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Implications of Certain Algebraic Aspects of Dual Resonance Models*

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Certain algebraic aspects present in the dual resonance models led Susskind and Frye to the construction of n-point pion amplitudes in the tree-graph approximation which are a sum of terms, each one coming from a distinct permutation and satisfying the Adler's condition separately. It was also shown that this line of reasoning picks up a definite form of chiral symmetry breaking (in $SU2\otimes SU2$). In this work we show that the extension to the pseudoscalar multiplet forces one to include the whole nonet. The resulting amplitudes coincide with those obtained from an $U(3)\otimes U(3)$ model with the chiral symmetry broken in a way that preserves the complete nonet degeneracy.

I. INTRODUCTION: FORMAL CONSTRUCTION OF n-POINT FUNCTION

The dual theory of strong interactions replaces the usual sum of Feynman tree graphs by sums of contributions coming from different permutations of the external particles. The term corresponding to a definite permutation depends only on the generalized Mandelstam variables associated with that arrangement of the external legs.

The hypothetical world considered by Susskind and Frye[1] containing only pions is extended here to include the pseudoscalar mesons.

As was pointed out by them, the one-loop Feynman diagram can be put into a one-to-one correspondence with distinct permutations.[2] This, and the absence of exotic resonances, strongly suggests that an internal quark loop to which the external mesons couple will indicate the appropriate $SU(3)$-invariant amplitude.

The $SU(3)$-invariant coupling of quarks with a pseudoscalar octet

$$\tilde{q}\gamma_5\lambda_k q\phi_k \tag{1}$$

contributes to the amplitude corresponding to the permutation 1, 2, ..., n with a factor

$$\mathrm{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_n}).$$

α_i is the $SU(3)$ index of the ith particle, and the λ's are the usual Gell-Mann matrices[3]; but this is not invariant under anticyclic rearrangements. We shall consider a symmetrized version of it, namely,

$$\mathfrak{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_n})\equiv\mathrm{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_n})+\mathrm{Tr}(\lambda_{\alpha_n}\cdots\lambda_{\alpha_2}\lambda_{\alpha_1}). \tag{2}$$

The whole amplitude will be given by the expression

$$\sum_{\text{distinct permutations}}\mathfrak{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_n})F_n(s^{1,2,\ldots,n}), \tag{3}$$

where $s^{1,2,\ldots,n}$ stands for the set of Mandelstam variables associated with the permutation 1, 2, ..., n.

The function F has the following properties: (a) It is invariant under cyclic and anticyclic rearrangements. (b) It goes to zero when the four-momentum of any one of the external particles goes to zero, the others being on the mass shell (Adler's condition[4]).

We shall show that (3) can be enforced in a way consistent with factorization. Let us consider a graph consisting of a certain number n of external lines entering a vertex, a number m entering a second vertex (n and m odd), and a single line connecting both vertices. The expression which gives the contribution of such a graph to the $(n+m)$-point function is

$$\sum_{\epsilon=1}^{8}\mathfrak{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_n}\lambda_\epsilon)\,\mathfrak{Tr}(\lambda_{\beta_1}\cdots\lambda_{\beta_m}\lambda_\epsilon)\frac{F_nF_m}{s_{1,2,\ldots,n}-m^2}. \tag{4}$$

Using the identity

$$\sum_{\epsilon=1}^{8}\mathrm{Tr}(X\lambda_\epsilon)\,\mathrm{Tr}(Y\lambda_\epsilon)=2\,\mathrm{Tr}(XY)-\tfrac{2}{3}\,\mathrm{Tr}X\,\mathrm{Tr}Y,$$

where X and Y are any 3×3 matrices, it is possible to prove for the symmetrized traces (2) (see Appendix)

$$\sum_{\mathrm{perm}\,\alpha_i}\sum_{\mathrm{perm}\,\beta_j}\sum_{\epsilon}\mathfrak{Tr}(\lambda_{\alpha_1}\cdots\lambda_{\alpha_n}\lambda_\epsilon)\,\mathfrak{Tr}(\lambda_\epsilon\lambda_{\beta_1}\cdots\lambda_{\beta_m})\times\frac{F_nF_m}{s_{1,2,\ldots,n}-m^2}=\sum_{\mathrm{perm}\,\alpha_i\beta_j}[4\,\mathfrak{Tr}(\lambda_{\alpha_1}\cdots\lambda_{\alpha_n}\lambda_{\beta_1}\cdots\lambda_{\beta_m})-\tfrac{2}{3}\,\mathfrak{Tr}(\lambda_{\alpha_1}\cdots\lambda_{\alpha_n})\,\mathfrak{Tr}(\lambda_{\beta_1}\cdots\lambda_{\beta_m})]\frac{F_nF_m}{s_{1,2,\ldots,n}-m^2}. \tag{5}$$

The first term on the right-hand side is of the desired form, but it is clear that the second term has no place in an $(n+m)$-point function of the form (3).

