PHYSICAL REVIEW D VOLUME 19, NUMBER 2 15 JANUARY 1979

Comments and Addenda

The section Comments and Addenda is for short communications which are not appropriate for regular articles. It includes only the following types of communications: (1) Comments on papers previously published in **The Physical Review** *or* **Physical Review Letters**. *(2) Addenda to papers previously published in* **The Physical Review** *or* **Physical Review Letters**, *in which the additional information can be presented without the need for writing a complete article. Manuscripts intended for this section must be accompanied by a brief abstract for information-retrieval purposes. Accepted manuscripts follow the same publication schedule as articles in this journal, and page proofs are sent to authors.*

Static axially symmetric solutions of self-dual SU(2) gauge fields in Euclidean four-dimensional space

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(Received 16 October 1978)

All solutions of the stationary axially symmetric Einstein equations are shown to correspond to static axially symmetric solutions of self-dual SU(2) gauge fields. Some simple examples are given.

Yang[1] has reduced the problem of finding self-dual SU(2) gauge fields on Euclidean four-dimensional flat space to solving a set of three Laplace-type equations for one real and one complex variable. In the R gauge, Yang's field equations for the variables ϕ, ρ, and $\bar\rho$ are

$$\begin{aligned}&\phi(\phi_{y\bar y}+\phi_{z\bar z})-\phi_y\phi_{\bar y}-\phi_z\phi_{\bar z}+\rho_y\bar\rho_{\bar y}+\rho_z\bar\rho_{\bar z}=0\,,\\&\phi(\rho_{y\bar y}+\rho_{z\bar z})-2\rho_y\phi_{\bar y}-2\rho_z\phi_{\bar z}=0\,,\\&\phi(\bar\rho_{y\bar y}+\bar\rho_{z\bar z})-2\bar\rho_{\bar y}\phi_y-2\bar\rho_{\bar z}\phi_z=0\,.\end{aligned}\tag{1}$$

The subscript denotes partial differentiation and

$$\begin{aligned}&\sqrt2\,y\equiv x_1+ix_2\,,\quad \sqrt2\,\bar y=x_1-ix_2\,,\\&\sqrt2\,z\equiv x_3-ix_4,\quad \sqrt2\,\bar z\equiv x_3+ix_4\end{aligned}\tag{2}$$

for the complexified Cartesian coordinates x_μ ($\mu=1,2,3,4$). For real values of x_μ (which is all we henceforth consider) $\bar\rho=\rho^*$ and ϕ is real. The coordinates of the self-dual potentials b_μ^i are given by

$$\begin{aligned}&\phi\vec b_y=(i\rho_y,\rho_y,-i\phi_{\bar y}),\quad \phi\vec b_{\bar y}=(-i\bar\rho_{\bar y},\bar\rho_{\bar y},i\phi_{\bar y})\,,\\&\phi\vec b_z=(i\rho_z,\rho_z,-i\phi_{\bar z}),\quad \phi\vec b_{\bar z}=(-i\bar\rho_{\bar z},\bar\rho_{\bar z},i\phi_{\bar z})\,.\end{aligned}\tag{3}$$

Look for solutions of Eqs. (1) of the form $\rho=\sigma e^{i\alpha}$ where σ is a real function and α is a real constant; transform to the space coordinates x_μ and consider static solutions $(\partial/\partial x_4)=0$. Equations (1) become

$$\begin{aligned}&\phi(\nabla^2\phi)=\vec\nabla\phi\cdot\vec\nabla\phi-\vec\nabla\sigma\cdot\vec\nabla\sigma\,,\\&\phi(\nabla^2\sigma)=2\vec\nabla\sigma\cdot\vec\nabla\sigma\,,\\&\sigma_{x_1}\phi_{x_2}-\phi_{x_1}\sigma_{x_2}=0\,,\\&\vec\nabla\equiv(\partial/\partial_{x_1},\partial/\partial_{x_2},\partial/\partial_{x_3})\,.\end{aligned}\tag{4}$$

Assuming axial symmetry about x_3, the third equation of the set (4) vanishes. With $\epsilon=\phi+i\sigma$, the first two equations become

$$\mathrm{Re}\,\epsilon(\nabla^2\epsilon)=\vec\nabla\epsilon\cdot\vec\nabla\epsilon\,.\tag{5}$$

This is the equation deduced by Ernst[2] for the axially symmetric gravitational field problem; ϵ is often called the Ernst potential. ϕ is the norm of the timelike Killing vector of the stationary space-time and σ is the twist potential.

Define

$$\epsilon\equiv\frac{(E-1)}{(E+1)}\,.\tag{6}$$

Then the equation for E is

$$(EE^*-1)\nabla^2E=2E^*\vec\nabla E\cdot\vec\nabla E\,,\tag{7}$$

and Eq. (7) is the Ernst equation. We have shown that any stationary axisymmetric gravitational field yields through ϕ and ρ $(=\sigma e^{i\alpha})$ a self-dual gauge field. A simple class of solutions is

$$\begin{aligned}&E=e^{-i\beta}\coth\psi\,,\\&\nabla^2\psi=0\,,\end{aligned}\tag{8}$$

where β is a real constant and ψ any function that satisfies the Laplace equation. In gravitation these solutions are of interest only if $\beta=0$ (modπ); otherwise space-time is not asymptotically flat.

