Multifactorizations and the Four-Reggeon Vertex Function in the Dual Resonance Models*

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By repeatedly factorizing the Koba-Nielsen representation of the N-point dual amplitude, we can obtain an amplitude with an arbitrary number of excited legs. In particular, we find an operator form of the four-Reggeon vertex function.

I. INTRODUCTION

It has been shown by Koba and Nielsen[1] that the N-point dual amplitude can be represented in a projectively invariant manner by associating with each external particle one Koba-Nielsen variable x_i, $i=1, 2, \ldots, N$ (Fig. 1). The N-point amplitude takes the form[2]

$$B_N(p_1,p_2,\ldots,p_N)=\int\Big(\prod_{i\neq(a,b,c)} dx_i\Big)\times\Big[1\Big/\Big|\prod_{i\neq a,a-2}(x_i-x_{i+2})\Big|\Big]\prod_{i,j(1\leqslant i<j\leqslant N)} u_{ij}^{-\alpha_{ij}-1}, \quad (1)$$

where $x_a=\infty$, $x_b=0$, $x_c=1$, and

$$u_{ij}\equiv(x_i,x_{i-1},x_j,x_{j+1})\equiv\frac{(x_i-x_j)(x_{i-1}-x_{j+1})}{(x_i-x_{j+1})(x_{i-1}-x_j)}. \quad (2)$$

The usual multiperipheral parametrization of the N-point function

$$B_N(p_1,\ldots,p_N)=\int_0^1\Big(\prod_{i=2}^{N-2} du_{1i}\Big)\times\Big(\prod_{i=2}^{N-2} u_{1i}^{-\alpha_{1i}-1}\Big)\Big[\prod_{i=2}^{N-1}(1-u_{1i})^{a-1}\Big]\times\prod_{i,j(2=i\leqslant j\leqslant N-2)}(1-u_{1i}u_{1i+1}\cdots u_{1j})^{-2bp_i\cdot p_{j+1}} \quad (3)$$

is then in the particular projective "frame" $x_1=\infty$, $x_2=1$, $x_N=0$. Another useful form can be obtained[2]

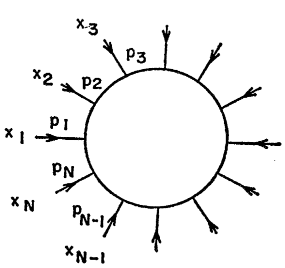

FIG. 1. N-point amplitude.

by choosing $x_1=\infty$, $x_b=0$, $x_c=1$; thus the N-point function has the form

$$B_N(p_1,\ldots,p_N)=\int\Big(\prod_{i\neq1,b,c} dx_i\Big)\times\Big\{\Big[\prod_{i,j(2\leqslant i<j\leqslant N)}(x_i-x_j)^{-2bp_i\cdot p_j}\Big]\times\Big[\prod_{2\leqslant i}^{N-1}(\theta(x_i-x_{i+1})(x_i-x_{i+1})^{-a-b(p_i^2+p_{i+1}^2)-1})\Big]\times\Big[\prod_{i=2(i\neq N-1)}^{N}(x_i-x_{i+2})^{a+bp_{i+1}^2}\Big]\Big\}. \quad (4)$$

For later convenience, we denote the integrand of Eq. (4) by

$$\{X_N\}\equiv\Big\{\Big[\prod_{i,j(2\leqslant i<j)}^{N}(x_i-x_j)^{-2bp_i\cdot p_j}\Big]\times\Big[\prod_{2\leqslant i}^{N-1}(\theta(x_i-x_{i+1})(x_i-x_{i+1})^{-a-b(p_i^2+p_{i+1}^2)-1})\Big]\times\Big[\prod_{i=2(i\neq N-1)}^{N}(x_i-x_{i+2})^{a+bp_{i+1}^2}\Big]\Big\}. \quad (5)$$

II. SINGLE FACTORIZATION

To do the single factorization of the N-point function, we divide the Koba-Nielsen variables x_i, $i=1, 2, \ldots, N$, into two sets, y_i, $i=1, 2, \ldots, M$, and z_j, $j=M+1, M+2, \ldots, N$ (Fig. 2). We explicitly choose three projective frames X, Y, and Z. They are defined by

$$\begin{aligned}
X:&\quad x_1=\infty,\quad x_2=1,\quad x_N=0;\quad x_{N+i}\equiv x_i,\quad i=1,2,\ldots,N\\
Y:&\quad y_{M+1}=\infty,\quad y_M=1,\quad y_1=0;\quad y_{M+1+i}\equiv y_i,\quad i=1,2,\ldots,M\\
Z:&\quad z_M=\infty,\quad z_{M+1}=1,\quad z_{N+i}\equiv z_{M-1+j},\quad j=1,2,\ldots,N-M+1.
\end{aligned} \quad (6)$$

The Y frame specifies the left-hand $(M+1)$-point amplitude, the Z frame specifies the right-hand $(N-M+1)$-point amplitude, and the X frame specifies the N-point amplitude as a whole. The following equations relate

* Work supported in part by the U. S. Atomic Energy Commission.
[1] Z. Koba and H. B. Nielson, Nucl. Phys. **B12**, 517 (1969).
[2] E. Donini and S. Sciuto, Istituto di Fisica dell'Università-Torino report, 1969 (unpublished).

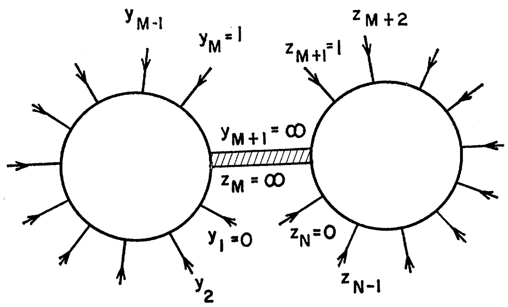

FIG. 2. Y, Z frames.

the X frame to the Y and Z frames:

$$\begin{aligned} x_i &= y_2/y_i\,, && i=1,2,\ldots,M \\ x_j &= y_2 t z_j\,, && j=M+1,\ldots,N \\ t &= x_{M+1}/x_M\,; && \end{aligned} \tag{7}$$

t is the variable associated with the propagator of the factorization channel. The y_2 is equal to the product of of all the usual multiperipheral parametrization variables belonging to the Y frame, i.e., $y_2=u_{12}u_{13}\cdots u_{1\,M+1}$. Substituting Eq. (7) into (4), we obtain[2]

$$B_N(p_1,\ldots,p_N)=\int\Big(\prod_{i\neq(M+1,M,1)} dy_i\Big)\times\{Y_{M+1}\}\int\Big(\prod_{j\neq(M,M+1,N)} dz_j\Big)\{Z_{N-M+1}\}\times\int_0^1 dt\, t^{-\alpha(s_1)-1}(1-t)^{a-1}\times\prod_{i=1}^{M}\prod_{j=M+1}^{N}(1-y_itz_j)^{-2bp_i\cdot p_j}\,, \tag{8}$$

where

$$s_1=(p_1+p_2+\cdots+p_M)^2=(p_{M+1}+\cdots+p_N)^2\,,$$

and $a+bp_i^2=0$, $i=1, 2, \ldots, N$. From now on, we will drop the $2b$ factor in $p_i\cdot p_j$. To factorize Eq. (8), we expand

$$\prod_{i=1}^{M}\prod_{j=M+1}^{N}(1-y_itz_j)^{-p_i\cdot p_j}$$

as the exponent of a logarithm[3] and introduce the harmonic-oscillator operators[4] $a_n^\dagger$, a_n. Hence

$$\prod_{i=1}^{M}\prod_{j=M+1}^{N}(1-y_itz_j)^{-p_i\cdot p_j}=\Big\langle 0\Big|\exp\Big[\sum_n\frac{a_n}{n^{1/2}}\Big(\sum_{j=M+1}^{M}p_jz_j^n\Big)\Big]t^{R_a}\times\exp\Big[\sum_n\frac{a_n^\dagger}{n^{1/2}}\Big(\sum_{i=1}^{M}p_iy_i^n\Big)\Big]\Big|0\Big\rangle\,. \tag{9}$$

Substituting (9) into (8) and defining

$$B_N(p_1\cdots p_N)=\langle 0|G_{(Z)}{}^{(1)}(a)D(R_a,s_1)G_{(Y)}{}^{(1)}(a^\dagger)|\rangle\,,$$

we then find the single factorization result (Fig. 3)

$$\langle 0|G_{(Z)}{}^{(1)}(a)|\lambda_a\rangle=\int\Big(\prod_j dz_j\Big)\{Z\}\times\Big\langle 0\Big|\exp\Big[\sum_n\frac{a_n}{n^{1/2}}\Big(\sum_{j=M+1}^{N}p_jz_j^n\Big)\Big]\Big|\lambda_a\Big\rangle\,, \tag{10a}$$

$$D(R_a,s_1)=\int_0^1 dt\, t^{-\alpha(s_1)-1}(1-t)^{a-1}t^{R_a}\,, \tag{10b}$$

$$\langle\lambda_a|G_{(Y)}{}^{(1)}(a^\dagger)|0\rangle=\int\Big(\prod_i dy_i\Big)\{Y\}\times\Big\langle\lambda_a\Big|\exp\Big[\sum_n\frac{a_n^\dagger}{n^{1/2}}\Big(\sum_{i=1}^{M}p_iy_i^n\Big)\Big]\Big|0\Big\rangle\,. \tag{10c}$$

In this and the following figures, to record the ordering of the particles on the side which has been removed, we keep a dot on each factorized Reggeon in such a way that the dot lies between x_M and x_{M+1}.

