PHYSICAL REVIEW D VOLUME 19, NUMBER 2 15 JANUARY 1979

Action at a distance and relativistic wave equations for spinless quarks

Philippe Droz-Vincent
Collège de France, Département de Mécanique Analytique, Tour 65-66, 4 Place Jussieu, 75230 Paris, Cedex 05, France
(Received 3 April 1978)

Within a manifestly covariant formulation of classical dynamics, we characterize the relativistic two-body wave equations admitting a classical analog in which the interaction is realized by means of action at a distance. As an application (and for example) the mass spectrum of Gunion and Li can be obtained from such a classical system, by quantization.

I. INTRODUCTION

In the most naive pictures, mesons are considered as bound states of two spinless quarks, and described by a coupled pair of equations. One is actually a wave equation in terms of the relative operator $\Box_r$ and involves additional terms which carry the interaction (Refs. 1–6). The other one is usually considered as a subsidiary condition. Of course these two equations have to be mutually consistent (compatibility). Although intuition is often claimed to be present in the construction of such models, little care, if any, is generally devoted to the possible existence of a classical analog.

Our purpose is to show the following:

(i) The condition which permits consideration of the model as the quantized version of a classical two-particle system interacting through action at a distance is much more restrictive than the compatibility condition.

(ii) When this stronger condition is fulfilled, the "subsidiary" equation as well as the wave equation are both deducible from the classical model by the quantization rules.

We may first notice that, unlike the Bethe-Salpeter equation, the various two-particle equations more recently proposed are not *a priori* connected with field theory in a definite way. Then we recall that action-at-a-distance theory, although perhaps compatible with fields, does not require *explicitly* the mediation of a field. Moreover, the version of this theory that we have in mind,[7] mathematically simpler to handle than those based on the Fokker action principle, admits a symplectic Hamiltonian formulation providing a classical approach towards a relativistic potential theory. That is why we attempt an action-at-a-distance interpretation for the two-body wave equations.

As we shall see below, such an interpretation implies severe restrictions on the potentials. This could be a principle of classification, and indicates that the models violating these restrictions must receive their interpretation from a theory which is not reducible to action at a distance. Whether realistic and elaborate theories can actually be cast into action-at-a-distance form is beyond the scope of this paper, which is simply devoted to improve the comprehension of naive models.[8]

Our argument mainly rests on a typical feature of the multi-time formalism.[9-12] N interacting particles have, in the relativistic sense, *not one but* N Hamiltonians, each one providing by quantization the left-hand side of a wave equation.

In this formulation, the Hamiltonians are just the generating functions leading to the equations of motion. These equations admit a Poisson-bracket form which involves N independent parameters (the proper times or a suitable generalization of them). This point of view may look a little unusual; however, it is very close to the philosophy contained in an early approach by Dirac.[13]

Here we consider the case $N=2$. The masses are not taken as constant *a priori*, but rather considered as constants of the motion. Accordingly, our two-particle phase space has 16 dimensions. We consider the canonical coordinates $q^\alpha, p^\beta, q'^\alpha, p'^\beta$, where q and q' are points in Minkowski space, while p and p' are four-vectors. These canonical coordinates satisfy the standard Poisson-bracket relations. Beware that, owing to the so-called "no-interaction theorem,"[14] the q, q' cannot be confused with the positions x, x' when interaction is present.[15] When possible, Greek indices running from 0 to 3 are omitted. For instance, p stands for p^α, etc. Scalar products are written in compact form:

$$p^2 = p\cdot p = p^\alpha p_\alpha, \text{ etc.}$$

We take $c=\hbar=1$,

$$\partial_\alpha = \partial/\partial x^\alpha, \quad \partial_{\alpha'} = \partial/\partial x^{\alpha'}.$$

We separate *external* from *internal variables* by setting

$$r = x - x',$$
$$z = q - q',$$
$$P = p + p', \quad y = \tfrac{1}{2}(p - p').$$

Application of the projector

$$\Pi^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - P^{\alpha}P_{\beta}/P^2$$

to any object will be denoted by a tilde. For instance,

$$\tilde{z}^{\alpha} = \Pi^{\alpha}_{\beta} z^{\beta}.$$

The energy is the time component of P. But the Hamiltonians (which generate the motion in manifestly covariant form) are phase-space scalar functions H and H'.

Interacting models are obtained when the free Hamiltonians

$$H_0 = \tfrac{1}{2}p^2, \quad H_0' = \tfrac{1}{2}p'^2 \tag{1.1}$$

are completed with additional terms. They are constants of the motion and identified with $\frac{1}{2}$ of the squared masses. Thus, on the orbits of the system, we have numerically

$$H = \tfrac{1}{2}m^2, \quad H' = \tfrac{1}{2}m'^2. \tag{1.2}$$

But, as phase-space functions, the Hamiltonians can be written in a general way as

$$H = H_0 + V + W, \quad H' = H_0' + V - W. \tag{1.3}$$

In contrast with the nonrelativistic case, the "relativistic potentials" $V+W$, $V-W$ have the dimensions of a squared mass and *cannot be chosen arbitrarily*.

