Scale Invariance and Current-Algebra Sigma Terms*

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It is shown that, in a scale-invariant theory in which the vacuum does not realize the symmetry, the dominant contribution to the sigma terms arising in current-algebra determinations of the meson-baryon scattering lengths can be estimated.

THE concept of scale invariance and its possible relevance to theories describing the elementary particles has attracted some attention in recent years.[1] The papers of Kastrap, Mack, and Wilson, in particular, consider the usefulness of scale invariance as a broken symmetry in the hadronic world. All the authors listed in Ref. 1 consider scale invariance as a symmetry realized by the vacuum, in which case—as is well known—all the particles in any theory having this symmetry must be massless. There is, however, the possibility that the vacuum does not realize the scale-invariance symmetry even though the Lagrangian has such an invariance (with all bare-mass and dimensional coupling constants being absent). This is, of course, the famous Goldstone symmetry limit, and the Goldstone theorem tells us that in this limit there must appear a set of massless particles associated with the generators of the symmetry group. In this paper we consider the latter possibility where the scale-invariance limit of a theory is realized in the Goldstone manner. Then, we are not necessarily confined to having zero *physical* masses for *all* the particles in the symmetry limit.[2] In fact, one could get instead relations between the physical masses of the particles and the coupling of the "Goldstone" boson to the particles. (These are entirely analogous to the Goldberger-Treiman relations, which are exact relations in the limit of chiral symmetry when the pseudoscalar mesons are considered as Goldstone bosons.) Of course, these relations cannot be tested at present, for even if we identify our Goldstone boson with the σ reported at 730 MeV, we are still very far from determining its coupling to hadrons other than the pion. We shall also make an assumption about the form of the terms which break the scale invariance, and following Mack[1] relate the Σ terms which arise in current-algebra calculations to the divergence of the dilatation current, i.e., the current associated with the scaling symmetry. This allows us to determine the dominant contribution of these terms to the current-algebra estimates of meson-baryon scattering lengths.

We assume the existence of a fundamental Lagrangian, $L=L(\boldsymbol{\phi}\partial_\mu\boldsymbol{\phi})$, which describes the strong interactions. Here $\boldsymbol{\phi}$ denotes a set of fields assembled, for convenience, into a vector.[3] A scale transformation is defined by

$$\tau:\ \boldsymbol{\phi}(x)\rightarrow e^{\mathbf{D}\tau}\boldsymbol{\phi}(e^\tau x)\,, \tag{1}$$

with $\mathbf{D}$ some matrix usually called "the dimension matrix" and τ some real number.

In an infinitesimal transformation, we have

$$\delta\boldsymbol{\phi}=\mathbf{D}\boldsymbol{\phi}+x^\mu\partial_\mu\boldsymbol{\phi}\,. \tag{2}$$

For the theory to be invariant under such transformations, we must have for the Lagrangian

$$\delta L=4L+x^\mu\partial_\mu L=\partial_\mu(x^\mu L)\,. \tag{3}$$

In the usual manner, we obtain a conserved current associated with this symmetry,

$$J_\mu=\boldsymbol{\pi}^\mu\cdot\mathbf{D}\boldsymbol{\phi}+x_\lambda T_c{}^{\mu\lambda}\,,$$

where

$$T_c{}^{\mu\lambda}=\boldsymbol{\pi}^\mu\partial^\lambda\boldsymbol{\phi}-g^{\mu\lambda}L$$

is the canonical energy-momentum tensor and $\boldsymbol{\pi}^\mu=\partial L/\partial(\partial_\mu\boldsymbol{\phi})$.

When scale invariance is broken, we have

$$\delta L=\partial_\mu(x^\mu L)+\Delta\,,$$

and the current-conservation equation is replaced by

$$\partial^\mu J_\mu=\Delta\,. \tag{4}$$

For convenience, we will also introduce the new energy-momentum tensor defined by Callan *et al.*,[3] in terms of which J_μ takes the simple form

$$J^\mu(x)=x_\nu\Theta^{\mu\nu}(x)\,. \tag{5}$$

$\Theta_{\mu\nu}$ is symmetric and conserved and gives rise to the same Poincaré generators as $T_{\mu\nu}{}^c$. The sufficient conditions for (5) to be valid are given in Ref. 3, and we will assume that they are satisfied by our theory. (In particular, the condition is satisfied by all renormalizable