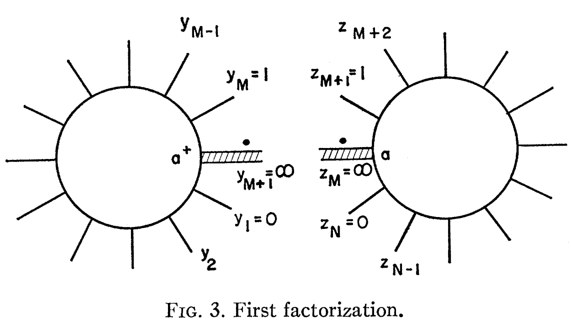

FIG. 3. First factorization.

III. SECOND FACTORIZATION

A. Planar Case

We start from Eq. (10a); let $M=0$, and divide z_i, $i=1, 2, \ldots, N$, into w_i, η, and v_j so that

$$\begin{aligned} &w_{L+1}=\infty\,, && w_L=1\,, && w_1=0\,, \\ &v_L=\infty\,, && v_{L+1}=1\,, && v_N=0\,, \\ &z_1=\infty\,, && z_2=1\,, && z_N=0\,, \end{aligned}$$

and so

$$\begin{aligned} z_i &= w_2/w_i\,, && i=1,2,\ldots,L \\ z_j &= w_2\eta v_j\,, && j=L+1,\ldots,N \\ \eta &= z_{L+1}/z_L\,. && \end{aligned} \tag{11}$$

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[3] K. Bardakci and S. Mandelstam, Phys. Rev. **184**, 1640 (1969); S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969).

[4] S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters **29B**, 679 (1969).

Substituting Eq. (11) into Eq. (10a), we find

$$\langle 0|G_{(Z)}{}^{(1)}(a)|\lambda_a\rangle=\int(\prod_i dw_i)\{W_{L+1}\}\int(\prod_j dv_j)\{V\}\int_0^1 d\eta\,\eta^{-\alpha(s_2)-1}(1-\eta)^{a-1}[\prod_{i=1}^{L}\prod_{j=L+1}^{N}(1-w_i\eta v_j)^{-p_i\cdot p_j}]$$
$$\times\left\langle 0\left|\exp\left\{\sum_n\frac{a_n}{n^{1/2}}\left[\sum_{i=2}^{L}p_i\left(\frac{w_2}{w_i}\right)^n+\sum_{j=L+1}^{N}p_j(w_2\eta v_j)^n\right]\right\}\right|\lambda_a\right\rangle. \quad (12)$$

To introduce the operator $a_n^\dagger$, we observe from Eq. (12) that

$$[\prod_{i=1}^{L}\prod_{j=L+1}^{N}(1-w_i\eta v_j)^{-p_i\cdot p_j}]\left\langle 0\left|\exp\left\{\sum_n\frac{a_n}{n^{1/2}}\left[\sum_{i=2}^{L}p_i\left(\frac{w_2}{w_i}\right)^n+\sum_{j=L+1}^{N}p_j(w_2\eta v_j)^n\right]\right\}\right|\lambda_a\right\rangle$$
$$=\left\langle 0\left|\exp\left[\sum_n\frac{a_n}{n^{1/2}}(\sum_{j=L+1}^{N}p_j(v_j)^n)\right]\eta^{R_a}\left\{\exp\left[\sum_n\frac{a_n^\dagger}{n^{1/2}}(\sum_{i=1}^{L}p_iw_i^n)\right]w_2^{R_a}\exp\left[\sum_n\frac{a_n}{n^{1/2}}\sum_{i=2}^{L+1}p_i\left(\frac{w_2}{w_i}\right)^n\right]\right\}\right|\lambda_a\right\rangle. \quad (13)$$

Therefore, if we define, from Eqs. (12) and (13),

$$\langle 0|G_{(Z)}{}^{(1)}(a)|\lambda_a\rangle=\langle 0|G_{(V)}{}^{(1)}(a)D(R_a,s_2)\times G_{(W)}{}^{(2)}(a^\dagger,a)|\lambda_a\rangle\equiv\langle\mu_a|G_{(W)}{}^{(2)}(a^\dagger,a)|\lambda_a\rangle,$$

we find the second-factorization result (Fig. 4)

$$\langle\mu_a|G_{(W)}{}^{(2)}(a^\dagger,a)|\lambda_a\rangle=\int(\prod dw_i)\{W_{L+1}\}\times\left\langle\mu_a\left|\exp\left[\sum_n\frac{a_n^\dagger}{n^{1/2}}\sum_{i=1}^{L}p_iw_i^n\right]w_2^{R_a}\times\exp\left[\sum_n\frac{a_n}{n^{1/2}}\sum_{i=2}^{L+1}p_i\left(\frac{w_2}{w_i}\right)^n\right]\right|\lambda_a\right\rangle, \quad (14)$$

where $w_{L+1}=\infty$, $w_L=1$, and $w_1=0$. Expression (14) is identical to the formula of Fubini *et al.*,[4] since, in terms of Chan variables, $w_i=u_{1\,i}u_{1\,i+1}\cdots u_{1\,L-1}$, and

$$w_2=\prod_{i=2}^{L-1}u_{1\,i}.$$

By employing a technique similar to that used in Eq. (13), we can also introduce b_m operator instead of $a_n^\dagger$, and we find the second-factorization expression

$$\left\langle 0\left|G_{(W)}{}^{(2)}(a,b)\right|\begin{matrix}\lambda_a\\ \lambda_b\end{matrix}\right\rangle=(\prod dw_i)\{W_{L+1}\}\times\left\langle 0\left|\exp(ab)_0\exp\left[\sum_m\frac{b_m}{m^{1/2}}\sum_{i=1}^{L}p_i\left(\frac{w_i}{w_2}\right)^m\right]w_2^{R_a}\times\exp\left[\sum_n\frac{a_n}{n^{1/2}}\sum_{i=2}^{L+1}p_i\left(\frac{w_2}{w_i}\right)^n\right]\right|\begin{matrix}\lambda_a\\ \lambda_b\end{matrix}\right\rangle, \quad (15)$$

where

$$(ab)_0=\sum_{n=1}^{\infty}a_nb_n$$

and

$$\langle 0|G_{(Z)}{}^{(1)}(a)|\lambda_a\rangle=\left\langle 0\left|G_{(W)}{}^{(2)}(a,b)D(R_b,s_2)G_{(V)}{}^{(1)}(b)\right|\begin{matrix}\lambda_a\\ 0_b\end{matrix}\right\rangle.$$

B. Nonplanar Case

From Eq. (10.1), with $M=0$, we change to the projective frame (Fig. 5)

$$z_L'=\infty,\quad z_{L+1}'=1,\quad z_{L-1}'=0\quad(1<L<N-1)$$

and relabel z_i''s so that

$$z_1''=z_L'=\infty,\quad z_2''=z_{L+1}'=1,\quad z_N''=z_{L-1}'=0,$$

or

$$z_i'=z_{N+i}'=z_{N+1-L+i}''=z_{1-L+i}''.$$

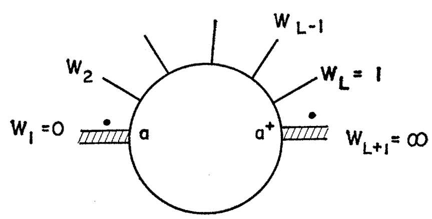

FIG. 4. Second factorization (planar).

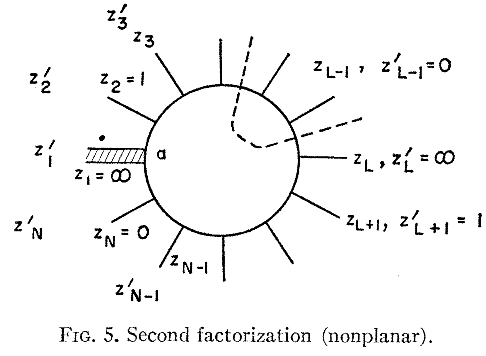

FIG. 5. Second factorization (nonplanar).

Then Eq. (10a) becomes

$$\langle 0|G_{(Z'')}{}^{(1)}(a)|\lambda_a\rangle=\int(\prod dz_i'')\{Z_N''\}\Big\langle 0\Big|\exp\Big[\sum_n \frac{a_n}{n^{1/2}}\sum_{i=1(i\neq N-L+2)}^{N} p_i \times\Big(\frac{z_{N+2-L}''-z_{N-L+3}''}{z_{N+1-L}''-z_{N+3-L}''}\Big)^n\Big(\frac{z_{N+1-L}''-z_i''}{z_{N+2-L}''-z_i''}\Big)^n\Big]\Big|\lambda_a\Big\rangle. \quad (16)$$

Carrying out the factorization as before, we find the second-factorized $(M+1)$-point tree (Fig. 6)

$$\Big\langle 0\Big|\tilde{G}_{(W)}{}^{(2)}(a,b)\Big|{\lambda_a \atop \lambda_b}\Big\rangle=\int(\prod dw_i)\{W_{M+1}\}\Big\langle 0\Big|\exp\Big[\sum_{nm}\frac{a_nb_m}{(nm)^{1/2}}\Big(\frac{w_{L+1}-w_L}{w_{L+1}-w_{L-1}}\Big)^n B_{nm}\Big(\frac{w_{L-1}}{w_L}\Big)w_L{}^m\Big]$$
$$\times\exp\Big[\sum_m \frac{b_m}{m^{1/2}}\Big(\sum_{i=1}^{M}p_iw_i{}^m\Big)+\sum_n\frac{a_n}{n^{1/2}}\sum_{i=1(i\neq L)}^{M+1}p_i\Big(\frac{w_{L+1}-w_L}{w_{L+1}-w_{L-1}}\Big)^n\Big(\frac{w_{L-1}-w_i}{w_L-w_i}\Big)^n\Big]\Big|{\lambda_a \atop \lambda_b}\Big\rangle, \quad (17)$$

where

$$B_{nm}(x)=\sum_{l=0}^{m}\binom{n}{l}\binom{-n}{m-l}(-)^m mx^l, \quad (18)$$

and

$$w_{M+1}=\infty,\quad w_M=1,\quad w_1=0.$$

As a consistency check, we may remove all bottom legs in Eq. (17) by setting $L=1$ in Eq. (17); we find

$$B_{nm}\Big(\frac{w_{L-1}}{w_L}\Big)w_L{}^m\Big(\frac{w_{L+1}-w_L}{w_{L+1}-w_{L-1}}\Big)^n\to\delta_{nm}mw_2{}^n,$$

$$\Big(\frac{w_{L+1}-w_L}{w_{L+1}-w_{L-1}}\Big)\Big(\frac{w_{L-1}-w_i}{w_L-w_i}\Big)\to\frac{w_2}{w_i}.$$

