It is now a trivial, if somewhat tedious, exercise to compute the integrals and traces in Eqs. (A2) and (A5); the results to order 1 are

$$
\begin{aligned}
B^{\mu\nu\lambda\sigma}(q-\tfrac{1}{2}(k+k'),k,k') \sim{}& (4\alpha^2/12)\{q^{-4}[8q^\mu q^\nu q^\lambda q^\sigma-2q^2(g^{\mu\nu}q^\lambda q^\sigma+g^{\nu\lambda}q^\sigma q^\mu+g^{\lambda\sigma}q^\mu q^\nu+g^{\sigma\mu}q^\nu q^\lambda)\\
&+q^4(g^{\mu\nu}g^{\lambda\sigma}+g^{\nu\lambda}g^{\mu\sigma}-g^{\mu\lambda}g^{\nu\sigma})]+q^{-2}[10g^{\mu\nu}q^\lambda q^\sigma-8g^{\nu\lambda}q^\mu q^\sigma-2g^{\lambda\sigma}q^\mu q^\nu-8g^{\mu\sigma}q^\nu q^\lambda\\
&+2g^{\mu\lambda}q^\nu q^\sigma+2g^{\nu\sigma}q^\mu q^\lambda-q^2(g^{\mu\nu}g^{\lambda\sigma}+3g^{\mu\lambda}g^{\nu\sigma}-5g^{\mu\sigma}g^{\nu\lambda})]-[4\ln(m^2/q^2)+6+8/3]\\
&\times(g^{\mu\nu}g^{\lambda\sigma}+g^{\mu\sigma}g^{\nu\lambda}-2g^{\mu\lambda}g^{\nu\sigma})+2[3(g^{\mu\nu}g^{\lambda\sigma}+g^{\sigma\mu}g^{\nu\lambda})-5g^{\mu\lambda}g^{\nu\sigma}]\}\\
={}&(4\alpha^2/12)\{8q^\mu q^\nu q^\lambda q^\sigma q^{-4}-10(g^{\nu\lambda}q^\mu q^\sigma+g^{\mu\sigma}q^\nu q^\lambda)q^{-2}-4g^{\lambda\sigma}q^\mu q^\nu q^{-2}\\
&+2(g^{\mu\lambda}q^\nu q^\sigma+g^{\nu\sigma}q^\mu q^\lambda)q^{-2}+8g^{\mu\nu}q^\lambda q^\sigma+6g^{\mu\sigma}g^{\nu\lambda}-2g^{\mu\lambda}g^{\nu\sigma}-[4\ln(m^2/q^2)+8/3](g^{\mu\nu}g^{\lambda\sigma}+g^{\mu\sigma}g^{\nu\lambda}-2g^{\mu\lambda}g^{\nu\sigma})\}
\end{aligned}
\tag{A6}
$$

and

$$
\begin{aligned}
B^{\mu\lambda\nu\sigma}\sim{}&(4\alpha^2/12)\{-q^{-4}[16q^\mu q^\nu q^\lambda q^\sigma-4q^2(g^{\mu\lambda}q^\nu q^\sigma+g^{\lambda\nu}q^\sigma q^\mu+g^{\nu\sigma}q^\mu q^\lambda+g^{\sigma\mu}q^\lambda q^\nu)\\
&+2q^4(g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}-g^{\mu\nu}g^{\lambda\sigma})]+q^{-2}[-16g^{\mu\nu}q^\lambda q^\sigma+4(g^{\mu\lambda}q^\nu q^\sigma+g^{\mu\sigma}q^\nu q^\lambda+g^{\nu\lambda}q^\mu q^\sigma+g^{\nu\sigma}q^\mu q^\lambda)\\
&+8g^{\lambda\sigma}q^\mu q^\nu-4q^2(g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda})]-[4\ln(m^2/q^2)+12-16/3]\\
&\times(g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}-2g^{\mu\nu}g^{\lambda\sigma})+2[3(g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda})-5g^{\mu\nu}g^{\lambda\sigma}]\}\\
={}&(4\alpha^2/12)\{-16q^\mu q^\nu q^\lambda q^\sigma q^{-4}+8(g^{\mu\lambda}q^\nu q^\sigma+g^{\mu\sigma}q^\nu q^\lambda+g^{\nu\lambda}q^\mu q^\sigma+g^{\nu\sigma}q^\mu q^\lambda)q^{-2}-16g^{\mu\nu}q^\lambda q^\sigma q^{-2}+8g^{\lambda\sigma}q^\mu q^\nu q^{-2}\\
&-4(g^{\mu\lambda}g^{\nu\sigma}+g^{\nu\lambda}g^{\mu\sigma})-[4\ln(m^2/q^2)+8/3](g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}-2g^{\mu\nu}g^{\lambda\sigma})\}\,.
\end{aligned}
\tag{A7}
$$