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[1] J. Wess, Nuovo Cimento **18**, 1086 (1960); D. M. Greenberger, Ann. Phys. (N.Y.) **25**, 290 (1963); H. A. Kastrup, Nucl. Phys. **58**, 561 (1964); G. Mack, *ibid.* **B5**, 499 (1968); K. G. Wilson, Phys. Rev. **179**, 1499 (1969); S. Ciccariello, R. Gatto, G. Sartori, and M. Tonin, Phys. Letters **30B**, 546 (1969); D. J. Gross and J. Wess, Phys. Rev. D **2**, 753 (1970).

[2] G. Mack and A. Salam, Ann. Phys. (N.Y.) **53**, 174 (1969); see also M. Gell-Mann, Hawaii Conference, 1969 (unpublished). If the generator of scale transformations does not annihilate the vacuum in the symmetry limit, its matrix elements would not, in general, be defined. Hence, the usual proof of $m_i=0$ in the scale-invariance limit would not be valid.

[3] C. Callan, S. Coleman, and R. Jackiw, M.I.T. report (unpublished). We will follow the notation and development of Sec. 5 of this paper.

theories.) Thus, (4) would read

$$\partial^\mu J_\mu = \Theta_\mu{}^\mu = \Delta . \tag{6}$$

Let us now write the Lagrangian as $L = L_0 + \epsilon L'$, where L_0 is the scale-invariant part and L' is the part which breaks the scale invariance, with ϵ a parameter characterizing the "strength" of the breaking.

Assuming that L' has a unique dimension, we note first that

$$\Delta = \epsilon(d-4)L' , \tag{7}$$

where d is the dimension of L'. Now since $\Theta_{\mu\nu}$ is symmetric and conserved, we may write its matrix element between spin-$\frac{1}{2}$ one-particle baryon states as

$$\begin{aligned}\langle p_1\sigma_1 | \Theta_{\mu\nu}(0) | p_2\sigma_2\rangle &= \bar{u}(p_1\sigma)[\tfrac{1}{4}(\gamma_\mu P_\nu + \gamma_\nu P_\mu)F_1(k^2) + \tfrac{1}{2}P_\mu P_\nu F_2(k^2) \\ &\quad + (g_{\mu\nu}k^2 - k_\mu k_\nu)F_3(k^2)]u(p_2,\sigma) ,\end{aligned} \tag{8}$$

where $P_\mu = (p_1+p_2)_\mu$ and $k_\mu = (p_1-p_2)_\mu$, normalizing the fermion states $\bar{u}u = 1$, and

$$\langle p_1\sigma_1 | p_2\sigma_2\rangle = \delta_{\sigma_1\sigma_2}(p_0/M)(2\pi)^3\delta(\mathbf{p}_1 - \mathbf{p}_2) .$$

Since the space integrals of $\Theta_{0\mu}$ and $x^\mu\Theta_{\nu 0} - x^\nu\Theta_{\mu 0}$ define the momentum and angular momentum operators, we have $F_1(0) = 1$ and $F_2(0) = 0$.

Let us now assume the existence of a scalar particle σ with the quantum numbers of the vacuum, and define the constant f_σ by

$$\langle 0 | \Theta_{\mu\nu} | \sigma(k)\rangle = f_\sigma(g_{\mu\nu}k^2 - k_\mu k_\nu) , \tag{9}$$

with $f_\sigma \sim O(1)$, i.e., $f_\sigma \nrightarrow 0$ as $\epsilon \to 0$. Then

$$\langle 0 | \Theta_\mu{}^\mu | \sigma(k)\rangle = 3f_\sigma m_\sigma{}^2 . \tag{10}$$

Now from (6) and (7), $\Theta_\mu{}^\mu \sim O(\epsilon)$, and since we assume that in the symmetry limit the σ becomes a Goldstone particle, we have, for $f_\sigma \sim O(1)$, $m_\sigma{}^2 \sim O(\epsilon)$.