Hence Eq. (17) does reduce to the planar case, Eq. (15). However, we can equally well remove all "top" legs in Eq. (17) by setting $L=M$ in Eq. (17); we then find an expression with both dots on the other sides of the two excited legs (Fig. 7), as compared with Fig. 4,

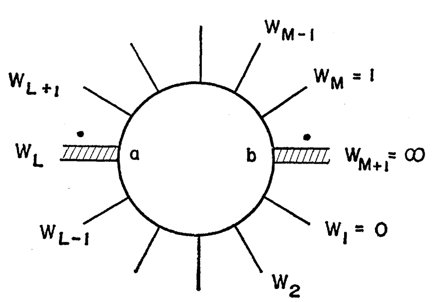

FIG. 6. Second factorization (nonplanar).

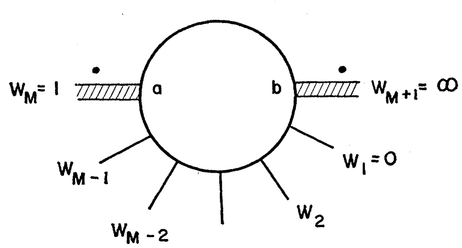

FIG. 7. Removing top legs.

$$\Big\langle 0\Big|\hat{G}_{(W)}{}^{(2)}(a,b)\Big|{\lambda_a \atop \lambda_b}\Big\rangle=\int(\prod dw_i)\{W_{M+1}\}$$
$$\times\Big\langle 0\Big|\exp\Big[\sum_{nm}\frac{a_nb_m}{(nm)^{1/2}}B_{nm}(w_{M-1})\Big]$$
$$\times\exp\Big[\sum_m\frac{b_m}{m^{1/2}}\sum_{i=1}^{M}p_iw_i{}^m+\sum_n\frac{a_n}{n^{1/2}}$$
$$\times\sum_{i=1(i\neq M)}^{M+1}p_i\Big(\frac{w_{M-1}-w_i}{1-w_i}\Big)^n\Big]\Big|{\lambda_a \atop \lambda_b}\Big\rangle, \quad (19)$$

where $w_{M+1}=\infty$, $w_M=1$, and $w_1=0$. We can verify that Eq. (19) is related to Eq. (15) simply by a projective transformation $w_i=1-w_i'$ in Eq. (19) plus two "twisting" operations on both of the excited legs. To see this, we first change the frame in Eq. (19) from $w_{M+1}=\infty$, $w_M=1$, and $w_1=0$ to $w_{M+1}'=\infty$, $w_1'=1$, and $w_M'=0$, i.e., $w_i=1-w_i'$. We then use the following identities[5,6]:

$$B_{nm}(1-x)=\sum_{l=0}^{m}\binom{n}{l}\binom{-n}{m-l}(-)^m m(1-x)^l$$
$$=\sum_l\binom{n}{l}\binom{m}{l}lx^l, \quad (20a)$$

$$\sum_{n=0}^{\infty}\binom{i}{n}\binom{n}{l}(-)^{n+l}=\delta_{il}, \quad (20b)$$

$$\sum_{nm}\frac{a_nb_m}{(nm)^{1/2}}B_{nm}(1-x)=\sum_n\bar{a}_n\bar{b}_nx^n, \quad (20c)$$

[5] C. B. Chiu, S. Matsuda, and C. Rebbi, Phys. Rev. Letters **23**, 1526 (1969).

[6] L. Caneschi, A. Schwimmer, and G. Veneziano, Phys. Letters **30B**, 356 (1969).

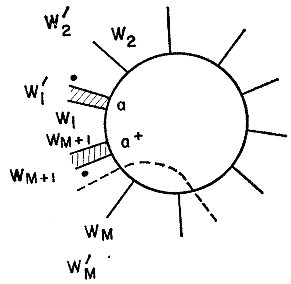

FIG. 8. Third factorization (planar).

$$\bar{a}_n \equiv (-1)^n \sum_{m=n}^{\infty} a_m \binom{m}{n} \left(\frac{n}{m}\right)^{1/2}, \tag{20d}$$

$$\bar{a}_n = (T^\dagger)^{-1} a_n T^\dagger, \tag{20e}$$

$$\Omega(\pi) = \exp\left(\sum_n \frac{\pi a_n^\dagger}{n^{1/2}}\right) T,$$

$$T = \left\{ \prod_{i=1}^{\infty} \exp\left[\sum_{j>i} \left(\frac{i}{j}\right)^{1/2} \binom{j}{i} a_j^\dagger a_i \right] \right\} (-)^R, \tag{20f}$$

$$\langle 0 | (T^\dagger)^{-1} = \langle 0 |, \tag{20g}$$

$$T^\dagger | \lambda \rangle = | \bar{\lambda} \rangle. \tag{20h}$$

Then, clearly, Eq. (19) becomes

$$\left\langle 0 \left| \hat{G}_{(W)}{}^{(2)}(a,b) \right| \begin{matrix} \lambda_a \\ \lambda_b \end{matrix} \right\rangle = \int (\prod dw_i') \{ W_{M+1}' \}$$

$$\times \left\langle 0 \right| \exp(ab)_0 \exp\left[\sum_m \frac{b_m}{m^{1/2}} \sum_{i=1}^{M} p_i w_i'^m \right] w_{M-1}{}^{R_a}$$

$$\times \exp\left[\sum_n \frac{a_n}{n^{1/2}} \sum_{i=1 (i \neq M)}^{M+1} p_i \left(\frac{w_{M-1}'}{w_i'} \right)^n \right] \left| \begin{matrix} \bar{\lambda}_a \\ \bar{\lambda}_b \end{matrix} \right\rangle, \tag{21}$$

which is analogous to Eq. (15), except that the states $|\lambda_a\rangle$, $|\lambda_b\rangle$ are twisted. This is the expected result, since excited-particle lines with the dots on opposite sides are related to each other by the twisting operator.

IV. THIRD FACTORIZATION

We start from the second-factorized tree, Eq. (14), and change from the frame $w_{M+1} = \infty$, $w_M = 1$, and $w_1 = 0$ to the new frame, $w_{M+1}' = \infty$, $w_1' = 1$, and $w_M' = 0$ (Fig. 8). From Eq. (14) we get

$$\langle \mu_a | G_{(W')}{}^{(2)}(a^\dagger, a) | \lambda_a \rangle = \int (\prod dw_i') \{ W' \}$$

$$\times \left\langle \mu_a \right| \exp\left[\sum_n \frac{a_n^\dagger}{n^{1/2}} \sum_{i=1}^{M} p_i (1 - w_i')^n \right] (1 - w_2')^{R_a}$$

$$\times \exp\left[\sum_n \frac{a_n}{n^{1/2}} \sum_{i=2}^{M+} p_i \left(\frac{1 - w_2'}{1 - w_i'} \right)^n \right] \left| \lambda_a \right\rangle. \tag{14'}$$

We then divide w_i' into r_i, s_j, and η as before:

$$r_{L+1} = \infty, \quad r_L = 1, \quad r_0 = 0,$$
$$s_L = \infty, \quad s_{L+1} = 1, \quad s_M = 0,$$
$$w_{M+1}' = w_0' = \infty, \quad w_1' = 1, \quad w_M' = 0$$

and

$$w_i' = r_1 / r_i, \quad i = 1, 2, \ldots, L$$
$$w_j' = r_1 \eta s_j, \quad j = L+1 \cdots M$$
$$\eta = w_{L+1}' / w_L'.$$

Therefore, in terms of r_i, s_j, and η, Eq. (14′) becomes

$$\langle \mu_a | G_{(W')}{}^{(2)}(a^\dagger, a) | \lambda_a \rangle = \int (\prod dr_i) \{ R_{L+2} \} \int \prod ds_j \{ S \} \int_0^1 d\eta \, \eta^{-\alpha(s_3)-1} (1-\eta)^{a-1} \left[\prod_{i=1}^{L} \prod_{j=L+1}^{M} (1 - r_i \eta s_j)^{-p_i \cdot p_j} \right]$$

$$\times \left\langle \mu_a \right| \exp\left\{ \sum_n \frac{a_n^\dagger}{n^{1/2}} \left[\sum_{i=1}^{L} p_i \left(1 - \frac{r_1}{r_i} \right)^n + \sum_{j=L+1}^{M} p_j \sum_{l=0}^{n} \binom{n}{l} (-)^l (r_1 \eta s_j)^l \right] \right\}$$

$$\times \exp\left\{ \sum_n \frac{a_n}{n^{1/2}} \left[\sum_{i=2}^{L} p_i \left(1 - \frac{r_1}{r_i} \right)^{-n} + \sum_{j=L+1}^{M+1} p_j \sum_{l=0}^{\infty} \binom{-n}{l} (-)^l (r_1 \eta s_j)^l \right] \right\} \left(1 - \frac{r_1}{r_2} \right)^{R_a} \left| \lambda_a \right\rangle. \tag{14''}$$

