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Diophantine Quantization: Application of the Methods of Algebraic Number Theory to the Theory of Elementary Particles

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The Gell-Mann–Okubo broken-$SU(3)$ meson mass formula is treated as a quadratic Diophantine equation and solved in integers. The relative masses of π, η, K, many known K^*, and the probable lowest L may be predicted unambiguously. The requirement of least-integer solutions mandates a unit mass of 70 MeV ($137\, m_e c^2$). Connections between rest masses and angular momenta can be made, and there is evidence of a general linear group.

IN this paper, we consider the Gell-Mann–Okubo meson mass relation[1]

$$\pi^2+3\eta^2=4K^2 \tag{1}$$

as a Diophantine equation and seek solutions in integers, similar to the sets (3,4,5), (5,12,13), etc., for the Pythagorean equation $x^2+y^2=z^2$. From standard methods of algebraic number theory[2] we may deduce the result

$$\pi=\tfrac{1}{2}\,|3a^2-b^2|\,, \tag{2a}$$

$$\eta=ab\,, \tag{2b}$$

$$K=\tfrac{1}{4}(3a^2+b^2)\,, \tag{2c}$$

with a and b integers. To limit ourselves to nontrivial and least-integral solutions, we require, respectively,

$$b\neq a\,,\quad b\neq 3a \tag{3}$$

and

$$\text{either } (a,b)=1\,,\quad a+b\equiv 0 \pmod 2 \tag{4a}$$

$$\text{or } (a,b)=2\,,\quad a+b\equiv 2 \pmod 4 \tag{4b}$$

where (a,b) is the greatest common divisor of a and b.

A very simple solution of Eq. (1), with $a=2$, $b=4$, corresponds to $\pi=2$, $\eta=8$, and $K=7$. If we compare these numbers with the experimental masses[3] of the basic meson octet [$m(\pi)=135$–140 MeV, $m(\eta)=549$ MeV, $m(K)=494$–498 MeV], we see at once that (a) this simple solution in integers happens to give very closely the relative masses of the basic meson octet, and (b) if the masses are divided by this simplest set of integers, we get quite closely the unit mass 70 MeV $=137 m_e c^2$ originally proposed by Nambu.[4]

To go further, we first look at the K masses, since experimentally there are fewer of them, and theoretically it is very easy to compute a complete spectrum since Eq. (2c) is additive, there is a triple degeneracy in a and b for the K's, and probably most important of all, there is little mixing. Column 2 of Table I lists all permitted values of K through 31; the corresponding sets of a and b are listed in column 1. Column 3 lists $70K$, the predicted masses in MeV. The last columns list the established[3] experimental masses, together with J^P assignments, and two numbers J_1 and J_2 which will be discussed below. From Table I we can see that all established K masses except 1775 can be fitted with the Diophantine scheme; and, in fact, $K=31$ can be fitted to the bump $K^*(2240)$.

Consider further mesons with $I=\frac{3}{2}$ (and assumed $Y=1$) which from recent convention[3] we shall call L mesons. If we go back to the original Okubo formula[1] for mesons,

$$m^2=m_0^2+m_2^2[I(I+1)-\tfrac{1}{4}Y^2]\,, \tag{5}$$

we see that for a broken-$SU(3)$ multiplet which includes $I=\frac{3}{2}$, we could derive three more equations like Eq. (1), of which two more are independent. With the values of a and b in Table I, the only integral value of L under 34 satisfying all four equations in integers turns out to be $L=17$ (1190 MeV) corresponding to $a=3$, $b=1$. It is interesting that the only mesons discovered so far[3] with

TABLE I. Comparison with experiment for K mesons ($I=\frac{1}{2}$, $Y=1$).

Basic integers a,b	Mass number K	Predicted mass (MeV) $70K$	Experimental mass (MeV) m	J^P	J-related quantum Nos. J_1	J_2
1, 5					$-\frac{1}{2}$	$\frac{1}{2}$
2, 4	7	490	K 494–498	0^-	0	0
3, 1					0	0
1, 7					0	1
3, 5	13	910	K^*892–899	1^-	1	0
4, 2					1	0
2, 8					1	1
3, 7	19	1330	K_A 1200–1350	$1^+(2^-)$	$\frac{3}{2}$	$\frac{1}{2}$
5, 1					$\frac{3}{2}$	$\frac{1}{2}$
1, 9					$\frac{1}{2}$	$\frac{3}{2}$
4, 6	21	1470	K_N 1420	$2^+(3^-)$	2	0
5, 3					2	0
			$K_{A,L}$1775			
1, 11					1	2
5, 7	31	2170	a	b	3	0
6, 4					3	0

a Bump at 2240; see text.
b 3 predicted.

[1] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962); M. Gell-Mann, California Institute of Technology Report No. CTSL-20, 1961 [unpublished but reprinted (as is Okubo's paper) in M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (Benjamin, New York, 1964)]. The use of squared masses for mesons was suggested by R. P. Feynman.