The sum $B^{\mu\nu\lambda\sigma}+B^{\mu\nu\sigma\lambda}+B^{\mu\lambda\nu\sigma}$ is then identically zero to terms of order $1/q$.

We will not calculate all terms of order $1/q$, but simply observe that the terms in Eq. (A3) are not obviously of the required order. For these terms, the limit as $q\to\infty$ cannot be interchanged with the δ integral. It is straightforward to verify that

$$
\int_0^1 d\delta\frac{f(\delta)}{[\delta m^2+(1-\delta)\mu^2+\delta(1-\delta)Q^2]^2}\underset{Q^2\to\infty}{\sim}\frac{f(0)}{\mu^2Q^2}+\frac{f(1)}{m^2Q^2}+O\left(\frac{1}{Q^4}\ln Q^2\right)
$$

and

$$
\int_0^1 d\delta\frac{f(\delta)}{[\delta m^2+(1-\delta)\mu^2+\delta(1-\delta)Q^2]}\underset{Q^2\to\infty}{\sim}\frac{f(0)}{Q^2}\ln\frac{Q^2}{\mu^2}+\frac{f(1)}{Q^2}\ln\frac{Q^2}{m^2}+\frac{1}{Q^2}\int_0^1 d\delta\frac{f(\delta)-(1-\delta)f(0)-\delta f(1)}{\delta(1-\delta)}+O\left(\frac{1}{Q^4}\ln Q^2\right).
\tag{A8}
$$

Such nonuniform limits do not contribute to the terms of order 1, and hence are not relevant to the question of Schwinger terms. They are, however, essential in exhibiting the canonical commutators.

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Remarks on a Test of the Padé-Approximant Approach in the Coulomb Bound-State Problem

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We examine a model of scalar positronium for the purpose of testing Padé approximants in the Coulomb problem and find that the (1,1) Padé approximant for the amplitude comprised of the Born term, the box, and the cross-box diagrams predicts an infinite number of bound states. Although the $O(4)$ symmetry is preserved, the spectrum obtained is not the relativistic Balmer formula derived by the eikonal approximation or other conventional methods. A recent method of extracting a relativistic Balmer formula (with all recoil effects) from the eikonal approximation is found to be equivalent to summing up definite portions of the (crossed) ladder diagrams. The (1,1) Padé approximant deduced from the first two terms of this approximate series yields exactly the same result.

RECENTLY Padé approximants have been applied to various calculations of perturbation series. Loeffel *et al.*[1] have shown that the diagonal Padé approximants formed from the asymptotically divergent perturbation-series calculations of Bender and Wu[2] for the ground-state energy of the anharmonic oscillator converge to the correct answer. Other applications of

[1] J. J. Loeffel, A. Martin, B. Simon, and A. S. Wightman, Princeton University report, 1969 (unpublished).

[2] C. M. Bender and T. T. Wu, Phys. Rev. **184**, 1231 (1969).