To remove s_j, $j = L+1, \ldots, M+1$, and η, we expand

$$\prod_{i=1}^{L} \prod_{j=L+1}^{M} (1 - r_i \eta s_j)^{-p_i \cdot p_j}$$

in terms of $c_l^\dagger$, c_l operators:

$$\prod_{i=1}^{L} \prod_{j=L+R}^{M} (1 - r_i \eta s_j)^{-p_i \cdot p_j} = \left\langle 0 \right| \exp\left[\sum_l \frac{c_l}{l^{1/2}} \sum_{j=L+1}^{M} p_j (\eta s_j)^l \right] \exp\left[\sum_l \frac{c_l^\dagger}{l^{1/2}} \sum_{i=1}^{L} p_i r_i^l \right] \left| 0 \right\rangle,$$

and use the identity

$$\langle 0 | \exp(cf) \exp(fg) = \langle 0 | \exp(cf) \exp(c^\dagger g),$$

which amounts to replacing the scalar f by the operator $c^\dagger$. We therefore replace

$$\sum_{j=L+1}^{M} p_j(\eta s_j)^l$$

in the exponent term of Eq. (14″). We can further write

$$\langle\mu_a|G_{(W)}{}^{(2)}(a^\dagger,a)|\lambda_a\rangle=\left\langle\begin{matrix}\mu_a\\0_c\end{matrix}\right|G_{(S)}{}^{(1)}(c)D(R_c,s_3)G_{(R)}{}^{(3)}(a^\dagger,a,c)\left|\begin{matrix}\lambda_a\\0_c\end{matrix}\right\rangle.$$

Therefore, we find the following formula for the thrice-factorized tree [Fig. 9(a)]:

$$\left\langle\begin{matrix}\mu_a\\0_c\end{matrix}\right|G_{(R)}{}^{(3)}(a^\dagger,a,c)\left|\begin{matrix}\lambda_a\\\lambda_c\end{matrix}\right\rangle=\int(\prod dr_i)\{R_{L+2}\}\left\langle\begin{matrix}\mu_a\\0_c\end{matrix}\right|\exp\left[\sum_l\frac{c_l}{l^{1/2}}\sum_{i=1}^{L}p_ir_i{}^l\right]$$
$$\times\exp\left\{\sum_n\frac{a_n{}^\dagger}{n^{1/2}}\left[\sum_{i=1}^{L+1}p_i\left(1-\frac{r_1}{r_i}\right)^n+\sum_{l=1}^{n}\binom{n}{l}(-)^l l r_1{}^l\frac{c_l}{l^{1/2}}\right]\right\}$$
$$\times\exp\left\{\sum_n\frac{a_n}{n^{1/2}}\left[\sum_{i=2}^{L+1}p_i\left(1-\frac{r_1}{r_i}\right)^{-n}+\sum_{l=1}^{\infty}\binom{-n}{l}(-)^l l(r_1)^l\frac{c_l}{l^{1/2}}\right]\right\}\left(1-\frac{r_1}{r_2}\right)^{R_a}\left|\begin{matrix}\lambda_a\\\lambda_c\end{matrix}\right\rangle,\quad(22)$$

where $r_{L+1}=\infty$, $r_L=1$, and $r_0=0$. We can also start from Eq. (15) and factorize as before; we find the thrice-factorized tree [Fig. 9(b)]

$$\left\langle0\right|G_{(R)}{}^{(3)}(a,b,c)\left|\begin{matrix}\lambda_a\\\lambda_b\\\lambda_c\end{matrix}\right\rangle=\int(\prod dr_i)\{R_{L+2}\}\left\langle0\right|\exp\left[\sum_{l=1}^{\infty}\frac{c_l}{l^{1/2}}\left(\sum_{i=1}^{L}p_ir_i{}^l\right)\right]\exp\left[\sum_{n=1}^{\infty}a_nb_n\left(1-\frac{r_1}{r_2}\right)^n\right]$$
$$\times\exp\left\{\sum_m\frac{b_m}{m^{1/2}}\left[\sum_{i=1}^{L+1}p_i\left(1-\frac{r_1}{r_i}\right)^m+\sum_{l=1}^{m}\binom{m}{l}(-)^l l(r_1)^l\frac{c_l}{l^{1/2}}\right]\right\}$$
$$\times\exp\left\{\sum_n\frac{a_n}{n^{1/2}}\left(1-\frac{r_1}{r_2}\right)^n\left[\sum_{i=2}^{L+1}p_i\left(1-\frac{r_1}{r_i}\right)^{-n}+\sum_{l=1}^{\infty}\binom{-n}{l}(-)^l l r_1{}^l\frac{c_l}{l^{1/2}}\right]\right\}\left|\begin{matrix}\lambda_a\\\lambda_b\\\lambda_c\end{matrix}\right\rangle,\quad(23)$$

where, again, $r_{L+1}=\infty$, $r_L=1$, and $r_0=0$.

It is interesting to see how Eq. (23) can reduce to Sciuto's[7] three-Reggeon vertex. Since we are unable to factorize the c_l leg without assuming the factor $1-r_1/r_2$ to be in the R frame (opposite the S frame), we therefore consider the case $L=2$ (Fig. 10). Substituting $r_3=\infty$, $r_2=1$, and $r_0=0$ in Eq. (23), we get

$$\left\langle0\right|G_{(R)}{}^{(3)}(a,b,c)\left|\begin{matrix}\lambda_a\\\lambda_b\\\lambda_c\end{matrix}\right\rangle=\int_0^1dr_1\,r_1{}^{-\alpha_{01}-1}(1-r_1)^{-\alpha_{12}-1}\left\langle0\right|\exp\left[\sum_l\frac{c_l}{l^{1/2}}(p_1r_1{}^l+p_2)\right]$$
$$\times\exp\left[\sum_n a_nb_n(1-r_1)^n\right]\exp\left\{\sum_m\frac{b_m}{m^{1/2}}\left[[p_3+p_2(1-r_1)^m]+\sum_{l=1}^{m}\binom{m}{l}(-)^l l r_1{}^l\frac{c_l}{l^{1/2}}\right]\right\}$$
$$\times\exp\left\{\sum_n\frac{a_n}{n^{1/2}}(1-r_1)^n\left[p_3+p_2(1-r_1)^{-n}+\sum_{l=1}^{\infty}\binom{-n}{l}(-)^l l r_1{}^l\frac{c_l}{l^{1/2}}\right]\right\}\left|\begin{matrix}\lambda_a\\\lambda_b\\\lambda_c\end{matrix}\right\rangle.\quad(24)$$

In order to remove the external scalar leg (i.e., r_2) in Fig. 10, we then go to the pole position of the α_{01} variable and look at the residue given by Eq. (24). We can rewrite Eq. (24) in the following form:

$$\left\langle\begin{matrix}\lambda_a\\\lambda_b\\0_c\end{matrix}\right|G_{(R)}{}^{(3)}(a^\dagger,b^\dagger,c)\left|\begin{matrix}0_a\\0_b\\\lambda_c\end{matrix}\right\rangle=\left\langle\begin{matrix}\lambda_a\\\lambda_b\\0_c\end{matrix}\right|\left[\exp\left(\sum_l\frac{c_l{}^\dagger}{l^{1/2}}p_2\right)\exp\left(\sum_l\frac{c_l}{l^{1/2}}p_2\right)\right]$$
$$\times\left[\int_0^1dr_1\,r_1{}^{-\alpha_{01}-1}(1-r_1)^{-a-b(p_1{}^2+p_2{}^2)-1}r_1{}^{R_c}(1-r_1)^{R_a}(1-r_1)^{-2bp_1\cdot p_2}\right]$$
$$\times\exp\left\{\sum_l\frac{c_l}{l^{1/2}}p_1+(a^\dagger,b^\dagger)_0+\sum_m\frac{b_m{}^\dagger}{m^{1/2}}[p_3+p_2(1-r_1)^m]+(b^\dagger,c)_-+\sum_n\frac{a_n}{n^{1/2}}[p_3+p_2(1-r_1)^{-n}]+(a^\dagger,c)_+\right\}\left|\begin{matrix}0_a\\0_b\\\lambda_c\end{matrix}\right\rangle.\quad(25)$$

---

[7] S. Sciuto, Istituto di Fisica dell'Università-Torino report, 1969 (unpublished).

We observe that the last factor { } in Eq. (25) is a three-Reggeon vertex if $p_2 \to 0$. To get the three-Reggeon vertex $W^{(3)}(a,b,c)$ from Eq. (25), we thus take the limit $p_2 \to 0$, and write

$$\lim_{p_2\to 0} \left\langle \begin{matrix}\lambda_a\\ \lambda_b\\ 0_c\end{matrix} \right| G^{(3)}(a^\dagger,b^\dagger,c) \left| \begin{matrix}0_a\\ 0_b\\ \lambda_c\end{matrix} \right\rangle = \lim_{p_2\to 0} \left\langle \begin{matrix}\lambda_a\\ \lambda_b\\ 0_c\end{matrix} \right| V(p_2)D'(R_a,R_c)W^{(3)}(a^\dagger,b^\dagger,c) \left| \begin{matrix}0_a\\ 0_b\\ \lambda_c\end{matrix} \right\rangle . \tag{26}$$

But under this limit the discussion of the factorization must then be modified, since the mass of leg 2 is no longer equal to that of the other legs. The relevant modification is given in the Appendix. As $p_2 \to 0$, the ordinary vertex function $V(p_2)$ becomes the unit operator and so $D'W^{(3)}(a^\dagger,b^\dagger,c)$ becomes a well-defined quantity. From Eqs. (25) and (26), we obtain the c_l-leg propagator

$$D'(R_c,s_3) = \int_0^1 dr_1\, r_1^{-\alpha_{01}-1}(1-r_1)^{-bp_1^2-1}\, r_1^{R_c}(1-r_1)^{R_a}\,, \tag{27}$$

and Sciuto's three-Reggeon vertex (Fig. 11)

$$\left\langle 0 \right| W^{(3)}(a,b,c) \left| \begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\end{matrix} \right\rangle = \left\langle 0 \right| \exp[p_1 c + (a,b)_0 + p_2' b + (b,c)_- + p_2' a + (a,c)_+] \left| \begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\end{matrix} \right\rangle , \tag{28}$$

where $p_2' \equiv p_3$.