[2] R. D. Carmichael, *Diophantine Analysis* (Wiley, New York, 1915; reprinted by Dover, New York, 1959). Equations (2) may be derived by a slight extension of the rational-solution method on p. 3. For a more sophisticated and rigorous treatment of the general field of algebraic number theory, see G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed. (Oxford U. P., Oxford, England, 1960).

[3] A. Barbaro-Galtieri *et al.*, Rev. Mod. Phys. **42**, 87 (1970).

[4] Y. Nambu, Progr. Theoret. Phys. (Kyoto) **7**, 595 (1952).

a possible $I=\frac{3}{2}$ are bumps with masses of 1175 and 1265 MeV, respectively.

Angular momenta. We should try to associate a and b with J and a second related quantum number. Two likely candidates for this second number, parity and the $\mathbf{L}\cdot\mathbf{S}$ from quark spectroscopy, appear to be unhelpful. However, the quantities

$$J_1\equiv\tfrac{1}{4}(3a+b-10)\,, \tag{6a}$$

$$J_2\equiv\tfrac{1}{4}(|a-b|-2) \tag{6b}$$

serve to put the K mass formula (2c) in the simple form

$$K=1+(J_1+2)(J_1+3)+3J_2(J_2+1)\,. \tag{7}$$

Values of J_1 and J_2 are shown in Table I. The second quantum number J_2 enters only for $K=19$ or for the first set of a and b. Note also that $L=17$ is assigned $J_1=J_2=0$ on this basis. It is also interesting that in all cases

$$\mathbf{J}=\mathbf{J}_1+\mathbf{J}_2 \quad \text{(vector addition)}\,, \tag{8}$$

which incidentally would allow a mixture of $J=1$ and $J=2$ for $K=19$ and might help explain the broad spectrum. From Eqs. (1), (2), and (5) one may obtain the mass formula

$$m^2=a^2b^2+\tfrac{1}{8}(9a^2-b^2)(a^2-b^2)[I(I+1)-\tfrac{1}{4}Y^2], \tag{9}$$

which from (6) may be expressed in terms of J's if desired.

Baryons. The mass equation analogous to (1) for the basic baryon octet,[1] cleared of fractions, is

$$2(N+\Xi)=3\Lambda+\Sigma\,. \tag{10}$$

A set of numbers consistent with (10) and with the experimental masses is $N=27/2$ (945 MeV), $\Xi=19$ (1330 MeV), $\Lambda=16$ (1120 MeV), $\Sigma=17$ (1190 MeV). But the argument is much less compelling from the standpoint of number theory, since a Diophantine equation with $n=1$ has a plethora of solutions, and the above set of numbers is by no means the simplest solution of Eq. (10)—for example, all solutions could be diminished by the same constant. In general, it would appear preferable to seek theoretical mandates from a study of the mesons.

Mesons with $I=1$ and $I=0$. In contrast to the rather tight predictions for the K and L mesons, Eqs. (2a) and (2b), for π mesons ($I=1$) and η mesons ($I=0$), respectively, predict large numbers of masses probably not found in nature. The best available scapegoat would appear to be mixing, which would be expected to destroy Eq. (1) in many cases. One can hope that the problem will be cleared up when an underlying group is found. It is encouraging that most of the low-lying mesons with $I=1$ or $I=0$ do have masses close to integral multiples of 70 MeV.

Leptons. One of the bases of the original hypothesis of Nambu[4] was that, on the 70-MeV scale, the muon has a mass number very close to $\frac{3}{2}$. It would be tempting to speculate that the muon be included in any underlying group (along with assignments $e=\nu=0$), but this is extremely premature. If this happened to be so, it would be an argument against the generality of the quark scheme, since the muon ($B=0$, $J=\frac{1}{2}$) is not constructible from quarks and antiquarks; in fact, Gell-Mann's original triplets[1] might be more consistent with the over-all group.

Electromagnetic effects. As pointed out by Nambu,[4] electromagnetic effects would constitute a fine structure. An interesting hypothesis would be that the departures in mass values from precise multiples of 70 MeV are due to electromagnetic effects; unfortunately, it appears to be untestable by an appeal to $SU(3)$, as it is tautologically consistent with, e.g., the Coleman-Glashow relation.[5] While the order of magnitude is certainly correct, an inspection of the experimental masses[3] appears to give no immediately obvious orderly conclusions, and at present such a hypothesis remains more a Procrustean bed than a useful guide. A further point, which is of extreme interest in view of an apparently stronger position for mass quantization, is that, in terms of a unit of 70 MeV, the rest mass of the electron is *precisely*, not just approximately, α.

An underlying group. Equations (7) and (8) (and the simpler fact that, except for $K=19$, $J=J_1+J_2$ arithmetically) remind one of expressions associated with the decomposition of $SO(4)$ or (the Lorentz group) $SO(1,3)$ into two rotation groups $SO(3)$. One might strongly suspect that there would be, for the K mass structure, an underlying linear group which is a sufficient generalization of $SO(4)$ or (more likely, the noncompact) $SO(1,3)$ to take the factor of 3 in Eq. (7) into account. Then such a group would have to be further enlarged to allow for other values of isospin.

