Radiative corrections to β and μ decays in $SU(2)_L \times U(1)$ models involving $2n$ flavors

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The analysis of the radiative corrections to β and μ decays is generalized to a class of $SU(2)_L \times U(1)$ models with n left-handed doublets on the basis of the currrent-algebra formulation. With some rather mild qualifications, the corrections turn out to be the same as in the four-flavor case. They are important in studying the tenability of the generalized as well as the standard models. We also discuss the question whether the radiative corrections may distinguish between $SU(2)_L \times U(1)$ and $SU(2)_L \times SU(2)_R \times U(1)$ models with the same low-energy phenomenology.

The radiative corrections to μ and β decays and the related problem of Cabibbo universality have been discussed in detail on the basis of the standard $SU(2)_L \times U(1)$ model involving four flavors.[1-4] In particular, this problem has been recently analyzed on the basis of a current-algebra formulation which allows us to take into account to a considerable degree the effects of the strong interactions.[4] The results agree quite well with the hypothesis of universality of the weak interactions in the sense of Cabibbo. On the other hand, the strong indications of the existence of heavy leptons and new flavors suggest the need for suitably enlarging the model.

In this short paper we consider the problem of the radiative corrections to β and μ decays on the basis of a simple extension of the standard $SU(2)_L \times U(1)$ model in which the quark and lepton sectors involve n left-handed doublets, and all right-handed fields are singlets. Specifically we assume that in the basis of mass-matrix eigenstates, the quark fields transform according to

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L, \ldots$$

where $u, c, t, \ldots$ have charge $\hat{q}+1$ and d', s', b' are suitable linear combinations of the eigenstates $d, s, b, \ldots$ of the mass matrix with charge $\hat{q}$. Analogously the left-handed leptons transform according to

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau' \end{pmatrix}_L, \ldots$$

where we have assumed muon- and electron-number conservation. All right-handed fields transform as singlets. For brevity this theory will be referred to as the generalized model.

The main purpose of this work is to show that the analysis of Ref. 4 can be readily extended to this more general model. Specifically, we will show that with the possible exception of the nonasymptotic photonic corrections induced by the axial-vector current and the small corrections in the Wilson coefficients induced by the strong interactions in the asymptotic domain, the corrections of order $G_F\alpha$ to the Fermi transitions and μ decay are the same as in the standard four-flavor model, provided that the masses of all new quarks and leptons are much smaller than the generic mass M_W of the intermediate bosons and that otherwise the underlying theory satisfies the assumptions of Ref. 4. To understand this result it is most convenient and economical to make use of the current-algebra formalism. We first note that the Lagrangian density describing the interactions of the quarks with the intermediate bosons can be written as

$$\mathcal{L}_{\text{int}} = -eA_\mu J_\gamma^\mu - \frac{g}{\sqrt{2}}(W_\mu^\dagger J_W^\mu + \text{H.c.}) - (g^2 + g'^2)^{1/2} Z_\mu J_Z^\mu, \tag{1}$$

where $W_\mu^\dagger$ stands for the field which creates a W^+ vector meson. We introduce a column vector $\psi_{\alpha,i}$ where the quark flavors of charge $\hat{q}+1$ are labeled by $\alpha = 1, \ldots, n$ those with charge $\hat{q}$ are labeled by $\alpha = n+1, \ldots, 2n$ and i represents the color index. Explicitly, $\psi_i^T = (u, c, t, \ldots, d, s, b, \ldots)_i$. In terms of ψ the hadronic currents can be expressed as

$$J_\gamma^\mu = \bar{\psi}\gamma^\mu Q\psi, \tag{2a}$$

$$J_W^\mu = \bar{\psi}\gamma^\mu a_- C_- \psi, \tag{2b}$$

$$J_Z^\mu = \tfrac{1}{2}\bar{\psi}C_3\gamma^\mu a_- \psi - \sin^2\theta_W \bar{\psi}\gamma^\mu Q\psi, \tag{2c}$$