Another interesting thrice-factorized tree (Fig. 12) can also be obtained easily by directly factorizing Eq. (15) in a general projective frame, similarly to what we did in Eqs. (16) and (17). It takes the following form (Fig. 12):

$$\begin{aligned}
\left\langle 0 \right| \tilde{G}_{(R)}(a,b,c) \left| \begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\end{matrix} \right\rangle &= \int \Big(\prod_{i\neq(R+1,R,1)} dr_i \Big) \{R_{R+1}\} \left\langle 0 \right| \exp\Big[\sum_l \frac{c_l}{l^{1/2}} \sum_{i=1}^{R} p_i r_i^{\,l} \Big] \\
&\times \exp(a,b)_0 \exp\Big\{ \sum_m \frac{b_m}{m^{1/2}} \Big(\frac{r_{L-1}-r_L}{r_{L-1}-r_{L+1}} \Big)^m \Big[\sum_{i=1(i\neq L)}^{R+1} p_i \Big(\frac{r_{L+1}-r_i}{r_L-r_i} \Big)^m + \sum_l \frac{c_l}{l^{1/2}} B_{ml}\Big(\frac{r_{L+1}}{r_L} \Big) r_L^{\,l} \Big] \Big\} \\
&\times \exp\Big\{ \sum_n \frac{a_n}{n^{1/2}} \Big(\frac{r_{L-1}-r_L}{r_{L-1}-r_{L+1}} \Big)^{-n} \Big[\sum_{i=1(i\neq L+1)}^{R+1} p_i \Big(\frac{r_{L+1}-r_i}{r_L-r_i} \Big)^{-n} + \sum_{l=1}^{\infty} \frac{c_l}{l^{1/2}} B_{-nl}\Big(\frac{r_{L+1}}{r_L} \Big) r_L^{\,l} \Big] \Big\} \\
&\times \Big[\Big(\frac{r_{L-1}-r_L}{r_{L-1}-r_{L+1}} \Big) \Big(\frac{r_{L+2}-r_{L+1}}{r_{L+2}-r_L} \Big) \Big]^{R_a} \left| \begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\end{matrix} \right\rangle ,
\end{aligned} \tag{29}$$

where

$$B_{nm}(x) = \sum_{i=0}^{m} \binom{n}{i} \binom{-n}{m-i} (-)^m m x^i = \sum_i \binom{n}{i} \binom{m}{i} i (1-x)^i . \tag{30}$$

The reason why we are interested in Eq. (29) will be clear in Sec. V. As a consistency check, we let $R = L-2$ in Eq. (29). We get

$$\Big(\frac{r_{L-1}-r_L}{r_{L-1}-r_{L+1}} \Big) \Big(\frac{r_i-r_{L+1}}{r_i-r_L} \Big) \to \Big(1 - \frac{r_{L+1}}{r_i} \Big) , \qquad r_L^{\,m} B_{nm}\Big(\frac{r_{L+1}}{r_L} \Big) \to \binom{n}{m} (-)^m m r_{L+1}^{\,m} ;$$

therefore Eq. (29) does reduce to (23), as it should.

V. FOURTH FACTORIZATION

Beginning with Eq. (23) [Fig. 9(b)], we change to the new frame

$$r_1' = \infty , \quad r_2' = 0 , \quad r_0' = 1 \quad (N \equiv L+1) ,$$

i.e.,

$$r_i = \Big(\frac{r_N' - r_{N-1}'}{r_N' - r_i'} \Big) \Big(\frac{1 - r_i'}{1 - r_{N-1}'} \Big) .$$

Substituting this in Eq. (23), and factorizing as before, we then obtain the four-times-factorized tree (Fig. 13)

$$
\begin{aligned}
&\left[0\left|G_{(W)}{}^{(4)}(a,b,c,d,)\right|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\right] = \int (\prod dw_i)\{W_{M+1}\}\left[0\right| \exp\left[\sum_k \frac{d_k}{k^{1/2}} \sum_{i=1}^{M} p_i w_i{}^k\right] \exp\left[\sum a_n b_n \left(1-\frac{w_2}{w_3}\right)^n\right] \\
&\quad\times\exp\left\{\sum_l \frac{c_l}{l^{1/2}}\left[\sum_{i=1(i\neq 3)}^{M+1} p_i \left(\frac{w_3-w_4}{w_2-w_4}\right)^l \left(\frac{w_2-w_i}{w_3-w_i}\right)^l + \left(\frac{w_3-w_4}{w_2-w_4}\right)^l \sum_{k=1}^{l} \frac{d_k}{k^{1/2}} B_{lk}\left(\frac{w_2}{w_3}\right) w_3{}^k\right]\right\} \\
&\quad\times\exp\left\{\sum_m \frac{b_m}{m^{1/2}}\left[\sum_{i=1(i\neq 2)}^{M+1} p_i \left(\frac{w_2-w_3}{w_2-w_i}\right)^m \left(\frac{w_i}{w_3}\right)^m + \left(1-\frac{w_2}{w_3}\right)^m \sum_{k=1}^{\infty} \binom{-m}{k} (-)^k k w_2{}^k \frac{d_k}{k^{1/2}}\right.\right. \\
&\quad\left.\left.+\sum_{l=1}^{M} \binom{m}{l} (-)^l l \frac{c_l}{l^{1/2}} \left(\frac{w_3-w_4}{w_2-w_4}\right)^l \left(\frac{w_2}{w_3}\right)^l\right]\right\} \exp\left\{\sum_n \frac{a_n}{n^{1/2}} \left(1-\frac{w_2}{w_3}\right)^n \left[\sum_{i=2}^{M+1} p_i \left(\frac{w_2-w_3}{w_2-w_i}\right)^{-n} \left(\frac{w_i}{w_3}\right)^{-n}\right.\right. \\
&\quad\left.\left.+\left(1-\frac{w_2}{w_3}\right)^{-n} \sum_k \binom{n}{k} (-)^k k w_2{}^k \frac{d_k}{k^{1/2}} + \sum_{l=1}^{\infty} \binom{-n}{l} (-)^l l \frac{c_l}{l^{1/2}} \left(\frac{w_2}{w_2}\right)^l \left(\frac{w_3-w_4}{w_3-w^2}\right)^l\right]\right\} \left|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\right], \qquad (31)
\end{aligned}
$$

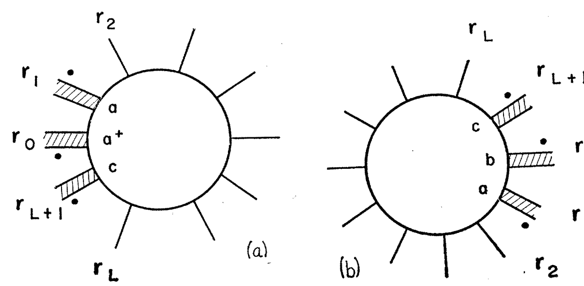

FIG. 9. Third factorization, (a) planar and (b) planar.

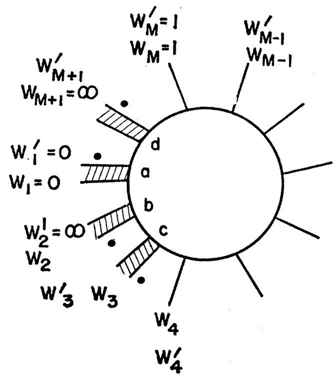

FIG. 13. Fourth factorization planar).

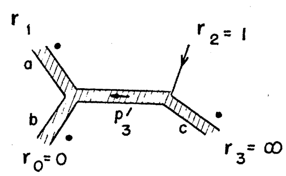

FIG. 10. Removing the r_2 leg.

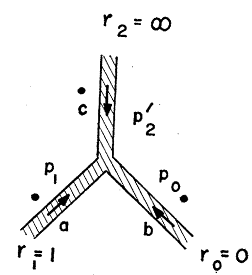

FIG. 11. Sciuto's three-Reggeon vertex.

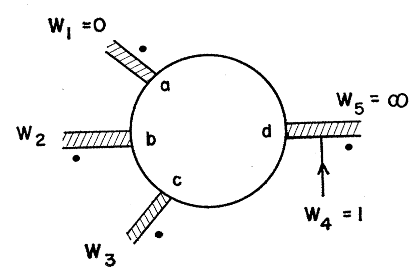

FIG. 14. Removing the w_4 leg.

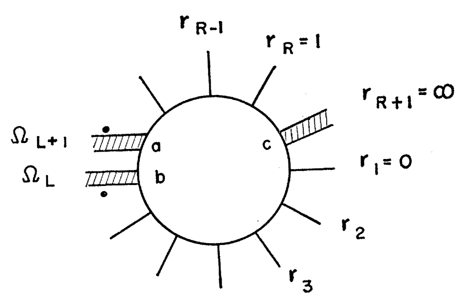

FIG. 12. Third factorization (nonplanar).