If we now take the trace in (8) and separate the σ pole term in both sides of the equation, we obtain

$$\begin{aligned}&[3f_\sigma m_\sigma{}^2/(m_\sigma{}^2 - k^2)]g_{\sigma NN} + O(\epsilon) \\ &\quad = M_N F_1(k^2) + \tfrac{1}{2}P^2F_2(k^2) \\ &\qquad + [3k^2 f_\sigma/(m_\sigma{}^2 - k^2)]g_{\sigma NN} + 3k^2\tilde{F}_3(k^2) .\end{aligned}$$

Here $\tilde{F}_3(k^2)$ is that part of F_3 not containing the σ pole, and $g_{\sigma NN}$ is the coupling of the σ to the spin-$\frac{1}{2}$ particle. Thus at $k^2 = 0$,

$$3f_\sigma g_{\sigma NN} = M_n + O(\epsilon) . \tag{11}$$

Hence, if the scale-invariance limit is realized in the Goldstone fashion, the masses of the particles are determined to $O(\epsilon)$ by the strength of their coupling to the σ. We would also have similar relations for the scalar and pseudoscalar meson, e.g.,

$$3f_\sigma g_{\sigma\pi\pi} = m_\pi{}^2 + O(\epsilon) , \tag{11$'$}$$

but, of course, the corrections would usually be larger than the square of the masses of the pseudoscalar particles. Note that although the bare 3-boson coupling is zero in the scale-invariant limit, there is no reason for the physical coupling to be zero (for instance σ, π, and π could hook onto a nucleon loop). However, if we postulate that chiral- and scale-invariance breaking are due to the same source, (11′) would just tell us that $g_{\sigma\pi\pi} \sim O(\epsilon)$; i.e., it is small compared to $M_n g_{\sigma NN}$ ($g_{\sigma\pi\pi}/M_n g_{\sigma NN} \sim m_\sigma{}^2/M_n{}^2$). One could also derive "soft" σ theorems which are analogous to the soft-pion theorems, but they do not seem to be of much use at present.

If L' has a unique dimension as we assumed to derive Eq. (7) and the symmetry limit as $\epsilon \to 0$ is realized in the Goldstone fashion, then we can prove that the dimension d of L' is unity. For if we wrlte

$$\Theta_{00} = \bar{\Theta}_{00} - \epsilon L' ,$$

where $\bar{\Theta}_{00}$ is the scale-invariant part of the energy-momentum tensor, then we have

$$\begin{aligned}M_n &= \langle N(\mathbf{p}=0) | \Theta_{00} | N(\mathbf{p}=0)\rangle \\ &= \langle N(\mathbf{p}=0) | \bar{\Theta}_{00} | N(\mathbf{p}=0)\rangle \\ &\qquad - \langle N(\mathbf{p}=0) | \epsilon L' | N(\mathbf{p}=0)\rangle .\end{aligned}$$

Now from Eq. (8), with ϵ set to zero,

$$\begin{aligned}&\langle N(\mathbf{p}) | \bar{\Theta}_{00} | N(\mathbf{p}'=0)\rangle \\ &\quad = \bar{u}(p,\sigma)\{\tfrac{1}{2}\gamma_0 P_0 F_1(k^2) + \tfrac{1}{2}P_0{}^2F_2(k^2) \\ &\qquad - \mathbf{p}^2[f_\sigma g_{\sigma NN}/(-k^2) + \tilde{F}_3(k')]\}u(\mathbf{0},\sigma) ,\end{aligned}$$

where we have separated the σ pole occurring at $k^2 = 0$ since $m_\sigma{}^2$ is zero in the symmetry limit.

If we now take the limit as $\mathbf{p} \to 0$ in this equation and note that

$$\begin{aligned}k_0{}^2 &= [(\mathbf{p}^2 + M_n{}^2)^{1/2} - M_n]^2 \\ &\to O(\mathbf{p}^4)\end{aligned}$$

and

$$\begin{aligned}k^2 &= k_0{}^2 - \mathbf{p}^2 \\ &\to -\mathbf{p}^2 ,\end{aligned}$$

then we obtain

$$\langle N(\mathbf{p}=0) | \bar{\Theta}_{00} | N(\mathbf{p}'=0)\rangle = \bar{M}_n - f_\sigma g_{\sigma NN} .$$

Here $\bar{M}_n$ is the nucleon mass in the limit of scale invariance. Now

$$\begin{aligned}M_n &= \langle N | \Theta_\mu{}^\mu | N\rangle \\ &= (d-4)\langle N | \epsilon L' | N\rangle ,\end{aligned}$$

by Eqs. (6) and (7). Hence, collecting these results, we have

$$M_n = \bar{M}_n - f_\sigma g_{\sigma NN} - M_n/(d-4) .$$

If we now use Eq. (11) and note that $M_n - \bar{M}_n = O(\epsilon)$, we find that, correct to zeroth order in ϵ, d is 1.