The additional interchange of a singlet will provide an extra term of the form

$$\mathfrak{Tr}(\lambda_{\alpha_1}\cdots\lambda_{\alpha_n}\lambda_0)\,\mathfrak{Tr}(\lambda_{\beta_1}\cdots\lambda_{\beta_m}\lambda_0)\frac{F_n{}^0F_m{}^0}{s_{1,2,\ldots,n}-m^2} \tag{6}$$

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[1] L. Susskind and G. Frye. Phys. Rev. D **1**, 1682 (1970).

[2] Permutations which are related by cyclic or anticyclic rearrangements are considered equal.

[3] M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (Benjamin, New York, 1964).

[4] S. L. Adler, Phys. Rev. **139**, B1638 (1965).

on both sides of (5). Such a term will exactly cancel the undesirable term, provided that the singlet is completely degenerate with the octet [recalling that $\lambda_0 \equiv (\sqrt{\frac{2}{3}})\mathbf{1}$]. The expression is finally given by

$$\sum_{\text{perm}\,\alpha_i} \sum_{\text{perm}\,\beta_j} \sum_{\epsilon=0}^{8} \mathfrak{Tr}(\lambda_{\alpha_1}\cdots\lambda_{\alpha_n}\lambda_\epsilon) \times \mathfrak{Tr}(\lambda_\epsilon\lambda_{\beta_1}\cdots\lambda_{\beta_m}) \frac{F_n F_m}{s_{1,2,\ldots,n}-m^2} = \sum_{\text{perm}\,\alpha_i} \sum_{\text{perm}\,\beta_j} 4\,\mathfrak{Tr}(\lambda_{\alpha_1}\cdots\lambda_{\alpha_n}\lambda_{\beta_1}\cdots\lambda_{\beta_m}) \frac{F_n F_m}{s_{1,2,\ldots,n}-m^2}.$$

II. FOUR-POINT AND SIX-POINT FUNCTIONS

In a model with only the pseudoscalar mesons present, the four-point function has no poles. The necessity of avoiding essential singularities at infinity restricts $F_4(s^{1,2,3,4})$ to a polynomial, which on the other hand must satisfy Adler's condition. The simplest choice

$$F_4(s^{1,2,3,4}) = (s_{12}+s_{23}-2m^2) \tag{7}$$

has been successfully used in Ref. 1.

Making the same assumption here, the total expression for the four-point amplitude is

$$A_4 \propto \mathfrak{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\lambda_{\alpha_3}\lambda_{\alpha_4})(s_{12}+s_{23}-2m^2) + \text{perm}(1243) + \text{perm}(1423). \tag{8}$$

The contribution to the six-point amplitude coming from tree graphs with one internal line is obtained using (6).

The part corresponding to the permutation 123456 is

$$\mathfrak{Tr}(\lambda_{\alpha_1}\cdots\lambda_{\alpha_6}) \left[\frac{(s_{12}+s_{23}-2m^2)(s_{56}+s_{45}-2m^2)}{s_{123}-m^2} + \frac{(s_{23}+s_{34}-2m^2)(s_{16}+s_{56}-2m^2)}{s_{234}-m^2} + \frac{(s_{34}+s_{45}-2m^2)(s_{61}+s_{12}-2m^2)}{s_{345}-m^2} \right]. \tag{9}$$

The contact term corresponding to this permutation is determined, imposing the validity of Adler's condition. In fact, it is required to be

$$-\mathfrak{Tr}(\lambda_{\alpha_1}\cdots\lambda_{\alpha_6})(s_{12}+s_{23}+s_{34}+s_{45}+s_{56}+s_{61}-6m^2). \tag{10}$$

This procedure can be indefinitely extended, providing the amplitude corresponding to an arbitrary even number of external legs.

III. $U(3)\otimes U(3)$ LAGRANGIAN MODEL

We are going to show now the existence of a Lagrangian in nonlinear $U(3)\otimes U(3)$ which agrees with the results obtained in the previous section. Our approach is a modified version of the one taken by Cronin[5] and by Bardeen and Lee.[6]

The part of the Lagrangian $U(3)\otimes U(3)$ symmetric is taken to be

$$\mathcal{L}_0 = -(1/8f^2)\,\mathrm{Tr}\partial_\mu M^\dagger \partial_\mu M. \tag{11}$$

f has the dimensions of mass^{-1} and is chosen to be real. M is the meson coupling matrix which is a function of the pseudoscalar meson matrix ϕ.

$$\phi = \frac{1}{\sqrt{2}} \sum_{i=0}^{8} \lambda_i \phi_i. \tag{12}$$

$M^\dagger M$ is an $U(3)\otimes U(3)$ invariant that we shall take as 1.

