thus the average transverse momentum grows with $\mathbf{q}^2$, but not as fast as $\mathbf{q}^2$; the momentum transfer at each pion vertex is damped at least by a factor $\ln(\mathbf{q}^2/m^2)$ compared to the momentum transfer at the photon vertex.

Finally, the average charge of the produced particles has been of interest in the original parton model of Bjorken.[2] We shall calculate this quantity in our model at the end of Sec. VI C.

B. Scaling

The original suggestion of Bjorken[18] was that νW_2 would depend only on the ratio $\omega=2m\nu/\mathbf{q}^2$. The only appearance of the fractional longitudinal momentum left to the nucleon after the collision is through $\delta(\mathbf{q}^2/x-2m\nu)$. This so-called scaling law, which is at least approximately obeyed by the data, is a fundamental property of the original parton model. It is also a property of a superrenormalizable field theory, as in $\lambda\phi^3$ and the cutoff neutral pseudoscalar theory of Drell, Levy, and Yan. In our model, this scaling law is violated. The partial W_2 for N pions, both for the outer and inner rainbow, has $\mathbf{q}^2$ dependence through $(\ln\mathbf{q}^2)^N$. Of course, the full W_2 has $\mathbf{q}^2$ dependence; it has a simple $(\mathbf{q}^2)^{\text{power}}$ behavior for W_2^{inner} and a rather complicated $\mathbf{q}^2$ dependence [see Eq. (4.28)] for W_2^{outer}. As we have stated previously, the $\mathbf{q}^2$ dependence for W_2^{outer} has simple power behavior in the longitudinal impact-parameter space. Although we shall discuss below why the particular $\mathbf{q}^2$ dependence of our model should not perhaps be taken too seriously, the data at this moment certainly do not forbid some $\mathbf{q}^2$ dependence for W_2, and thereby a breakdown of the scaling law.

C. Form Factors

From Eq. (2.4) it is clear that in our model, in which W_2 grows as a power of $\mathbf{q}^2$, the cross section also shows an increase with $\mathbf{q}^2$. Physically, this too rapid growth of W_2 must be damped by form factors at the various vertices. There are two general types of corrections one might want to consider: those coming from electromagnetic corrections and those coming from strong-interaction corrections. We have seen [see Eq. (6.18) *et seq.*] that the momentum transfer at the photon vertex is larger than the momentum transfer at the pion vertices by at least an order of $\ln(\mathbf{q}^2/m^2)$. Therefore, we only consider corrections to the photon vertex, ignoring the off-mass-shell effect of the nucleon.

One can then calculate pure electromagnetic photon-vertex corrections by replacing γ_μ at the vertex by

$$\gamma_\mu F_1(q^2)+(i\sigma_{\mu\nu}q^\nu/2m)F_2(q^2).$$

Note that this substitution is not strictly correct because the proton is not on the mass shell. However, this should be a good approximation if the off-shell-mass effect is small compared with q^2. In our case, γ_+ in the matrix elements $\mathfrak{M}$ which are squared would be replaced by $\gamma_+F_1+(i\sigma_{+\nu}q^\nu/2m)F_2$. The trace which appears in $|\mathfrak{M}|^2$ is then

$$\mathrm{Tr}\left[(\text{outer})\left(\gamma_+F_1+\frac{i\sigma_{+\nu}q^\nu}{2m}F_2\right)\times(\text{inner})\left(\gamma_+F_1+\frac{i\sigma_{+\mu}q^\mu}{2m}F_2\right)\right] =\mathrm{Tr}\left[(\text{outer})\left(\gamma_+F_1-\frac{i\sigma_{+1}q_1}{2m}F_2\right)\times(\text{inner})\left(\gamma_+F_1-\frac{i\sigma_{+1}q_1}{2m}F_2\right)\right]. \quad (6.19)$$

We have here explicitly taken $\sigma_{+\nu}q^\nu$. There are two direct terms and a cross term to consider in (6.19). The $(\gamma_+F_1)^2$ term is just our old result multiplying $F_1^2(q^2)$. The cross term is of the form $(\sigma_{+\perp}=i\gamma_+\gamma_\perp)$

$$-(i/2m)q_\perp F_1F_2 \times \mathrm{Tr}\{(\text{outer})\gamma_+[(\text{inner})\gamma_+\gamma_\perp+\gamma_\perp(\text{inner})\gamma_+]\}.$$