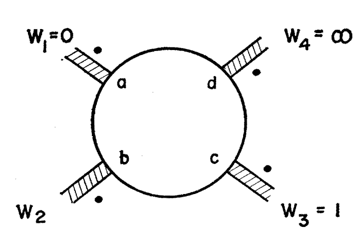

FIG. 15. Four-Reggeon amplitude.

where $w_{M+1}=\infty$, $w_M=1$, and $w_1=0$. We can simplify Eq. (31) by changing to the projective frame $w_1'=0$, $w_2'=\infty$, and $w_M'=1$, i.e.,

$$w_i=(w_{M+1}',w_1',w_M',w_i')=\frac{(1-w_{M+1}')w_i'}{w_i'-w_{M+1}'}\,.$$

Then Eq. (31) becomes the following expression (Fig. 13):

$$\begin{aligned}
&\Bigl(0\Bigl|G_{(W')}{}^{(4)}(a,b,c,d)\Bigr|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\Bigr)=\int(\prod dw_i')\{W_{M+1}'\}\Bigl(0\Bigl|\exp\Bigl[\sum_k\frac{d_k}{k^{1/2}}\sum_{i=1}^{M}p_i\Bigl(\frac{1-w_{M+1}'}{w_i'-w_{M+1}'}\Bigr)^k w_i'^k\Bigr]\\
&\times\exp\Bigl[\sum_n a_nb_n\Bigl(\frac{w_{M+1}'}{w_i'}\Bigr)^n\Bigr]\exp\Bigl\{\sum_l\frac{c_l}{l^{1/2}}\Bigl[\sum_{i=1(i\neq3)}^{M+1}p_i\Bigl(\frac{w_3'-w_4'}{w_3'-w_i'}\Bigr)^l+\Bigl(\frac{w_3'-w_4'}{w_3'-w_{M+1}'}\Bigr)^l\sum_{k=1}^{l}\frac{d_k}{k^{1/2}}\\
&\times B_{lk}\Bigl(1-\frac{w_{M+1}'}{w_3'}\Bigr)\Bigl(\frac{(1-w_{M+1}')w_3'}{w_3'-w_{M+1}'}\Bigr)^k\Bigr]\Bigr\}\exp\Bigl\{\sum_m\frac{b_m}{m^{1/2}}\Bigl[\sum_{i=1(i\neq2)}^{M+1}p_i\Bigl(\frac{w_i'}{w_3'}\Bigr)^m\\
&+\Bigl(\frac{w_{M+1}'}{w_3'}\Bigr)^m\sum_{k=1}^{\infty}\binom{-m}{k}(-)^kk(1-w_{M+1}')^k\frac{d_k}{k^{1/2}}+\sum_{l=1}^{m}\binom{m}{l}(-)^ll\frac{c_l}{l^{1/2}}\Bigl(1-\frac{w_4'}{w_3'}\Bigr)^l\Bigr]\Bigr\}\\
&\times\exp\Bigl\{\sum_n\frac{a_n}{n^{1/2}}\Bigl(\frac{w_{M+1}'}{w_3'}\Bigr)^n\Bigl[\sum_{i=2}^{M+1}p_i\Bigl(\frac{w_i'}{w_3'}\Bigr)^{-n}+\Bigl(\frac{w_{M+1}'}{w_3'}\Bigr)^{-n}\sum_{k=1}^{n}\binom{n}{k}(-)^kk\frac{d_k}{k^{1/2}}(1-w_{M+1}')^k\\
&\qquad+\sum_{l=1}^{\infty}\binom{-n}{l}(-)^ll\frac{c_l}{l^{1/2}}\Bigl(1-\frac{w_4'}{w_3'}\Bigr)^l\Bigr]\Bigr\}\Bigl|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\Bigr)\,. \qquad (32)
\end{aligned}$$

Since the minimum number of legs for the fourth factorization to be feasible is five, i.e., $M=4$ in Eq. (31), to get the four-Reggeon vertex, we consider the case $M=4$ in Eq. (31) (Fig. 14). Set $M=4$ in Eq. (31); we get

$$\begin{aligned}
&\Bigl(0\Bigl|G_{(W)}{}^{(4)}(a,b,c,d)\Bigr|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\Bigr)=\int dw_2dw_3\{w_2^{-\alpha_{12}-1}w_3^{a+bp_2^2-2bp_1\cdot p_3}(w_3-w_2)^{-\alpha_{23}-1}(1-w_2)^{a+bp_3^2-2bp_2\cdot p_4}(1-w_3)^{-\alpha_{34}-1}\}\\
&\times\Bigl(0\Bigl|\exp\Bigl\{\sum_n\frac{a_n}{n^{1/2}}\Bigl[p_3\Bigl(1-\frac{w_2}{w_3}\Bigr)^n+p_5+p_4\Bigl(1-\frac{w_2}{w_3}\Bigr)^n\Bigl(\frac{w_3-w_2}{1-w_2}\Bigr)^{-n}w_3^n\Bigr]+\sum_k\frac{d_k}{k^{1/2}}(p_4+p_3w_3^k+p_2w_2^k)+\sum_m\frac{b_m}{m^{1/2}}\\
&\times\Bigl[p_3+p_4\Bigl(\frac{1}{w_3}\Bigr)^m\Bigl(\frac{w_2-w_3}{w_2-1}\Bigr)^m+p_5\Bigl(1-\frac{w_2}{w_3}\Bigr)^m\Bigr]+\sum_l\frac{c_l}{l^{1/2}}\Bigl[p_4+p_5\Bigl(\frac{1-w_3}{1-w_2}\Bigr)^l+p_1\Bigl(\frac{1-w_3}{1-w_2}\Bigr)^l\Bigl(\frac{w_2}{w_3}\Bigr)^l\Bigr]\Bigr\}\\
&\times\exp\Bigl\{\sum_n a_nb_n\Bigl(1-\frac{w_2}{w_3}\Bigr)^n+\sum_{nl}\frac{a_nc_l}{(nl)^{1/2}}\Bigl(1-\frac{w_2}{w_3}\Bigr)^l\binom{-n}{l}(-)^ll\Bigl(\frac{w_2}{w_3}\Bigr)^l\Bigl(\frac{1-w_3}{1-w_2}\Bigr)^l+\sum_{nk}\frac{a_nd_k}{(nk)^{1/2}}\binom{n}{k}(-)^kkw_2^k\\
&+\sum_{ml}\frac{b_mc_l}{(ml)^{1/2}}\binom{m}{l}(-)^ll\frac{w_2}{w_3}\Bigl(\frac{1-w_3}{1-w_2}\Bigr)^l+\sum_{mk}\frac{b_md_k}{(mk)^{1/2}}\binom{-m}{k}(-)^kkw_2^k\Bigl(1-\frac{w_2}{w_3}\Bigr)^m\\
&\qquad+\sum_{lk}\frac{c_ld_k}{(lk)^{1/2}}\Bigl(\frac{1-w_3}{1-w_2}\Bigr)^lB_{lk}\Bigl(\frac{w_2}{w_3}\Bigr)w_3^k\Bigr\}\Bigl|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\Bigr)\,. \qquad (33)
\end{aligned}$$

Notice, in terms of Chan variables, that $w_2=u_{12}u_{13}$ and $w_3=u_{13}$, We are interested in the poles of α_{13} variables. Analogously to the three-Reggeon case, we therefore take the limit $p_4\to0$ in Eq. (33) and write

$$\lim_{p_4\to0}\Bigl(0\Bigl|G_{(W)}{}^{(4)}(a,b,c,d)\Bigr|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\Bigr)=\lim_{p_4\to0}\Bigl(0\Bigl|W^{(4)}(a,b,c,d)D''(R_d,R_c,s_4)V_{(p_4)}{}^{(d)}\Bigr|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\Bigr)\,. \qquad (34)$$

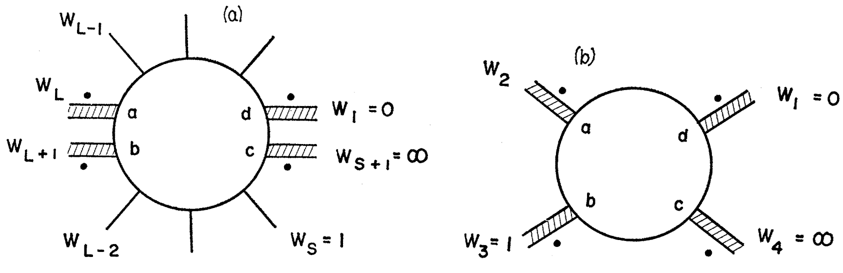

FIG. 16. (a) Fourth factorization (nonplanar) and (b) four-Reggeon amplitude.