We break the chiral symmetry by considering as the whole Lagrangian

$$\mathcal{L} = -\frac{1}{8f^2}\mathrm{Tr}(\partial_\mu M^\dagger \partial_\mu M) + \frac{m^2}{8f^2}\mathrm{Tr}(M+M^\dagger). \tag{13}$$

We expand M as a power series in ϕ:

$$M = \sum_{n=0}^{\infty} a_n (if\phi)^n. \tag{14}$$

The coefficients a_n are real because of parity invariance. Without loss of generality, we can put $a_0=1$; a_1 may be absorbed into f and we choose $a_1=2$. The condition $MM^\dagger=1$ gives some relations between the coefficients.

Up to sixth order, we can express a_4 and a_6 in terms of a_3 and a_5.

$$a_4 = 2(a_3-1), \quad a_6 = 2a_5 - 4(a_3-1) + \tfrac{1}{2}a_3^2. \tag{15}$$

Cronin showed that Adler's condition for the four-point function is satisfied only if $a_3=0$. So the expression for M is

$$M = 1 + 2if\phi - 2(f\phi)^2 - 2(f\phi)^4 + ia_5(f\phi)^5 - (2a_5+4)(f\phi)^6 + \cdots. \tag{16}$$

Using (16), we can rewrite the Lagrangian (13) up to sixth order in the fields:

$$\mathcal{L} = -\tfrac{1}{2}f^2[\mathrm{Tr}(\partial_\mu\phi^2\partial_\mu\phi^2) + m^2\,\mathrm{Tr}(\phi^4)] - f^4[\tfrac{1}{2}a_5\,\mathrm{Tr}(\partial_\mu\phi\partial_\mu\phi^5) + \mathrm{Tr}(\partial_\mu\phi^2\partial_\mu\phi^4) + \tfrac{1}{4}m^2\,\mathrm{Tr}(\phi^6)]. \tag{17}$$

In order to illustrate how this Lagrangian is able to reproduce the amplitudes given in Sec. II, we shall explicitly work out some steps in the four-point case.

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[5] J. A. Cronin, Phys. Rev. **169**, 1483 (1967).
[6] W. A. Bardeen and B. W. Lee, Phys. Rev. **177**, 2389 (1969).

Using (12), we can write

$$\mathcal{L}^{(4)}=-\tfrac{1}{8}f^2[\![\sum_{ijlm}\mathrm{Tr}[\lambda_i\lambda_j\lambda_l\lambda_m]\{[\phi_i(\partial_\mu\phi_j)\phi_l(\partial_\mu\phi_m)+\phi_i(\partial_\mu\phi_j)(\partial_\mu\phi_l)\phi_m+(\partial_\mu\phi_i)\phi_j\phi_l(\partial_\mu\phi_m)+(\partial_\mu\phi_i)\phi_j(\partial_\mu\phi_l)\phi_m]+[m^2\phi_i\phi_j\phi_l\phi_m]\}]\!] . \quad (18)$$

Let us consider the matrix elements of the first term in (18), taking all four particles as incoming. The total factor multiplying $\mathrm{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\lambda_{\alpha_3}\lambda_{\alpha_4})$ (recalling the cyclic invariance property of the regular traces) is

$$\mathrm{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\lambda_{\alpha_3}\lambda_{\alpha_4})(2p_1p_3+2p_2p_4).$$

In fact, the same factor appears multiplied by $\mathrm{Tr}(\lambda_{\alpha_4}\lambda_{\alpha_3}\lambda_{\alpha_2}\lambda_{\alpha_1})$.

So, using definition (2), we can write the whole contribution coming from the first term in (18) to the permutation (1234):

$$\mathfrak{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\lambda_{\alpha_3}\lambda_{\alpha_4})(2p_1p_3+2p_2p_4). \quad (19)$$

A similar work provides the total expression for this permutation, namely,

$$-\tfrac{1}{4}f^2\,\mathfrak{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\lambda_{\alpha_3}\lambda_{\alpha_4})[2p_1p_3+2p_2p_4+p_2p_3+p_3p_4+p_4p_1+p_1p_2+2m^2]=\tfrac{1}{4}f^2\,\mathfrak{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\lambda_{\alpha_3}\lambda_{\alpha_4})\times(s_{12}+s_{23}-2m^2), \quad (20)$$

which is precisely the amplitude proposed in Sec. II.