In order to ensure the existence of world lines[10,11] it is geometrically necessary that

$$\{H, H'\} = \text{const.} \tag{1.4}$$

Moreover, the physical interpretation of H and H', as well as, independently, the requirement of symmetry under particle exchange, demands that in fact

$$\{H, H'\} = 0. \tag{1.5}$$

In practice, Eq. (1.5) is a condition on V and W.

In order to make the forthcoming calculations easier we shall use the combinations $4(H+H') - P^2$ and $H - H'$. Assuming that V and W are Poincaré invariant, $\{H, P^2\}$ and $\{H', P^2\}$ vanish. Thus we can replace Eq. (1.5) by the equivalent condition

$$\{4(H+H') - P^2, H - H'\} = 0. \tag{1.6}$$

Using the obvious identities

$$4(H_0 + H_0') - P^2 = 4y^2, \tag{1.7}$$
$$H_0 - H_0' = y \cdot P, \tag{1.8}$$

we get

$$4(H+H') - P^2 = 4y^2 + 8V, \tag{1.9}$$
$$H - H' = y \cdot P + 2W. \tag{1.10}$$

Inserting (1.9) and (1.10) into (1.6), performing the calculation, we can finally write the condition

$$\{y^2, W\} + \{V, y \cdot P\} + 2\{V, W\} = 0, \tag{1.11}$$

which is equivalent to (1.5).

A large class of solutions can be obtained by setting $W = 0$. This particular situation will be referred to as the *single-potential case*. This case deserves special interest because of its simplicity: then the condition (1.11) is completely solved[12] by requiring that V depends on $\tilde{z}^2$, P^2, $\tilde{y}^2$, $\tilde{z}\cdot\tilde{y}$, $y \cdot P$ but *does not depend* on $z \cdot P$ which is just, in the rest frame of the system,[12] the relative coordinate time, up to a factor $|P|$. Let us now go back to the general case.

II. WAVE EQUATIONS

As in previous papers,[11,16] we apply the most straightforward quantization procedure. The wave function Ψ depends on the two space-time points x, x'.

The operators $-i\partial$, $-i\partial'$ are substituted for the momenta p, p'. But the multiplicative operators x and x' cannot represent the positions, except in the free case, since we have the usual correspondence

$$\{\,,\,\} \to [\,,\,],$$

while the Poisson brackets $\{x, p\}$, $\{x', p'\}$, $\{x, x'\}$, $\{x, p'\}$, $\{x', p\}$, $\{x, x\}$, $\{x', x'\}$ cannot have the standard values.[17]

In contrast, the brackets $\{q, p\}$, $\{q', p'\}$, $\{q, q'\}$, $\{q, p'\}$, $\{q', p\}$, $\{q, q\}$, $\{q', q'\}$ are standard, so the multiplicative operators x, x' represent the canonical variables q, q'. This defines the correspondence rule, up to well-known problems, such as factor ordering and locality, which are not seriously relevant for the kind of interactions we are going to consider. For example, a classical function $f(\tilde{z}^2, P^2)$ will become an operator according to the scheme

$$f_{\text{class}}(\tilde{z}^2, P^2) \to f(\tilde{r}^2, P^2),$$

where $P = -i(\partial + \partial')$ and r is understood as a multiplicative operator, which means that

$$\tilde{r}^2 = r^2 - (r \cdot P)^2/P^2 \tag{2.1}$$

is a nonlocal operator, due to the denominator.

As we are going to consider eigenfunctions of P, no trouble will arise.

From now on, we mean that $y^\alpha = -i\partial/\partial r^\alpha$; thus $y^2 = -\Box_r$. Each Hamiltonian becomes an operator, and by analogy with their classical interpretation [see Eq. (1.2)] we postulate the wave equations

$$H\Psi = \tfrac{1}{2}m^2\Psi\,, \quad H'\Psi = \tfrac{1}{2}m'^2\Psi\,. \tag{2.2}$$

This principle obviously permits us to recover two Klein-Gordon equations in the trivial case $V = W = 0$.

In the general case, we now have a pair of coupled wave equations. Whereas the classical Hamiltonians satisfy Eq. (1.5), the corresponding operators fulfill its quantum-mechanical counterpart

$$[H, H'] = 0 \tag{2.3}$$

or, equivalently, the operator form of Eq. (1.11)

$$[y^2, W] + [V, y\cdot P] + 2[V, W] = 0\,. \tag{2.4}$$

In order to make comparisons with various models found in the literature, we prefer to consider the combinations $[4(H+H') - P^2]\Psi$ and $(H-H')\Psi$. Using the operator form of (1.7), (1.8) yields finally

$$[\Box_r + \tfrac{1}{2}(m^2 + m'^2) - \tfrac{1}{4}P^2 - 2V]\Psi = 0\,, \tag{2.5}$$

$$[y\cdot P + 2W - \tfrac{1}{2}(m^2 - m'^2)]\Psi = 0\,, \tag{2.6}$$

which is equivalent to Eq. (2.2).