We find

$$D''(R_d,R_c,s_4)=\int_0^1 du_{13}u_{13}^{-\alpha_{123}-1+R_d}(1-u_{13})^{-bp_3^2-1+R_c}(1-u_{12}u_{13})^{\alpha(p_3)-R_c}, \tag{35}$$

and the four-Reggeon vertex function

$$\left[0\left|W^{(4)}(a,b,c,d)\right|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\right]=\int_0^1 du_{12}u_{12}^{-\alpha_{12}-1}(1-u_{12})^{-\alpha_{23}-1}$$

$$\times\left[0\right|\exp\left\{\sum_n\frac{a_n}{n^{1/2}}[p_4'+p_3(1-u_{12})^n]+\sum_m\frac{b_m}{m^{1/2}}[p_3+p_4'(1-u_{12})^m]+\sum_l\frac{c_l}{l^{1/2}}(p_4'+p_1u_{12}^l)+\sum_k\frac{d_k}{k^{1/2}}(p_2+p_2u_{12}^k)\right\}$$

$$\times\exp\left\{\sum_n a_nb_n(1-u_{12})^n+\sum_{nl}\frac{a_nc_l}{(nl)^{1/2}}(1-u_{12})^n\binom{-n}{l}(-)^l lu_{12}^l+\sum_{nk}\frac{a_nd_k}{(nk)^{1/2}}\binom{n}{k}(-)^k ku_{12}^k+\sum_{ml}\frac{b_mc_l}{(ml)^{1/2}}\binom{m}{l}(-)^l lu_{12}^l\right.$$

$$\left.+\sum_{mk}\frac{b_md_k}{(mk)^{1/2}}(1-u_{12})^m\binom{-m}{k}(-)^k ku_{12}^k+\sum_{lk}\frac{c_ld_k}{(lk)^{1/2}}B_{lk}(u_{12})\right\}\left|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\right], \tag{36}$$

where $p_4'=p_5$ and $u_{12}=w_2/w_3$. We can then reintroduce Koba-Nielsen variables $w_4=\infty$, $w_3=1$, $w_2=u_{12}$, and $w_1=0$ into Eq. (36) (Fig. 15). Equation (36) looks all right, but we have to keep in mind the factor $[(1-u_{13})/(1-u_{12}u_{13})]^{R_c-\alpha(p_3)}$ in the propagator corresponding to the fourth leg (d_k leg).

Another interesting four-Reggeon vertex [Fig. 16(b)] can be derived directly from Eq. (29). Directly factorizing Eq. (29) again, we obtain the four-times-factorized tree [Fig. 16(a)]

$$\left[0\left|G_{(W)}{}^{(4)}(a,b,c,d)\right|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\right]=\int(\prod dw_i)\{W_{S+1}\}\left[0\right|\exp\left\{\sum_n\frac{a_n}{n^{1/2}}\sum_{i=1(i\neq L)}^{S}p_i\left(\frac{w_{L+2}-w_{L+1}}{w_{L+1}-w_L}\right)^{-n}\left(\frac{w_L-w_i}{w_{L+1}-w_i}\right)^{-n}\right.$$

$$\left.+\sum_m\frac{b_m}{m^{1/2}}\sum_{i=1(i\neq L+1)}^{S+1}p_i\left(\frac{w_{L+2}-w_{L+1}}{w_{L+2}-w_L}\right)^m\left(\frac{w_L-w_i}{w_{L+1}-w_i}\right)^m+\sum_l\frac{c_l}{l^{1/2}}\sum_{i=1}^{S}p_iw_i^l+\sum_k\frac{d_k}{k^{1/2}}\sum_{i=2}^{S+1}p_i\left(\frac{w_2}{w_i}\right)^k\right\}$$

$$\times\exp\left\{(a,b)_0+\sum_{nl}\frac{a_nc_l}{(nl)^{1/2}}\left(\frac{w_{L+2}-w_{L+1}}{w_{L+2}-w_L}\right)^{-n}B_{-nl}\left(\frac{w_L}{w_{L+1}}\right)w_{L+1}{}^l+\sum_{nk}\frac{a_nd_k}{(nk)^{1/2}}\left(\frac{w_{L+2}-w_{L+1}}{w_{L+2}-w_L}\right)^{-n}\left(\frac{w_L}{w_{L+1}}\right)^{-n}\right.$$

$$\times B_{-nk}\left(\frac{w_{L+1}}{w_L}\right)\left(\frac{w_2}{w_{L+1}}\right)^k+\sum_{nl}\frac{b_mc_l}{(ml)^{1/2}}\left(\frac{w_{L+2}-w_{L+1}}{w_{L+2}-w_L}\right)^mB_{ml}\left(\frac{w_L}{w_{L+1}}\right)w_{L+1}{}^l+\sum_{mk}\frac{b_md_k}{(mk)^{1/2}}\left(\frac{w_{L+2}-w_{L+1}}{w_{L+2}-w_L}\right)^m\left(\frac{w_L}{w_{L+1}}\right)^m$$

$$\left.\times B_{mk}\left(\frac{w_{L+1}}{w_L}\right)\left(\frac{w_2}{w_{L+1}}\right)^k+\sum_{lk}\frac{c_ld_k}{(lk)^{1/2}}w_2{}^ll\right\}\left[\left(\frac{w_{L+1}-w_{L+2}}{w_L-w_{L+2}}\right)\left(\frac{w_{L-1}-w_L}{w_{L-1}-w_{L+1}}\right)\right]^{R_a}\left|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\right], \tag{37}$$

where $w_{S+1}=\infty$, $w_S=1$, and $w_1=0$. By procedures similar to those used in Eqs. (33)–(36), we obtain the four-

Reggeon vertex [Fig. 16(a)]

$$\left[0\left|W^{(4)}(a,b,c,d)\right|\begin{matrix}\lambda_a\\\lambda_b\\\lambda_c\\\lambda_d\end{matrix}\right]=\int_0^1 dw_2 w_2^{-\alpha_{12}-1}(1-w_2)^{-\alpha_{23}-1}$$

$$\times\left[0\right|\exp\left\{\sum_n \frac{a_n}{n^{1/2}}(p_1+p_4w_2^n)+\sum_m \frac{b_m}{m^{1/2}}(p_4+p_1w_2^m)+\sum_l \frac{c_l}{l^{1/2}}(p_3+p_2w_2^l)+\sum_k \frac{d_k}{k^{1/2}}(p_2+p_3w_2^k)\right\}$$

$$\times\exp\left\{\sum_n a_nb_nw_2^n+\sum_l c_ld_lw_2^l+\sum_{nl}\frac{a_nc_l}{(nl)^{1/2}}B_{-nl}(w_2)w_2^n+\sum_{nk}\frac{a_nd_k}{(nk)^{1/2}}B_{nk}(w_2)+\sum_{ml}\frac{b_mc_l}{(ml)^{1/2}}B_{ml}(w_2)\right.$$

$$\left.+\sum_{mk}\frac{b_md_k}{(mk)^{1/2}}B_{-mk}(w_2)w_2^m\right\}\left|\begin{matrix}\lambda_a\\\lambda_b\\\lambda_c\\\lambda_d\end{matrix}\right] . \quad (38)$$

Again we must keep in mind the factor $[(1-u_{13})/(1-u_{12}u_{13})]^{R_b-\alpha(p_3)}$, where $u_{12}=w_2$, in the propagator corresponding to the fourth leg (c_l leg). The c_l-leg propagator is

$$D''(R_c,R_b,s_4)=\int_0^1 du_{13}u_{13}^{-\alpha_{123}-1+R_c}(1-u_{13})^{-\alpha(p_3)-1+R_b+a}(1-u_{12}u_{13})^{\alpha(p_3)-R_b}. \quad (39)$$

It is interesting to observe that if we directly twist[6] c_l and d_k legs in Eq. (36) without worrying about the asymmetrical fourth leg (d_k leg in Fig. 15), we get, from Eq. (36),

$$\left[0\left|W^{(4)}(a,b,c,d)\right|\begin{matrix}\lambda_a\\\lambda_b\\\lambda_c\\\lambda_d\end{matrix}\right]=\int_0^1 dw_2 w_2^{-\alpha_{12}-1}(1-w_2)^{-\alpha_{23}-1}$$

$$\times\left[0\right|\exp\left\{\sum_n \frac{a_n}{n^{1/2}}[p_4+p_3(1-w_2)^n]+\sum_m \frac{b_m}{m^{1/2}}[p_3+p_4(1-w_2)^m]+\sum_l \frac{\bar{c}_l}{l^{1/2}}[p_2+p_1(1-w_2)^l]\right.$$

$$\left.+\sum_k \frac{\bar{d}_k}{k^{1/2}}[p_1+p_2(1-w_2)^k]\right\}\exp\left\{\sum_n a_nb_n(1-w_2)^n+\sum_l \bar{c}_l\bar{d}_l(1-w_2)^l\right.$$

$$+\sum_{nl}\frac{a_n\bar{c}_l}{(nl)^{1/2}}B_{-nl}(1-w_2)(1-w_2)^n+\sum_{nk}\frac{a_n\bar{d}_k}{(nk)^{1/2}}B_{nk}(1-w_2)+\sum_{ml}\frac{b_m\bar{c}_l}{(ml)^{1/2}}B_{ml}(1-w_2)$$

$$\left.+\sum_{mk}\frac{b_m\bar{d}_k}{(mk)^{1/2}}B_{mk}(1-w_2)(1-w_2)^m\right\}\left|\begin{matrix}\lambda_a\\\lambda_b\\\lambda_c\\\lambda_d\end{matrix}\right] . \quad (40)$$

Comparing Eq. (40) with Eq. (38), we see that they are identical expressions, if p_1, p_2, p_3, p_4, and $1-w_2$ in Eq. (40) are replaced by p_2, p_3, p_4, p_1, and w_2, respectively, as required for different labeling of w_i's in Eq. (38). The reason why we can ignore the factor $[(1-u_{13})/(1-u_{12}u_{13})]^{R_c-\alpha(p_3)}$ in passing the twisting operator $\Omega_c^\dagger(-p_3)$ to the left-hand side of Eq. (36) is the same as in Caneschi and Schwimmer's[8] treatment on the three-Reggeon vertex case, since their derivation does not depend on the form of the z variable. In our case, $z=u_{13}(1-u_{12})/(1-u_{12}u_{13})$.