where

$$Q=\begin{pmatrix}(\hat{q}+1)\underline{1} & \\ & \hat{q}\underline{1}\end{pmatrix}, \quad (3a)$$

$$C_3=\begin{pmatrix}\underline{1} & \\ & -\underline{1}\end{pmatrix}, \quad (3b)$$

$$C_-=\begin{pmatrix}0 & 0\\ A^\dagger & 0\end{pmatrix}, \quad (3c)$$

$a_-=(1-\gamma_5)/2$, $\underline{1}$ is the $n\times n$ unit matrix and A is an $n\times n$ unitary matrix which arises from the biunitary transformations necessary for the diagonalization of the quark mass matrix. In Eqs. (2a)–(2c), the matrices Q, C_-, and C_3 act on the flavor indices while a summation over the color index i is understood. Equations (1)–(3) are an obvious generalization of the corresponding expressions in the standard four-flavor model. The hadronic currents satisfy the following equal-time algebra:

$$[J_W^0(x), J_Z^\mu(x')]_{x^0=x'^0}=\cos^2\theta_W J_W^\mu(x)\delta(\vec{x}-\vec{x}'), \quad (4a)$$

$$[J_W^0(x), J_\gamma^\mu(x')]_{x^0=x'^0}=J_W^\mu(x)\delta(\vec{x}-\vec{x}'), \quad (4b)$$

$$[J_W^0(x), J_W^{\dagger\mu}(x')]_{x^0=x'^0}=-J_3^\mu(x)\delta(\vec{x}-\vec{x}')+\text{S.t.}, \quad (4c)$$

$$J_3^\mu(x)\equiv\bar{\psi}\gamma^\mu a_- C_3\psi=2(\sin^2\theta_W J_\gamma^\mu+J_Z^\mu), \quad (4d)$$

where S.t. stands for a c-number Schwinger term. We observe that this algebra is formally identical to that of the standard four-flavor model, although the currents themselves are different as they involve additional fields.

To obtain our result it is then sufficient to make the following observations:

(i) Using the Ward identities associated with the above time-time and time-space algebra and invoking the assumed properties of the underlying theory such as the asymptotic freedom of the strong interactions and the partial conservation of the weak hadronic currents, it was shown in Ref. 4 that at zero momentum transfer the radiative corrections to the Fermi transitions are given by model-independent terms proportional to the zeroth-order amplitude plus two-current correlation functions (that is, amplitudes proportional to Fourier transforms of time-ordered products of two currents) involving either J_W^μ and J_Z^λ or J_W^μ and J_γ^λ.

(ii) With the exception of the nonasymptotic part of the photonic contributions induced by the axial-vector current and with the neglect of small corrections induced by the strong interactions in the asymptotic domain, it was shown in Ref. 4 that to order $G_F\alpha$ the two-current correlation functions are effectively given by the leading term in the short-distance expansion of the product of current operators with Wilson coefficients calculated as in the free field theory. Alternatively, they can be obtained by calculating the leading term in the Bjorken-Johnson-Low limit with canonical evaluation of commutators. It is easy to see that such contributions are proportional to the zeroth-order amplitude with identical coefficients in the standard and generalized models. In fact, if one uses the first line of arguments it is sufficient to point out that in both theories the leading operator in the short-distance expansion is a current involving the same quarks, namely, the u and d quarks (other pieces do not contribute to the $\Delta S=0$ β decay because they would violate the flavor-conservation law of the strong interactions) and in the limit of the free-field theory the Wilson coefficients in the expansion are not affected by the presence of additional quark flavors because both J_Z^λ and J_γ^λ are diagonal operators. If one uses the second line of arguments, the result follows from the observation that the leading term in the Bjorken-Johnson-Low limit is controlled by the equal-time commutators of the current components (both time-time and space-space) and these are formally identical in both theories.