Since both the inner and outer factors have an even number of γ matrices to leading order, this term contains an odd number of γ matrices and is zero. Finally, using $\gamma_\perp\gamma_+\gamma_\perp=2\gamma_+$, it is easy to see that the F_2^2 term is

$$(1/2m)^2\mathbf{q}^2F_2^2(q^2)\times(\text{old answer}).$$

Thus, the inclusion of these form factors has the property of replacing W_2 by

$$[F_1^2(q^2)-(1/2m)^2q^2F_2^2(q^2)]W_2(\text{old}) =\left(G_E^2-\frac{q^2}{4m^2}G_M^2\right)\Big/\left(1-\frac{q^2}{4m^2}\right)\times W_2(\text{old}),$$

where G_E and G_M are charge and magnetic form factors.[22] Of course, this has the effect of damping the q^2 dependence. In the case of $G_E=G_M/\mu_p=$ universal form factor $F(q^2)$, we have for large q^2,

$$W_2(q^2,\nu)=\int_0^1 dx\ 2mA_0(x,q^2)A_i(q^2)[F(q^2)]^2 \times\delta(\mathbf{q}^2/x-2m\nu), \quad (6.20)$$

where the above equation is valid up to a constant factor. Note that all our discussions on the longitudinal and transverse momentum distribution, number distribution, longitudinal impact representation, and on the scaling law are not affected by this modification. The only effect is the over-all q^2 dependence of the cross section.

We wish to point out that our result on inelastic ep scattering can be generalized to deep-inelastic electron-neutron scattering. All the features discussed in this paper persist, and the inelastic form factor $W_2(n)$ for the en scattering is related simply to the $W_2(p)$ for the ep scattering through

$$\frac{W_2(n)}{W_2(p)}=\frac{F_{1(n)}(q^2)^2-(q^2/4m^2)F_{2(n)}(q^2)^2}{F_{1(p)}(q^2)^2-(q^2/4m^2)F_{2(p)}(q^2)^2} =\frac{-(q^2/4m^2)\mu_n^2}{1-(q^2/4m^2)\mu_p^2}\approx\left(\frac{\mu_n}{\mu_p}\right)^2,$$

where μ_n and μ_p are the anomalous magnetic moments of the neutron and proton, respectively. The last expression depends also on the experimental fact that the nucleon form factors are governed by a universal form factor. Hence, for large q^2,ν, we predict that the deep-inelastic en and ep scatterings have comparable cross section as well as similar final hadron distribution. An experimental test on this conclusion is certainly desirable. The reader should note that the above calculation is not self-consistent, in the sense that the neutron form factor is due to charged pions. We have not, however, considered the inclusion of charged pions in our calculation. A correct treatment would include both neutron and proton as well as the triplet of pions, and hence a much larger class of graphs must be considered. At present we are unable to carry out the complete calculation, but we believe that the above tentative conclusion may still be relevant.

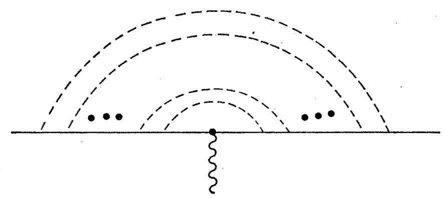

FIG. 6. Set of vertex correction diagrams considered in Ref. 23.

After completing this work, we came across an interesting report of Appelquist and Primack on the electromagnetic form-factor calculation.[23] Among many other diagrams, they considered diagrams such as those in Fig. 6 in a neutral pseudoscalar-meson theory. This calculation is clearly in the same spirit as ours if one keeps only the leading terms in $\ln(|q^2|/m^2)$. To this order, they find a form factor at the photon vertex of

$$F(q^2)=\exp\left(-\frac{g^2}{32\pi^2}\ln\frac{|q^2|}{m^2}\right) \quad (6.21)$$

in the region where q^2 is positive. Since we retain only the leading terms in our calculation, it is natural for us to use Eq. (6.21). Continued to our region of q^2, this result is just such as to cancel the q^2 dependence of $A_i(q^2)$. The q^2 dependence in $A_0(q^2,x)$, however, cannot be completely canceled, owing to the x dependence of the A_0. In other words, the scaling law is partially recovered, if one takes the form factors into account. This cancellation will be exact in processes such as $e^-+e^+\rightarrow$ hadrons in which no momenta of final hadrons are detected.[24]

If this form factor is included in our expression, we have

$$W_2(q^2,\nu)=\int_0^1 dx\ 2mA_0(x,q^2) \times\exp\left(-\frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)\delta\left(\frac{\mathbf{q}^2}{x}-2m\nu\right).$$

[22] D. R. Yennie, M. M. Lévy, and D. G. Ravenhall, Rev. Mod. Phys. 29, 144 (1957). For recent experimental results, see L. H. Chan *et al.*, Phys. Rev. 141, 1298 (1966).