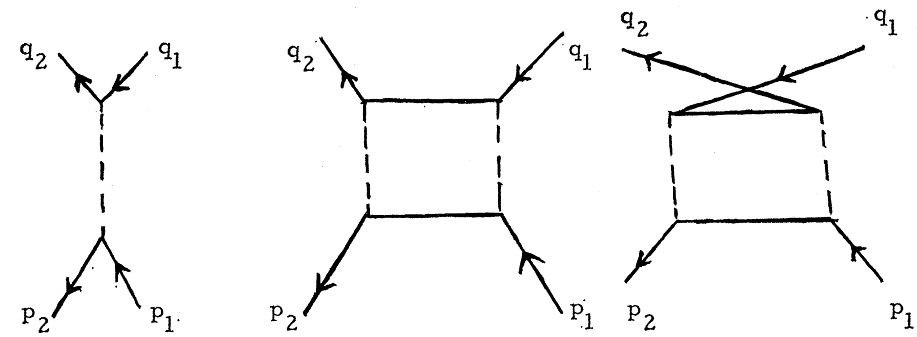

FIG. 1. Diagrams whose contribution to the scattering amplitude is examined.

Padé approximants have been made to the partial-wave decomposition of scattering amplitudes.[3] In this paper, we examine a model of scalar positronium for the purpose of testing this second type of application of the Padé approximants in an exactly solvable example.

Two scalar particles of mass M are assumed to interact by means of a long-range force mediated by a scalar photon of infinitesimal mass μ. The diagrams whose contribution to the scattering amplitude is to be examined are depicted in Fig. 1. These diagrams are among the Feynman series of all (crossed) ladder diagrams approximated by the eikonal method. The fact that this approximation method leads to the relativistic Balmer formula[4] motivates our restriction of fourth-order diagrams. Below threshold

$$\begin{aligned} s&=-(p_1+q_1)^2<4M^2\,,\\ t&=-(p_1-p_2)^2=(4M^2-s)\sin^2(\tfrac{1}{2}\theta)>0\,,\\ u&=-(p_1-q_2)^2=(4M^2-s)\cos^2(\tfrac{1}{2}\theta)>0\,. \end{aligned} \tag{1}$$

We assume that $\mu^2\ll 4M^2-s$. Under this assumption, the small-μ behavior of the diagrams in Fig. 1 is given by

$$T(s,t,u)=\frac{4g^2}{\mu^2-t-i\epsilon}+\frac{4g^4}{\pi^2}\left(\frac{\sin^{-1}[(s/4M^2)^{1/2}]}{(4M^2s-s^2)^{1/2}}+\frac{\sin^{-1}[(u/4M^2)^{1/2}]}{(4M^2u-u^2)^{1/2}}\right)\int_{4\mu^2}^{\infty}\frac{dt'}{(t'^2-4\mu^2t')^{1/2}}\frac{1}{t'-t-i\epsilon}. \tag{2}$$

We choose to expand this amplitude about the forward direction in a power series in t/μ^2.[5] Unlike the partial-wave decomposition, this prescription does not single out a fixed value for the angular momentum; rather, it limits the range of partial waves considered from the zeroth partial wave to the nth partial wave, where n is the power of t/μ^2 whose coefficient is chosen for study. Expanding (2) in such a power series yields

$$\frac{1}{\mu^2}\sum_{n=1}^{\infty}\left(\frac{t}{\mu^2}\right)^{n-1}\left[4g^2+\frac{g^4}{\pi}\frac{B(n,n)}{(4M^2s-s^2)^{1/2}}\right], \tag{3}$$

where $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Euler beta function. We have replaced u by $4M^2-s$ is in the expansion as $t<\mu^2\ll 4M^2-s$. The (1,1) Padé approximant in g^2 is

$$P(t,s)=\frac{1}{\mu^2}\sum_{n=1}^{\infty}\left(\frac{t}{\mu^2}\right)^{n-1}\frac{4g^2}{1-g^2B(n,n)/4\pi(4M^2s-s^2)^{1/2}}. \tag{4}$$