There are, as shown in Ref. 4, corrections of order $\bar{g}_s^2(\kappa^2)$ to the Wilson coefficients [$\bar{g}_s(\kappa^2)$ is the effective coupling constant of the strong interactions] and these corrections do depend on the number of flavors. There is also a nonasymptotic phononic contribution induced by the axial-vector current,[4,5] which is not governed by the short-distance behavior. In principle, this contribution may depend on the dynamical details of the strong interactions and may be different in the two models. The estimates of Refs. 4 and 5 for the latter and those of Ref. 4 for the $\bar{g}_s^2(\kappa^2)$ terms suggest, however, that these two classes of contributions are quite small in comparison with the others, especially if we assume that the number of flavors is not too close to the critical number of 17.[6] To the extent that they may be regarded as negligible, we conclude that the corrections of order $G_F\alpha$ to the Fermi transitions are the same in both models.

A completely analogous discussion can be carried out verbatim for μ decay with the simplification that strong-interaction effects need not be considered.[7]

We emphasize the fact that our conclusions can be traced to the formal equivalence of the equal-time algebras of current operators in the standard and generalized models.

As is well known, after absorbing all possible phases by suitable definition of the quark fields, the matrix A of Eq. (3c) is characterized by $n(n-1)/2$ rotation angles and $(n-1)(n-2)/2$ phase parameters.[8] For practical applications it is convenient to parametrize $A_{11}=\cos\theta$ so that θ can be determined by comparing μ decay and the Fermi transitions. The magnitude of A_{12} can be obtained from the usual $\Delta S=1$ semileptonic transitions but, because A is an $n\times n$ unitary matrix, we no longer have the equality $|A_{11}|^2+|A_{12}|^2=1$ but rather an inequality $|A_{11}|^2+|A_{12}|^2\leq 1$.

As a practical application, we consider the simplest version of the $SU(2)_L\times U(1)$ theory in which $M_W=M_Z\cos\theta_W$. Using as input the analysis of the β-decay data due to Wilkinson and Alburger[9] and setting $M_Z=87.3$ GeV (which corresponds to $\sin^2\theta_W=0.24$), and $\hat{q}=-\frac{1}{3}$, we obtain $A_{11}=\cos\theta=0.9733$. Inserting a value $A_{12}=0.226\pm 0.006$ compatible with both the K_{l3} and hyperon decay data, we find $|A_{11}|^2+|A_{12}|^2=0.998\pm 0.004$ (Ref. 10) which is nicely consistent with the standard model. As was pointed out by Ellis *et al.* and by Harari,[8] this analysis can be used to put a rough upper limit on other mixing angles in the generalized models. For instance in the six-flavor case, using the parametrization of Refs. 8 and 11, $A_{11}=\cos\theta_1=\cos\theta$, $A_{12}=\sin\theta\cos\theta_3$, $A_{13}=\sin\theta\sin\theta_3$, we see that $A_{13}^2=1-A_{11}^2-A_{22}^2\leq 0.006$ or $\sin\theta_3\leq 0.34$. The upper limit is not very restrictive because $\sin^2\theta_3$ is multiplied by a very small number and furthermore the answer is sensitive to small fluctuations and errors in the value of $1-A_{11}^2-A_{12}^2$.

We conclude with the following observations:

(i) It is important to note that the radiative corrections play a crucial role in determining the tenability of the generalized as well as the standard models. In fact, without applying these corrections, the values obtained from μ, β, and $\Delta S=1$ decays satisfy $|A_{11}|^2+|A_{12}|^2>1$ with a margin of about 4%, a conclusion which would rule out both the standard and generalized models.

(2) It was pointed out in Ref. 4 that the nonphotonic radiative corrections of order $G_F\alpha$ to arbitrary semileptonic decays are controlled by Ward identities and the leading terms in the short-distance expansions. But we have seen in this paper that, with the neglect of the small asymptotic corrections of order $\bar{g}_s^2(\kappa^2)$ induced by the strong interactions, such contributions are the same in the standard and generalized models. Thus, we conclude that radiative corrections of order $G_F\alpha$ to observables not affected by photonic exchanges do not distinguish between these two classes of models.