[23] T. Appelquist and J. R. Primack, Phys. Rev. D 1, 1144 (1970).

[24] One of the authors (S.-J. C.) wishes to thank Professor M. Gell-Mann for a stimulating conversation on this point.

Therefore, we have

$$\nu W_2(q^2,\nu)=xA_0(x,q^2)\exp\left(-\frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)\xrightarrow[q^2\to\infty]{}0,$$

$$x=\mathbf{q}^2/2m\nu$$

and, for the average charge for the hadrons,

$$\langle Q^2\rangle=\int dx\,\nu W_2(q^2,\nu)=\exp\left(-\frac{g^2}{48\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)\to 0$$

slowly as $q^2\to\infty$.

This is understandable. Since there is only one charged particle in our model, and the number of pions increase linearly as $\ln\mathbf{q}^2$ increases, the average Q^2 necessarily goes to zero as $q^2\to\infty$.

Another distribution of νW_2 which is of practical interest is

$$\int\frac{1}{x}\nu W_2(q^2,\nu)dx=\int_0^1 A_0(x,q)^2\exp\left(-\frac{g^2}{32\pi^2}\ln\frac{\mathbf{q}^2}{m^2}\right)dx$$

$$=\text{constant of } O(1) \text{ independent of } \mathbf{q}^2.$$

The experimental distributions for Q^2 and $\int x^{-1}\nu \times W_2(q^2,x)dx$ at present q^2, ν values are[9] 0.17 ± 0.01 and $>0.72\pm0.01$, respectively.

D. Multiphoton Exchange

The inclusion of multiphoton exchange in *ep* inelastic process is, strictly speaking, of only academic interest. However, multiparticle exchange processes are important if one tries to generalize the result of this calculation to the realm of strong interactions.

It turns out that the inclusion of multiphoton exchange does not affect the hadronic part of the amplitude at all. The only change in the *ep* scattering is to replace the one-photon propagator $e^2i/\mathbf{q}^2$ by the eikonal form

$$E'(q)=\int d^2b\, e^{-i\mathbf{b}\cdot\mathbf{q}}(e^{-i\chi(b)}-1),$$

whose driving term is the one-photon-exchange amplitude

$$\chi(b)=-e^2\int\frac{d^2q}{(2\pi)^2}\frac{e^{i\mathbf{q}\cdot\mathbf{b}}}{\mathbf{q}^2}.$$

The form of W_2 will be unchanged.

There are two crucial points in reaching the above conclusion. First, there is only one charged particle in the scattering process. Second, at very high energy photons tend to be exchanged as a single unit (i.e., as a bundle[25]) and interact simultaneously, as in the parton model. This last fact makes the description in terms of two structure functions possible. After proper renormalization, the resultant N-photon amplitude is identical, to within a simple kinematical factor, to the corresponding renormalized one-photon amplitude.[26] The summing of N-photon processes with photon vertices permuted in all possible ways is now well known,[27] and leads precisely to the eikonal form $E'(k)$ mentioned above.

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APPENDIX A

In this appendix we would like to give a brief outline of how a calculation of the type given for neutral pseudoscalar-meson theory—a renormalizable theory—can be given for $\lambda\phi^3$. $\lambda\phi^3$ is a superrenormalizable theory, and in that sense it is similar to the cutoff meson theory. In particular, the scaling law holds, a result which we speculate would be true for any superrenormalizable theory.

When we refer to diagrams in this appendix, we mean the diagram referred to with γ_5 of the meson-nucleon propagator replaced by 1 and with a "nucleon" spin of 0 rather than $\frac{1}{2}$.

In $\lambda\phi^3$ theory we cannot say that rainbow diagrams of the Fig. 5 type (for an appropriate power of λ) dominate over diagrams such as Fig. 3(c); that is, pure rainbow diagrams do *not* dominate over crossed rainbows. In this sense, $\lambda\phi^3$ differs in a very important way from our pseudoscalar-meson model. However, diagrams with "pions" crossing from the outer to the inner region, such as Fig. 3(b), *are* small.