Finally, we apply the twisting operators $\Omega_b^\dagger(-p_3)$ and $\Omega_d^\dagger(-p_1)$ to the b_m and d_k legs in Eq. (38); then we end up

[8] L. Caneschi and A. Schwimmer, Nuovo Cimento Letters **3**, 213 (1970).

with the completely symmetrical expression of the four-Reggeon vertex function (Fig. 17)

$$\left\{0\left|W^{(4)}(a,b,c,d)\right|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\right\}=\int_0^1 dw_2 w_2^{-\alpha_{12}-1}(1-w_2)^{-\alpha_{23}-1}$$

$$\times\left\{0\left|\exp\left\{\sum_n\frac{a_n}{n^{1/2}}(p_1+p_4w_2^n)+\sum_m\frac{\bar{b}_m}{m^{1/2}}[p_2+p_1(1-w_2)^m]+\sum_l\frac{c_l}{l^{1/2}}(p_3+p_2w_2^l)+\sum_k\frac{\bar{d}_k}{k^{1/2}}[p_4+p_3(1-w_2)^k]\right\}\right.\right.$$

$$\times\exp\left\{\sum_{nm}\frac{a_n\bar{b}_m}{(nm)^{1/2}}\binom{m}{n}(-)^n n w_2^n+\sum_{ml}\frac{\bar{b}_m c_l}{(ml)^{1/2}}\binom{l}{m}(-)^m m(1-w_2^m)+\sum_{lk}\frac{c_l\bar{d}_k}{(lk)^{1/2}}\binom{k}{l}(-)^l l w_2^l\right.$$

$$\left.\left.+\sum_{nk}\frac{\bar{d}_k a_n}{(kn)^{1/2}}\binom{n}{k}(-)^k k(1-w_2)^k\right\}\exp\left\{\sum_{nl}\frac{a_n c_l}{(nl)^{1/1}}B_{-nl}(w_2)w_2^n+\sum_{mk}\frac{\bar{b}_m\bar{d}_k}{(mk)^{1/2}}B_{-km}(1-w_2)(1-w_2)^k\right\}\left|\begin{matrix}\lambda_a\\ \lambda_b\\ \lambda_c\\ \lambda_d\end{matrix}\right\}\right. . \qquad (41)$$

To conclude, we see that the four-Reggeon vertex function has a rather simple form. However, the fourth-leg propagator has an asymmetrical expression relative to the other three legs. Similarly to the three-Reggeon vertex case, this assymetrical fourth leg again indicates the problem of linear dependence.[3] Unless we can find an N-point dual amplitude which automatically takes care of the problem of linear dependence, we will not be able to get a completely factorizable four-Reggeon vertex function. On the other hand, the vertex composed of at least one external scalar particle together with an arbitrary number of factorized Reggeons is an exact expression. This might give another view of the problem of linear dependence.

The generalization to N-Reggeon amplitude has been carried out, but is still to be published. After completion of this work, we understand that Carbone and Sciuto have also obtained the four-Reggeon amplitude independently.

Added Note. With an exactly similar technique, we obtain the completely symmetrical N-Reggeon amplitude. It takes the following form:

$$W^{(N)}(a^{(i)})=\int\prod_{i\neq(N,N-1,1)}dw_i\{W_N\}\exp\left\{\sum_{i=1}^{N}\left[\sum_n\frac{a_n^{(i)}}{n^{1/2}}\left(\sum_{j=1}^{N-2}p_{N+i-j}P^n(i,\,i+1,\,i-1,\,N+i-j)\right)\right]\right.$$

$$+\sum_{i=1}^{N}\left[\frac{a_n^{(i+1)}a_m^{(i)}}{(nm)^{1/2}}\binom{n}{m}(-1)^m mP^m(i,\,i+1,\,i-1,\,i+2)\right]$$

$$+\sum_{i=1}^{N}\left[\frac{a_n^{(i+1)}a_m^{(i-1)}}{(nm)^{1/2}}P^n(i-1,\,i,\,i+2,\,i+1)B_{nm}[P(i-1,\,i,\,i+1,\,i+2)]P^m(i-1,\,i,\,i-2,\,i+1)\right]$$

$$+\sum_{i=1}^{N}\left[\frac{a_n^{(i+1)}a_m^{(i-2)}}{(nm)^{1/2}}P^n(i-2,\,i,\,i+2,\,i+1)B_{nm}[P(i-2,\,i-1,\,i+1,\,i+2)]P^m(i-2,\,i-1,\,i-3,\,i+1)\right]$$

$$+\cdots$$

$$+\sum_{i=1}^{N}\left[\frac{a_n^{(i+1)}a_m^{(i-k)}}{(nm)^{1/2}}P^n(i-k,\,i,\,i+2,\,i+1)B_{nm}[P(i-k,\,i-k+1,\,i+1,\,i+2)]P^m(i-k,\,i-k+1,\,i-k-1,\,i+1)\right]$$

$$+\cdots$$

$$+\sum_{i=1}^{N}\left[\frac{a_n^{(i+1)}a_n^{(i-\frac{1}{2}N+1)}}{2(nm)^{1/2}}P^n(i-\tfrac{1}{2}N+1,\,i,\,i+2,\,i+1)\right.$$

$$\left.\left.\times B_{nm}[P(i-\tfrac{1}{2}N+1,\,i-\tfrac{1}{2}N+2,\,i+1,\,i+2)]P^m(i-\tfrac{1}{2}N+1,\,i-\tfrac{1}{2}N+2,\,i-\tfrac{1}{2}N,\,i+1)\right]\right\},$$

where $w_N=\infty$, $w_{N-1}=1$, and $w_1=0$,

$$P(x,y,z,w)\equiv\frac{(x-z)(y-w)}{(x-w)(y-z)},$$

and N is assumed to be even. The superscript index of the operator a_n coincides with the Koba-Nielsen labeling.

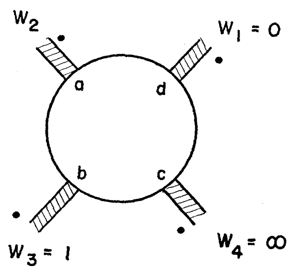

FIG. 17. Symmetrical four-Reggeon amplitude.

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APPENDIX

When the four-momentum goes to zero, say $p_M \to 0$, the bootstrap condition $a+bp_M^2=0$ is no longer satisfied, and the factor

$$\prod_{i=2\,(i\neq N-1)}^{N} (x_i - x_{i+2})^{a+bp_{i+1}^2}$$

in Eq. (4) gives an extra contribution[9] of

$$(1-y_{M-1}t)^{a+bp_M^2}$$

to Eq. (8). This extra factor will make the factorization of y_{M-1} and t unfeasible. To ensure that this factor is absent from Eq. (8), we have to modify the spectrum of the trajectories in Eq. (8). We therefore go back to Eq. (4) and concentrate on the troublesome factor $x_{M-1}-x_{M+1}$, which is raised to the power $\alpha_{M-1,M}-\alpha_{M-1,M+1}+\alpha_{M,M+1}$. If we assume that the trajectories $\alpha_{M-1,M}$, $\alpha_{M-1,M+1}$, and $\alpha_{M,M+1}$ have the intercepts $a_{M-1,M}$, $a_{M-1,M+1}$, and $a_{M,M+1}$, respectively, then

$$\alpha_{M-1,M}-\alpha_{M-1,M+1}+\alpha_{M,M+1}=a_{M-1,M}+a_{M,M+1}-a_{M-1,M+1}+bp_M^2-2bp_{M-1}\cdot p_{M+1}. \quad \text{(A1)}$$

It is the extra power

$$a_{M-1,M}+a_{M,M+1}-a_{M-1,M+1}+bp_M^2$$

which complicates the factorization. Therefore, we can demand

$$a_{M-1,M}+a_{M,M+1}-a_{M-1,M+1}+bp_M^2=0 \quad \text{(A2)}$$

for all values of p_M^2. In particular, we are interested in the case $p_M^2=0$. Thus, we obtain the condition

$$a_{M-1,M}+a_{M,M+1}-a_{M-1,M+1}=0. \quad \text{(A3)}$$

Further conditions in Eq. (A3) can be obtained by requiring the propagators associated with the factorized channels x_M/x_{M-1} and x_{M+1}/x_M to have the ordinary form $p_M^2 \to 0$. This leads to the requirement

$$a_{M-1,M}=a_{M,M+1}=a_{M-1,M+1}=0. \quad \text{(A4)}$$

Once we assign the intercept 0 to the trajectory $\alpha_{M,M+1}$, the next factor whose power contains the trajectory $\alpha_{M,M+1}$,

$$(x_M-x_{M+2})^{\alpha_{M,M+1}-\alpha_{M,M+2}+\alpha_{M+1,M+2}}, \quad \text{(A5)}$$

will have an extra power $-a_{M,M+2}$ in Eq. (4). We can eliminate this extra power by assigning an intercept 0 to the trajectory $\alpha_{M,M+2}$ while keeping the value a for the intercept of the trajectory $\alpha_{M+1,M+2}$ in Eq. (A5). Then the next factor which contains the trajectory $\alpha_{M,M+2}$ is

$$(x_M-x_{M+3})^{\alpha_{M,M+2}-\alpha_{M,M+3}-\alpha_{M+1,M+2}+\alpha_{M+1,M+3}}. \quad \text{(A6)}$$

We therefore again require $\alpha_{M,M+3}$ to have zero intercept. Continuing such reasoning, we conclude that the trajectories (assuming $x_N=0$, $x_1=\infty$)

$$\alpha_{i,j},\ i=2,3,\ldots,M\ (i\neq j),\ j=M,M+1,\ldots,N-1 \quad \text{(A7)}$$

must have the intercept zero, while all other trajectories have the intercept a. The requirement of Eq. (A7) does not affect the final result. Then our procedure of letting $p_M \to 0$ is justified.

[9] This is pointed out by Professor S. Mandelstam.