The coefficient of t^{n-1} has a pole for

$$s_n=2M^2\left[1+\left(1-\frac{\alpha^2}{f(n)^2}\right)^{1/2}\right], \tag{5}$$

where

$$\alpha=g^2/4\pi M^2\,,\quad f(n)=2/B(n,n)\,. \tag{6}$$

This is to be compared with the relativistic Balmer formula[6]

$$s_n=2M^2\left[1+\left(1-\frac{\alpha^2}{n^2}\right)^{1/2}\right]. \tag{7}$$

The amplitude (4) does display the usual l degeneracy imposed by the $O(4)$ symmetry, but the spectrum represented by (5) and (6) is quantitatively different from (7). The former spectrum of energy levels approach threshold much faster with increasing n than the Balmer formula (7).

One of the approximation techniques that yields the correct result (7) is the eikonal approximation. The sum of all Feynman graphs of the (crossed) ladder variety

[3] D. Bessis and M. Pusterla, Nuovo Cimento **54A**, 243 (1968); J. L. Basdevant and B. W. Lee, Nucl. Phys. **B13**, 182 (1969).

[4] E. Brezin, C. Itzykson, and J. Zinn-Justin, Phys. Rev. D **1**, 2349 (1970). Earlier related works include H. Suura and D. R. Yennie, Phys. Rev. Letters **10**, 69 (1963); M. Lévy, *ibid.* **5**, 235 (1962).

[5] Because of infrared divergences in the amplitude (2), the (1,1) Padé approximant applied to the partial-wave decomposition of (2) leads to bound states that are not independent of μ. This situation would also arise with the prescription presented in the text if we did not restrict our considerations of fourth-order terms to the box and cross-box diagrams.

[6] Brezin *et al.* (Ref. 4). This formula is also derived from the quasipotential approach. See C. Itzykson, V. G. Kadyshevsky, and I. T. Todorov, Phys. Rev. D **1**, 2823 (1970). The former paper derives the result corresponding to (7) in the case of scalar positronium and vector photons. Our discussion of the eikonal treatment of scalar positronium with scalar photons is drawn from this work and from private communications of C. Itzykson.

found by this method is proportional to[7]

$$s^{1/2}(4M^2+t-s)^{1/2}\int d^2b\,\left\{\exp\left[4ig^2\int_{-\infty}^{+\infty}d\sigma\right.\right.$$
$$\left.\left.\times\int_{-\infty}^{+\infty}d\tau\Delta_+(b-2p\sigma+2q\tau,\mu)\right]-1\right\}e^{ik\cdot b},\quad(8)$$

where

$$k^2=-t,\quad p=\tfrac{1}{2}(p_1+p_2),\quad q=\tfrac{1}{2}(q_1+q_2).\quad(9)$$

Integration yields

$$s^{1/2}(4M^2+t-s)^{1/2}$$
$$\times\int d^2b\,e^{ik\cdot b}\left[\exp\left(\frac{g^2}{\pi}\frac{K_0(\mu b)}{(4M^2s-st-s^2)^{1/2}}\right)-1\right],\quad(10)$$

and near the forward direction for $t<\mu^2\ll 4M^2-s$ this is proportional to

$$(4M^2s-s^2)^{1/2}\sum_{n=0}^{\infty}\int_0^{\infty}db\frac{(-t)^n}{(2n!)^2}b^{2n+1}(e^{zK_0(\mu b)}-1),\quad(11)$$

where

$$z=g^2/\pi(4M^2s-s^2)^{1/2}.\quad(12)$$

Scaling the variable $\mu b=x$ gives us

$$(4M^2s-s^2)^{1/2}\sum_{n=0}^{\infty}\int_0^1 dx\left(\frac{-t}{\mu^2}\right)^n\frac{x^{2n-1}}{\mu^2(2n!)^2}(e^{-z\ln x}-1)$$
$$+(4M^2s-s^2)^{1/2}\sum_{n=0}^{\infty}\int_1^{\infty}dx\left(\frac{-t}{\mu^2}\right)^n\frac{x^{2n+1}}{\mu^2(2n!)^2}$$
$$\times(e^{zK_0(x)}-1).\quad(13)$$