This proof assumes that the form factor $F_1(k^2)$ is changed only by $O(\epsilon)$ in going to the scale-invariant limit. Note that we cannot prove that $F_1(0) = 1$ in the scale-invariant limit, $\epsilon = 0$, just as in Eq. (8), owing to the presence of the zero-mass pole. For if we assume that $\int d^3x\, \bar{\Theta}_{00}(x)$ defines the Hamiltonian in the symmetry limit and that its matrix elements exist, then

$$\langle N(p_1)|\int d^3x\,\overline{\Theta}_{00}(x,0)|N(p_2)\rangle$$
$$=(2\pi)^3\delta(\mathbf{p}_1-\mathbf{p}_2)p_{20}{}^2/M$$
$$=(2\pi)^3\delta(\mathbf{p}_1-\mathbf{p}_2)\times[(p_{20}{}^2/M)F_1(0)+2p_{20}{}^2F_2(0)-f_\sigma g_{\sigma NN}],$$

where the last term comes from the zero-mass pole and clearly makes the result frame dependent. Thus, the matrix elements of $\int d^3x\,\overline{\Theta}_{00}(x)$ do not exist. This would seem to be related to the fact that the matrix element of the generator of scale transformations is not well defined.[2]

We now make three assumptions about L_0 and L': (i) L_0 is an $SU_3\times SU_3$ singlet; (ii) there are no $SU_3\times SU_3$ singlets in L'; and (iii) $L'=-u_0-cu_8$, where the u_i's form part of the basis for the $(3,\bar{3})+(\bar{3},3)$ representation of $SU_3\times SU_3$ and $c=-1.25+O(\epsilon)$, i.e., we assume the Gell-Mann–Oakes–Renner (GMOR) symmetry-breaking form.[4] Thus, we have a model in which both the chiral symmetry and the scale invariance are broken by the same terms. An explicit model Lagrangian is the SU_3 σ model of Lévy,[5] with

$$L_0=-q\{-i\gamma^\alpha\partial_\alpha-g_0[(\surd\tfrac{2}{3})\sigma_0+(\surd\tfrac{2}{3})i\pi_0\gamma_5+\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}+i\boldsymbol{\pi}\cdot\boldsymbol{\lambda}\gamma_5]\}q+\tfrac{1}{2}(\partial_\alpha\sigma_0\partial^\alpha\sigma_0+\partial_\alpha\pi_0\partial^\alpha\pi_0+\partial_\alpha\boldsymbol{\sigma}\cdot\partial^\alpha\boldsymbol{\sigma}+\partial_\alpha\boldsymbol{\pi}\cdot\partial^\alpha\boldsymbol{\pi})-\lambda_0(\sigma_0{}^2+\pi_0{}^2+\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}+\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2,$$
$$\epsilon L'=\alpha\sigma_0+\beta\sigma_8,$$

where q is the quark field, σ_0 (with $\langle\sigma_0\rangle\neq0$) and π_0 are unitary singlet scalar and pseudoscalar fields, and $\boldsymbol{\sigma}$, $\boldsymbol{\pi}$ are octet scalar and pseudoscalar fields, belonging to a $(3,\bar{3})+(\bar{3},3)$ representation of $SU_3\times SU_3$. In this model we have

$$\partial_\alpha A^\alpha{}_k=-(\surd\tfrac{2}{3})\alpha\pi_k-\beta d_{8km}\pi_m,\quad k\neq8$$
$$\partial_\alpha J^\alpha=-3(\alpha\sigma_0+\beta\alpha_8).$$