Another class of solutions of interest is the Tomimatsu-Sato[3] series of solutions; we state only the first two:

$$E=p\xi-iq\eta\,,\tag{9}$$

$$E = \frac{p^2\xi^4 + q^2\eta^4 - 1 - 2ipq\xi\eta(\xi^2 - \eta^2)}{2p\xi(\xi^2-1) - 2iq\eta(1-\eta^2)} . \tag{10}$$

p, q are parameters such that $p^2 + q^2 = 1$. ξ, η are prolate spheroidal coordinates

$$\begin{aligned} &x_3 = c\xi\eta, \quad x_1 = c(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\sin\theta , \\ &x_2 = c(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\cos\theta , \\ &\xi \geq 1, \quad -1 \leq \eta \leq 1, \quad 0 \leq \theta \leq 2\pi . \end{aligned} \tag{11}$$

Equation (9) is the Kerr solution of general relativity with the special case of the Schwarzschild solution when $p=1$, $q=0$. As specific examples we calculate the gauge potentials for two special cases of the Kerr solution (9). One is $p=1$, $q=0$ (Schwarzschild), the other is $p=0$, $q=1$. First we write the gauge potentials in the x_μ coordinate system, taking for simplicity $\alpha = 0$ (it can be shown that $\alpha \neq 0$ differs from $\alpha = 0$ by a gauge transformation):

$$\begin{aligned} &\phi\vec{b}_{x_1} = (\sigma_{x_2}, \sigma_{x_1}, -\phi_{x_2}) , \\ &\phi\vec{b}_{x_2} = (-\sigma_{x_1}, \sigma_{x_2}, \phi_{x_1}) , \\ &\phi\vec{b}_{x_3} = (0, \sigma_{x_3}, 0) , \\ &\phi\vec{b}_{x_4} = (\sigma_{x_3}, 0, -\phi_{x_3}) . \end{aligned} \tag{12}$$

Example I: $p=1$, $q=0$ [in Eq. (9)].

$$\begin{aligned} &\vec{b}_{x_1} = \left(0, 0, \frac{-x_2}{c(\xi^2-1)}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right) , \\ &\vec{b}_{x_2} = \left(0, 0, \frac{x_1}{c(\xi^2-1)}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right) , \\ &\vec{b}_{x_3} = (0, 0, 0) , \\ &\vec{b}_{x_4} = \left(0, 0, -\frac{1}{c(\xi^2-1)}\left(\frac{x_3+c}{r_2} + \frac{x_3-c}{r_1}\right)\right) . \end{aligned} \tag{13}$$

Example II: $p=0$, $q=1$ [in Eq. (9)].

$$\begin{aligned} &\vec{b}_{x_1} = \frac{1}{c(\eta^2+1)}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\left(-x_2, -x_1, +\frac{2\eta}{(\eta^2-1)}x_2\right) , \\ &\vec{b}_{x_2} = \frac{1}{c(\eta^2+1)}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\left(x_1, -x_2, -\frac{2\eta}{\eta^2-1}x_1\right) , \\ &\vec{b}_{x_3} = \frac{1}{c(\eta^2+1)}\left(0, \frac{x_3+c}{r_2} - \frac{x_3-c}{r_1}, 0\right) , \\ &\vec{b}_{x_4} = \frac{1}{c(\eta^2+1)}\left(\frac{x_3+c}{r_2} - \frac{x_3-c}{r_1}, 0, \frac{2\eta}{\eta^2-1}\left(\frac{x_3-c}{r_1} - \frac{x_3+c}{r_2}\right)\right) , \\ &r_1^2 \equiv (x_3-c)^2 + x_1^2 + x_2^2, \quad r_2^2 \equiv (x_3+c)^2 + x_1^2 + x_2^2 , \\ &\eta = (r_2 - r_1)/2c , \quad \xi = (r_2 + r_1)/2c . \end{aligned} \tag{14}$$

In example I, the potentials are singular at $r_1 + r_2 = 2c$, which corresponds to the Schwarzschild horizon. Example II is singular at $\eta = \pm 1$ which corresponds to $|x_3| \geq c$. The nature of these singularities is to be determined.

A whole industry has grown up in general relativity theory which generates solutions of the stationary axisymmetric field equations from other solutions. (I cite only some papers.[4]) Basically this industry arises from elementary considerations.

The first is that if ϵ is a solution of Eq. (5), so is ϵ' where

$$\epsilon' = \frac{a\epsilon + ib}{1 + ic\epsilon} . \tag{15}$$

a, b, c, are real constants and Eq. (15) represents a three-parameter group of transformations, G, which transform solutions into solutions. G does not commute with coordinate transformations C, so that by alternating operations C with G one can usually get an endless chain of solutions depending eventually on an infinite number of parameters. Perhaps the best catalog of these transformations has been made by Kinnersley.[4] The most interesting solutions in general relativity are those that are asymptotically flat; these do not necessarily map into the most interesting self-dual SU(2) gauge fields.

Ward[5] has shown how to construct self-dual gauge fields. Atiyah and Ward[6] and Corrigan, Fairlie, Yates, and Goddard[6] have applied the Ward construction to finding finite-action solutions in the R gauge. It does not seem difficult to apply Ward's construction to find the static axisymmetric SU(2) self-dual fields in the R gauge and to select the fields for which $\rho = \bar{\rho}$ (up to a constant phase factor). Thus Ward's construction and the isomorphism described in this paper should give a constructive method of finding the stationary axisymmetric solutions of Einstein's equations.

I am grateful to F. P. Esposito and to J. J. G. Scanio for discussions relating to these issues. The work was supported in part by the National Science Foundation under Grant No. Int. 75-18457.

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