In the interval $[0,1]$ we have approximated $K_0(x)\approx-\ln x$. The bound-state poles are determined from the singularity of the integrand for small x. Therefore we shall neglect the rest of the amplitude in the following considerations. Integrating the first part of (13) gives

$$(4M^2s-s^2)^{1/2}\sum_{n=0}^{\infty}\left(\frac{-t}{\mu^2}\right)^n\frac{1}{\mu^2}\frac{1}{(2n!)^2}$$
$$\times\left(\frac{1}{2n+2-z}-\frac{1}{2n+2}\right),\quad(14)$$

which has poles at

$$z=\frac{g^2}{\pi}\frac{1}{(4M^2s-s^2)^{1/2}}=2n+2,\quad(15)$$

$$s_{n+1}=2M^2\left[1+\left(1-\frac{\alpha^2}{(n+1)^2}\right)^{1/2}\right],n=0,1,2,\ldots\,.\quad(16)$$

[7] M. Lévy and J. Sucher, Phys. Rev. **186**, 1656 (1969); H. D. I. Abarbanel and C. Itzykson, Phys. Rev. Letters **23**, 53 (1969).

The correct degeneracy is exhibited by the factor t^n which contains all partial waves up to the nth.

This result can also be deduced by expanding (11) in a perturbation series. This is accomplished by expanding $e^{-z\ln x}$ in a power series in $z\ln x$ in the first part of (13). This gives

$$(4M^2s-s^2)^{1/2}\sum_{n=0}^{\infty}\left(\frac{-t}{\mu^2}\right)^n\frac{1}{\mu^2}\frac{1}{(2n!)^2}$$
$$\times\int_0^1 dx\,x^{2n+1}\sum_{l=1}^{\infty}\frac{(-z\ln x)^l}{l!}=(4M^2s-s^2)^{1/2}$$
$$\times\sum_{n=0}^{\infty}\left(\frac{-t}{\mu^2}\right)^n\frac{1}{\mu^2}\frac{1}{(2n!)^2}\frac{1}{2n+2}\sum_{l=1}^{\infty}\left(\frac{z}{2n+2}\right)^l.\quad(17)$$

The sum in l reproduces (14). Because it is a geometric series, the (1,1) Padé approximant in z obtained from the first two terms,

$$\frac{1}{\mu^2}\sum_{n=0}^{\infty}\left(\frac{-t}{\mu^2}\right)^n\frac{1}{(2n!)^2}\frac{1}{(2n+2)^2}$$
$$\times\left(\frac{g^2}{\pi}+\frac{g^4}{2\pi^2}\frac{1}{(4M^2s-s^2)^{1/2}}\frac{1}{n+1}\right),\quad(18)$$

gives the same result as the summation of the whole series.[8]

Although this latter result is of some interest, we must emphasize the failure of the (1,1) Padé approximant to give a correct quantitative account of the positronium spectrum when applied to the full amplitude considered. Apparently this failure of the (1,1) Padé approximant is not restricted to the Coulomb problem. The first s-wave bound state of the exponential potential is reproduced poorly by the (1,1) Padé approximant.[9] In that example, the higher-order Padé approximants give better results for the first s-wave bound state. It could be that higher-order Padé approximants inferred from the sixth-order (crossed) ladder diagrams would shift the predicted positronium spectrum in the right direction.

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[8] It should be emphasized that Eq. (18) represents only a *part* of the perturbation expansion of the eikonal equation (10). Thus, for example, the contributions in g^2 deduced from (10) differ by crucial numerical factors from those given in the expansion (18).

[9] J. L. Basdevant and B. W. Lee (Ref. 3).