Using assumptions (i)–(iii), we can now calculate the Σ terms correct to $O(\epsilon)$. The full Lagrangian is $L=L_0-\epsilon(u_0+cu_8)$, so that

$$\langle f|\Sigma_{\beta\alpha}(0)|i\rangle$$
$$=i\langle f|[Q^\alpha(0),\partial^\mu A_\mu{}^\beta(0)]|i\rangle$$
$$=\epsilon\{[\tfrac{2}{3}\delta_{\alpha\beta}+(\surd\tfrac{2}{3})cd_{\delta\alpha\beta}]\langle f|u_0|i\rangle+[(\surd\tfrac{2}{3})d_{\alpha\beta\gamma}+cd_{8\beta\sigma}d_{\sigma\alpha\gamma}+\tfrac{2}{3}\delta_{\beta8}\delta_{\alpha\gamma}]\langle f|u_\gamma|i\rangle\}.\quad(12)$$

The coupling of the "almost-Goldstone" boson σ gives a contribution of $O(\epsilon^{-1})$ to the matrix element $\langle f|u_0|i\rangle$, so that the first term dominates over the rest. We do not expect the σ to have an octet part which becomes a Goldstone boson, since we would like to retain the algebraic structure of $SU(3)$. That is, we would like to have the SU_3 generators annihilating the vacuum in the SU_3-symmetry limit. Also, from Eqs. (6) and (7) and our assumptions (i)–(iii), we have $\langle f|\Theta_\mu{}^\mu|i\rangle=(4-d)\epsilon\langle f|(u_0+cu_8)|i\rangle$. Choosing $|i\rangle$ and $|f\rangle$ to be one-nucleon states at the same momentum, we have, using Eq. (8),

$$(4-d)\epsilon\langle N|u_0|N\rangle=M_n+O(\epsilon)\quad(13)$$

[$\langle N|u_0|N\rangle$ being $O(\epsilon^{-1})$ because of the σ pole term]. Thus, we have from (12) and (13)

$$\langle N(p)|\Sigma_{\beta\alpha}(p)|N(p)\rangle=[\tfrac{2}{3}\delta_{\alpha\beta}+(\surd\tfrac{2}{3})cd_{8\alpha\beta}]\mu+O(\epsilon),$$

where

$$\mu=M_n/(4-d)+O(\epsilon).$$

Using our previous result that $d=1+O(\epsilon)$, we find $\mu=310$ MeV $+O(\epsilon)$, where we have used the observed nucleon mass. If we assume that d is exactly 1, then we can calculate the first correction to μ by using experimental baryon mass splittings. In this case $\epsilon c\langle N|u_8|N\rangle\approx-200$ MeV, which gives $\mu\approx510$ MeV.

An evaluation of μ has been made by von Hippel and Kim[6] by using the experimental values for the KN and $\bar{K}N$ scattering lengths and extrapolating to threshold by means of a dispersion-relation method. They find $\mu\sim215$ MeV, with large systematic errors. If this result is maintained when better data become available, then the assumption that L' does not have an $SU(3)\times SU(3)$ singlet would be in doubt.

After completing this work, we learned from Professor M. Gell-Mann that he and co-workers have independently arrived at similar results.[7]

Note added in proof. It has been shown by Gell-Mann and Brown[7] that when one assumes that the scale breaking term has a unique dimension and has *no* chiral ($SU_3\times SU_3$) scalars, then the dimension d of L' is 2. The argument they give depends on looking at the shift in the energy of a pseudoscalar-meson state (due to chiral symmetry breaking) in a frame in which the 3-momentum is tending to infinity. We can put this argument in a manifestly covariant form if we use a result of Brown.[8] He shows by doing first-order perturbation theory around a symmetry limit that the shift in the squared mass due to the perturbation is given by the one-particle matrix elements of the symmetry-breaking perturbation. Thus, assuming that the procedure of doing perturbation theory around a Goldstone-type symmetry is valid and noting that in the chiral-symmetry limit the pseudoscalar mesons have zero mass, we have[9]

$$m_i{}^2=\epsilon\langle i|u|i\rangle+O(\epsilon^2),\quad(14)$$

[4] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1969).
[5] M. Lévy, Nuovo Cimento **52A**, 23 (1967).
[6] F. von Hippel and J. K. Kim, Phys. Rev. Letters **22**, 740 (1969); Phys. Rev. D **1**, 151 (1970).
[7] M. Gell-Mann, Hawaii Summer School Lectures, 1969, Caltech Report No. CALT-68-244 (unpublished). See also P. Carruthers, Caltech Report Nos. CALT-68-265 and CALT-68-266 (unpublished).
[8] L. S. Brown, Phys. Rev. **187**, 2260 (1969).
[9] We normalize the states covariantly: $\langle p'|p\rangle=2p_0(2\pi)^3\delta^3(\mathbf{p}-\mathbf{9})$.