The six-point case can be worked out in a similar way. The contact term in this case has the expression

$$-\tfrac{1}{8}f^4\,\mathfrak{Tr}(\lambda_{\alpha_1}\cdots\lambda_{\alpha_6})[(p_1p_2+p_2p_3+p_3p_4+p_4p_5+p_5p_6+p_6p_1)(a_5+2)+(p_1p_3+p_2p_4+p_3p_5+p_4p_6+p_5p_1+p_6p_2)(a_5+4)+(p_1p_3+p_2p_5+p_4p_6)\times(a_5+4)+\tfrac{3}{2}(2a_5+4)m^2],$$

which contains the arbitrary parameter a_5. Putting it equal to 0, we get the expression desired:

$$\tfrac{1}{8}f^4\,\mathfrak{Tr}(\lambda_{\alpha_1}\lambda_{\alpha_2}\lambda_{\alpha_3}\lambda_{\alpha_4}\lambda_{\alpha_5}\lambda_{\alpha_6})\times(s_{12}+s_{23}+s_{34}+s_{45}+s_{56}+s_{61}-6m^2).$$

IV. CONCLUSIONS

It was the purpose of this work to continue investigating the apparent connection between dual theory and chiral symmetry, and the use of dual theory in choosing the way in which chiral symmetry is broken.

The natural extension from the pion isomultiplet and an $SU(2)\otimes SU(2)$ algebra to higher symmetries led us to include the whole pseudoscalar nonet as a nonlinear realization of the group $U(3)\otimes U(3)$.

This Lagrangian model is semirealistic in the sense that it includes the η' with the same mass as the octet (this is far from true in nature), but at the same time it deals with only physical particles.

Note added in proof. After completion of this work, we received a paper by J. Ellis and B. Renner [Cambridge University report (unpublished)], where some of the results obtained by us are discussed, though with a somewhat different spirit.

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APPENDIX

We know that the identity

$$\sum_\epsilon \mathrm{Tr}(X\lambda_\epsilon)\,\mathrm{Tr}(Y\lambda_\epsilon)=2\,\mathrm{Tr}(XY)-\tfrac{2}{3}\,\mathrm{Tr}X\,\mathrm{Tr}Y \quad (\mathrm{A1})$$

is valid in general for X and Y arbitrary matrices (3×3). We define

$$\mathfrak{Tr}(X)\equiv\mathrm{Tr}(X+X'),$$

where

$$X'=\lambda_{\alpha_n}\cdots\lambda_{\alpha_2}\lambda_{\alpha_1} \quad \text{if} \quad X=\lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_n}.$$

We are interested in

$$\sum_\epsilon \mathfrak{Tr}(X\lambda_\epsilon)\,\mathfrak{Tr}(Y\lambda_\epsilon)$$

$$=\sum_\epsilon \{[\mathrm{Tr}(X\lambda_\epsilon)+\mathrm{Tr}(\lambda_\epsilon X')][\mathrm{Tr}(Y\lambda_\epsilon)+\mathrm{Tr}(\lambda_\epsilon Y')]\}$$

$$=\sum_\epsilon \{\mathrm{Tr}[(X+X')\lambda_\epsilon]\,\mathrm{Tr}[(Y+Y')\lambda_\epsilon]\}$$

$$=2\,\mathrm{Tr}[(X+X')(Y+Y')]-\tfrac{2}{3}\,\mathrm{Tr}(X+X')\,\mathrm{Tr}(Y+Y'). \quad (\mathrm{A2})$$

Let us consider the first term in (A2), and

$$R\equiv\sum_{\mathrm{perm}\,X}\sum_{\mathrm{perm}\,Y}2\,\mathrm{Tr}[(X+X')(Y+Y')]F_XF_Y, \quad (\mathrm{A3})$$

where the F's are such that $F_X=F_{X'}$.

$$R=2\sum_{\mathrm{perm}\,X}\sum_{\mathrm{perm}\,Y}\{[\mathrm{Tr}(XY)+\mathrm{Tr}(Y'X')]+[\mathrm{Tr}(XY')+\mathrm{Tr}(X'Y)]\}F_XF_Y. \quad (\mathrm{A4})$$

It is trivial that the second term in (A4) will give the same contribution as the first one after summing over permutations. Then

$$R=4\sum_{\mathrm{perm}\,X}\sum_{\mathrm{perm}\,Y}\mathfrak{Tr}(XY)F_XF_Y. \quad (\mathrm{A5})$$

Finally, we have for the $\mathfrak{Tr}$'s the relation

$$\sum_{\mathrm{perm}\,X}\sum_{\mathrm{perm}\,Y}\sum_\epsilon \mathfrak{Tr}(X\lambda_\epsilon)\,\mathfrak{Tr}(Y\lambda_\epsilon)F_XF_Y=\sum_{\mathrm{perm}\,X}\sum_{\mathrm{perm}\,Y}[4\,\mathfrak{Tr}(XY)-\tfrac{2}{3}\,\mathfrak{Tr}(X)\,\mathfrak{Tr}(Y)]\times F_XF_Y. \quad (\mathrm{A6})$$