The unit mass. The significance of the unit mass is not necessarily that such a particle exists, nor that physical particles are made out of such a building block. The immediate conclusions, rather, would appear to deal more with physical theory: (a) A scale has now been set such that the fundamental length proposed by Heisenberg[6] can be assigned a numerical value, because the Diophantine approach identifies a set of unambiguously defined integers, the smallest whole integers satisfying a certain equation, with a set of experimental masses. (b) The broken-$SU(3)$ mass formulas of Gell-Mann and Okubo[1] are now asserted to be exact, rather than perturbations on something else. Hence, such mass formulas, and any underlying group which would yield them, should be accepted as a starting point, and not merely considered a result of field theory.

These conclusions may be a little premature, if not

[5] S. Coleman and S. L. Glashow, Phys. Rev. Letters **6**, 423 (1961).

[6] W. Heisenberg, Z. Physik **32**, 20 (1938).

simply wrong, but in any case I would suggest that both the hypothesis of Nambu[4] and the mass formulas of Gell-Mann and Okubo[1] are exact in integers, that a strong theoretical support can be given to them in terms of algebraic number theory, and that the methods of algebraic number theory should be used as a test of, or even an approach to, elementary-particle mass formulas.

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Unitary Nonplanar Closed Loops. II*

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Using the methods developed in a previous paper, we generalize the calculation of the dual amplitude for a nonplanar diagram with a single closed loop to the case with an arbitrary number of "twisted vertices." Just as with the four-point function discussed in the previous paper, we find that, if we write the amplitude $M=\int_\Sigma d^4\Pi\, \mathfrak{M}(\Pi)$, $\mathfrak{M}(\Pi)$ is periodic in Π, the period being given by the sum of the four-momenta of the "twisted vertices." We then show in the general case that the prescription of choosing Σ to range over just one period yields the imaginary part required by perturbative unitarity. We verify that M so defined is dual in three different ways. We show explicitly that our result is equivalent to the Kikkawa-Klein-Sakita-Virasoro (KKSV) prescription; we also prove duality directly from the Bardakci-Ruegg–like form without reference to the KKSV structure; and finally we show that duality is manifest within the operator formalism *before* the trace is performed.

I. INTRODUCTION

IN a recent paper,[1] hereafter referred to as I, we have developed techniques for calculating the dual amplitude corresponding to a nonplanar diagram with one closed loop. In I, we worked out explicitly the amplitude for the four-point diagram with one "twisted vertex." The result was found to be almost precisely the form predicted by the duality arguments of Kikkawa, Klein, Sakita, and Virasoro (KKSV).[2] When we wrote the single loop amplitude as $M=\int_\Sigma \mathfrak{M}(\Pi)d^4\Pi$, we found that, for the four-point function, $\mathfrak{M}(\Pi)$ was periodic in Π with the period given by the four-momentum of the twisted vertex, or equivalently, by the sum of the four-momenta of the "untwisted vertices." We then found that by choosing the region of integration, Σ, to range over precisely one period we obtain the correct result for the imaginary part.

In this paper we present the explicit calculation of the arbitrary N-point nonplanar amplitude with m untwisted vertices and $N-m$ twisted vertices. Again we find that, except for a factor

$$\prod_{r=1}\left(\frac{1}{1-x^r}\right)^4,$$

our result is in agreement with the KKSV prescription if one includes *all* lines in the dual diagram. In the arbitrary case we find that $\mathfrak{M}(\Pi)$ is periodic with period given by the sum of the four-momenta of the untwisted vertices.

In Sec. II we derive our result for $\mathfrak{M}(\Pi)$ and write it in Kikkawa-like form. In Sec. III we discuss the singularity structure of $\mathfrak{M}(\Pi)$ and show that by choosing the region Σ to range over just one period, we obtain precisely the result required by perturbative unitarity. In Sec. IV we discuss the duality of the final result; in particular, we discuss duality within the framework of the operator formalism. Most of the details of the calculations are relegated to the appendices.

II. CALCULATION OF UNITARY CLOSED LOOP

Writing the amplitude for Fig. 1 as $M=\int_\Sigma d^4\Pi\mathfrak{M}(\Pi)$, with the region Σ to be specified in accordance with

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† NSF Graduate Fellow.
[1] M. Kaku and C. B. Thorn, Phys. Rev. D 1, 2860 (1970). We refer to formulas from this reference by prefixing a Roman numeral I, e.g., (I,2.3).
[2] K. Kikkawa, S. A. Klein, B. Sakita, and M. A. Virasoro, Phys. Rev. D 1, 3258 (1970); K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. 184, 1701 (1969); K. Kikkawa, *ibid.* 187, 2249 (1969).