These facts indicate that a complete analysis of $\lambda\phi^3$ is rather complicated. We do not attempt this here. It is still possible, however, to learn the most relevant features of this theory. With F a phase-order factor and $\mathfrak{M}$ an appropriate amplitude, $W_2=\int F|\mathfrak{M}|^2$. The phase-space factor F is still given by Eq. (B4). The amplitude $\mathfrak{M}$ is given by, for diagrams of the class of Figs. 5 and 3(c) (inclusion of other diagrams with no crossing from inner to outer regions causes no difficulty),

$$\begin{aligned}\mathfrak{M}=\{[&\Delta_1\Delta_2\cdots\Delta_M\\&+(\text{terms involving other "outer" }\Delta\text{'s})]\\&\times[\Delta_{M+1}\Delta_{M+2}\cdots\Delta_N\\&+(\text{terms involving other "inner" }\Delta\text{'s})]\}^{-1}.\end{aligned}\qquad(\text{A1})$$

[25] S. J. Chang, Phys. Rev. D 1, 2977 (1970); H. Cheng and T. T. Wu, *ibid.* 1, 1069 (1970); 1, 1083 (1970); Y. P. Yao, *ibid.* 1, 2971 (1970).

[26] The program of renormalization is described briefly in Chang's article in Ref. 25.

[27] For an elementary treatment of multiphoton processes, see S. J. Chang and S. Ma, Phys. Rev. Letters 22, 1334 (1969); Phys. Rev. 188, 2385 (1969).

$\Delta_1\cdots\Delta_M$ are as in the text, and depend only on outer quantities, while $\Delta_{M+1}\cdots\Delta_N$ depend only on inner quantities. Since all the Δ_i are Lorentz invariants, the Lorentz transformation (4.2) on the inner rainbow variables accomplishes the factorization of W_2 into two pieces. Thus it is again quite natural that this model produces two jets. The decoupling will again take the form of Eq. (4.5).

In our renormalizable theory, the numerator function N provides an extra $\mathbf{k}_1{}^2\cdot\mathbf{k}_2{}^2\cdots\mathbf{k}_M{}^2$ factor. This makes the theory logarithmically divergent, with the minus-component δ function providing a $\mathbf{q}^2$ cutoff on the transverse integrals. In the superrenormalizable case, the transverse integrals converge without cutoff. Thus there is no $\mathbf{q}^2$ dependence at all introduced by the transverse integral. The transverse $\mathbf{k}_i$'s can be ignored in the minus-component δ function, leaving us with the parton δ function. Since there is no extra q^2 dependence, the Bjorken scaling law holds. Because of the way q^2 dependence enters in the renormalizable theory, we would speculate that any superrenormalizable theory gives the scaling law.

APPENDIX B

The phase-space factor (P.S.F.) for an n-particle final state is

$$\text{P.S.F.}=(2\pi)^4\delta^4(\sum_j p_j-P)\prod_{i=1}^{n}\delta(p_i{}^2-m_i{}^2)\frac{d^4p_i}{(2\pi)^3},\quad\text{(B1)}$$

where P_μ is the total energy-momentum 4-vector. In terms of $x=p_+$, $\mathbf{p}=(p^1,p^2)$, we have

$$\delta(p_i{}^2-m_i{}^2)\frac{d^4p_i}{(2\pi)^3}=\frac{dx_i}{4\pi x_i}\frac{d^2p_i}{(2\pi)^2},\quad\text{(B2)}$$

$$(2\pi)^4\delta^4(\sum p_i-P)=8\pi^2\delta(\sum x_i-P_+)(2\pi)^2\delta^2(\sum\mathbf{p}_i-\mathbf{P})\times\delta\left(\sum\frac{\mathbf{p}_i{}^2+m_i{}^2}{x_i}-P_-\right).\quad\text{(B3)}$$

Hence, the P.S.F. can be rewritten as

$$8\pi^2\delta(\sum x_i-P_+)(2\pi)^2\delta^2(\sum\mathbf{p}_i-\mathbf{P})\delta\left(\sum_j\frac{\mathbf{p}_j{}^2+m_j{}^2}{x_j}-P_-\right)\times\prod_i\frac{dx_i}{4\pi x_i}\frac{d^2p_i}{(2\pi)^2}.\quad\text{(B4)}$$

Note that the W_2 can be expressed simply as a product of the hadron P.S.F. and the square of an amplitude $|\mathfrak{M}|^2$ through

$$W_2=(m/4\pi)|\mathfrak{M}|^2\times(\text{P.S.F.}),$$

where $|\mathfrak{M}|^2$ is the imaginary part of the invariant amplitude of the forward plus component of the Compton scattering. This relation is represented graphically as in Fig. 2(c).

It is important to note that the choice of the variables p^1, p^2, and $p_+=p^0+p^3$ for phase space [see Eq. (B2)] introduces only the very simple p_+ into the denominator. The conventional choice of p^1, p^2, and p^3 introduces a p^0 into the denominator, with the attendant square-root difficulties. It is this fact which allows us to perform the $\int d^2k$ integrals in the text with no approximations on transverse quantities in denominators.