where ϵu is the chiral-symmetry-breaking term and i denotes some pseudoscalar meson. Recapitulating the rest of the argument, one has

$$2m_i^2 = \langle i | \Theta_\mu{}^\mu | i \rangle = (d-4) \langle i | \epsilon L' | i \rangle , \quad (15)$$

where we have used Eqs. (6) and (7) of the text. Now of course if $L' = -u$, i.e., if it has no chiral $SU_3 \times SU_3$ scalars, then comparison of (14) and (15) gives $d=2$. However, if we recollect that by taking matrix elements between nucleons we can show $d = 1 + O(\epsilon)$ (without making any assumptions about the $SU_3 \times SU_3$ properties of L'), we are forced to the conclusion that $L' \neq -u$. That is, L' must contain an $SU_3 \times SU_3$ scalar if the chiral-breaking part has a unique dimension. It is interesting that this is the conclusion one would come to if one accepted the scattering-length analyses of Kim and von Hippel.[6] Of course, if the chiral-breaking part of L' does not have a unique dimension, then nothing can be said about the existence of a chiral scalar in L'.

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Fully Reggeized Scattering Amplitudes*

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We investigate some restrictions imposed on scattering amplitudes of hadrons by Lorentz covariance and the requirement that external as well as virtual particles lie on Regge trajectories (fully Reggeized scattering amplitudes). A generalized Fourier-Mellin integral representation is proposed which expresses these requirements and plays the role of a generalized partial-wave expansion. In the pole approximation, the proposed representation automatically contains local duality and for spinless external particles it reduces to the Veneziano formula.

I. INTRODUCTION

THE object of this paper is to study some restrictions imposed on scattering amplitudes of hadronic processes by Lorentz covariance and by the compositeness of hadrons. The meaning of Lorentz covariance has been widely discussed in the literature, and by now its meaning is clear. For our purposes "compositeness" of hadrons means that all hadrons lie on Regge (or rather Lorentz) trajectories. Therefore, any scattering amplitude involving hadrons should be an analytic function with "sufficiently good" analyticity properties in the "internal" as well as "external" Lorentz angular momenta. Lacking any reasonably consistent theory, we do not know which analyticity properties in the Lorentz angular momenta are sufficiently good, but one's physical instinct suggests that—for example—a "good" scattering amplitude should not contain δ functions or θ functions in the external "spin" variables. It turns out that even such very modest requirements lead to some nontrivial results. In order to get a feeling about how such results can emerge, the reader may recall that *any* partial-wave expansion of the "usual" type [whether it is according to representations of the groups $SU(2)$, $SU(1,1)$, or $SL(2,C)$ does not matter very much] involves at some point the coupling of the "spins" of external particles in a scattering amplitude to each other and possibly to spins of internal particles. The usual types of couplings always involve the reduction of Kronecker products which have very different properties depending on whether the representations of the groups involved are finite- or infinite-dimensional; correspondingly, the coupling coefficients (Wigner coefficients) contain very unpleasant singularities. Therefore, such coupling schemes practically preclude the possibility of a smooth continuation of scattering amplitudes in the spins of external particles away from the physical points. Fortunately, for the Lorentz group there exists another coupling scheme (or rather a *class* of coupling schemes) which *does not involve the reduction of Kronecker products* and can be continued smoothly in any of the Lorentz angular momenta involved.[1] We thus propose that a scattering amplitude describing a reaction involving composite hadrons in the external as well as in the virtual states (a "fully Reggeized" scattering amplitude) should be constructed in terms of these "analytical invariants."

The material is arranged as follows. After describing briefly the mathematical apparatus necessary for our investigations (Sec. II), we proceed to construct an ansatz for the scattering amplitude, which at least has a chance of possessing the right properties (Sec. III). In Sec. IV some more-detailed investigations follow, this time on the four-point function, although we indicate how the results can be generalized for an arbitrary number of external particles. As a result of these in-

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[1] M. A. Naimark, Am. Math. Soc. Transl., Ser. 2, **36**, 100 (1954); W. Rühl, Nuovo Cimento **44**, 659 (1966).