(3) It is interesting to entertain the possibility that the $SU(2)_L\times U(1)$ models may be an effective approximation at not too high frequencies of an $SU(2)_L\times SU(2)_R\times U(1)$ theory,[12] as in this manner one could reconcile the phenomenological successes of the former with the theoretically appealing idea that parity violation may arise from spontaneous breakdown. As the $SU(2)_L\times U(1)$ and $SU(2)_L\times SU(2)_R\times U(1)$ models become in principle different at real or virtual momenta $\kappa^2\approx M_{W_R}{}^2$, it is interesting to inquire whether these two classes of models may be distinguished by the radiative corrections. If the masses of M_{W_R} and one of the neutral vector mesons Z_2 are sufficiently large, say $M_{W_R}\approx M_{Z_2}\approx 10M_{W_L}$, the answer in the case of the correction to $\cos\theta$ is negative. The reason is that W_R and Z_2 effectively decouple as their contributions are of order $G_F\alpha M_{W_L}{}^2/M_{W_R}{}^2$ or $G_F\alpha M_{W_L}{}^2/M_{Z_2}{}^2$ modulo logarithms or, otherwise, they are universal (i.e., the same in β and μ decays).

Note added in proof: After submitting this paper for publication, we have received an interesting report by R. Schrock and L. L. Wang [Phys. Rev. Lett. 41, 1692 (1978)] in which, among other subjects, a new fit to the Cabibbo theory and a detailed calculation of the mixing angle θ_3 are presented.

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[6]As the number of flavors approaches 17, $\bar{g}_s^2(\kappa^2)$ becomes a more slowly decreasing function of κ^2. Thus for a given value of $g_{sR}{}^2=\bar{g}_s^2(\mu^2)$ (g_{sR} is the renormalized coupling constant of the strong interactions, and μ is the momentum at which it is defined), the corrections of order $\bar{g}_s^2(\kappa^2)$ become more important as the number flavors increases. As an extreme example, for $2n=16$ and color SU(3),

$$\frac{\bar{g}_s{}^2(\kappa^2)}{4\pi^2}\approx\frac{g_{sR}{}^2}{4\pi^2}\left[1+\frac{g_{sR}{}^2}{48\pi^2}\ln\left(\frac{\kappa^2}{\mu^2}\right)\right]^{-1}$$

in the domain in which the right-hand side is small. If $\mu=\sim 1$ GeV and $g_{sR}{}^2/4\pi^2\sim 1$, this expression is ~ 1 even in the region $\kappa\approx M_W$ where important contributions to many of the Feynman integrals arise. Thus, in this extreme case, in order to justify the neglect of $\bar{g}_s{}^2(\kappa^2)/4\pi^2$ in the evaluation of the Feynman integrals, we would have to assume that $g_{sR}{}^2/4\pi^2$ itself is quite small. If the number of flavors ≥17, the asymptotic freedom of the strong interactions is lost and, in order to justify the neglect of effects of order $\bar{g}_s^2(\kappa^2)/4\pi^2$, we would have to appeal to other assumptions about the strong interactions as described, for example, in Sec. 8 of Ref. 4.

[7]For a discussion of the radiative corrections to μ decay in the current-algebra approach, see Sec. 7 of Ref. 4.

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[12]It is interesting to note that the situation we envisage can be obtained as a limiting case of the $SU(4)_L \times SU(4)_R \times U(1)$ model of Ref. 13. In fact, taking $\beta=0$ and $1+\cos\xi\tan\gamma\approx 0$, $M_{Z_2}{}^2$ becomes much larger than $M_{Z_1}{}^2$ and J_Z^λ becomes close to the Weinberg-Salam current provided we identify $\sin^2\xi=2\sin^2\theta_W$.

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