Equation (2.5) appears formally as a Klein-Gordon equation in terms of the relative variable r^α. By translation invariance, neither V nor W can depend on $(x + x')$, but they generally depend on P and y. Fortunately, further separation of the center-of-mass motion will permit us to replace P by a constant timelike vector K in (2.5) and (2.6).

Equation (2.6) plays the role of a subsidiary condition fixing the dependence of the wave function on the "relative time," since $y\cdot P\Psi$ reduces to $K^0\partial/\partial r^0\Psi$ in the rest frame. But we stress that, as opposed to a widespread habit, this equation has not been assumed for convenience, but derived.

When we consider a single-potential model, with equal masses, this equation completely removes the dependence of Ψ on the relative time. It is very important to notice that Eq. (2.3), which implies the integrability of (2.2), is stronger than the most general compatibility condition required for such a system. Whereas Eq. (1.5) has a geometrical meaning, Eq. (2.3) has an obvious physical significance.

To be consistent with the massive interpretation of the Hamiltonians, it is expected that the masses are simultaneously measurable. In order to describe relativistic bound states of spinless particles, many authors (Refs. 1–5) have directly worked out equations of the form of (2.5) and (2.6). In other words, they assume a pair of equations which can be identified with (2.5) and (2.6), provided that V and W are specified in terms of r^α, y^μ, P^ν. Let us emphasize that, unless V and W fulfill (2.4), the quantum-mechanical model defined by (2.5) and (2.6) is not deducible from a classical action-at-a-distance system by the straightforward quantization procedure defined above.

Then the following question arises: What if V and W still permit the integrability of the system (2.5) and (2.6), but in a way which violates the condition (2.4)? In this case it is still possible to consider H and H' defined by (1.3), but they satisfy the commutation law

$$[H, H'] = aH + bH'\,, \tag{2.7}$$

with a and b = const (not simultaneously vanishing). The corresponding classical Hamiltonians satisfy the Poisson-bracket analog of (2.7). Thus they define an integrable differential system, but the equations

$$\frac{\partial x}{\partial \tau'} = \{H, x'\} = 0\,, \quad \frac{\partial x'}{\partial \tau} = \{H', x\} = 0\,, \tag{2.8}$$

which enable us to interpret the solutions of this system in terms of world lines,[11] will not have solutions any more, as can be proved simply.[18] So one is left with solutions of the abstract form

$$q = q(\tau, \tau')\,, \quad q' = q'(\tau, \tau')\,,$$
$$p = p(\tau, \tau')\,, \quad p' = p'(\tau, \tau')\,. \tag{2.9}$$

They certainly define a couple of two-surfaces in phase space, but there is no longer any clue[19] for recovering one-dimensional world lines. Therefore we insist that, although coupled wave equations violating condition (2.4) are not necessarily empty, the condition (2.4) is required in order to find, as a classical analog, a system of two particles without spin interacting through action-at-a-distance forces.

III. DISCUSSION OF VARIOUS MODELS

We have previously discussed[11,12] the simplest solution of (2.4), namely

$$W = 0\,, \quad V = k\tilde{r}^2\,, \quad k = \text{const} > 0\,. \tag{3.1}$$

This potential, which does not depend on the relative time, is our version of the harmonic oscillator.

We have also briefly mentioned (classically) in Ref. 11 an alternative choice which corresponds to the quantum-mechanical operators

$$W=0\,,\quad V=a[r^2P^2-(r\cdot P)^2]\ \text{with}\ a=\text{const}>0\,. \tag{3.2}$$

This case is among a class of local potentials recently suggested by Leutwyler and Stern.[4] In contrast, the so-called "covariant harmonic oscillator" introduced by Feynman *et al.*[1] and further investigated by Kim and Noz[2] exhibits potentials depending on the relative time, and a very short calculation shows that it does not fulfill the condition (2.5); therefore, it cannot be derived from a classical two-particle system, *in the sense we consider* here (action-at-a-distance picture).

For a similar reason, in the bag model of Preparata and Craigie,[5] we are led to rule out the confinement in relative time; thus we take $R_t=\infty$, which retains confinement in space only.

Fortunately, an appealing model proposed by Gunion and Li[3] turns out to be completely compatible with action-at-a-distance theory. Actually their picture can be incorporated into our framework as follows: They start with a wave equation of the form

$$[(\partial/\partial r+\tfrac{1}{2}iP)^2+(m+U)^2]\Psi=0\,, \tag{3.3}$$

and they assume that

$$\Psi=\exp\left(iK\cdot\frac{x+x'}{2}\right)\psi(r)\,, \tag{3.4}$$

where K is a constant timelike vector. U depends only on

$$|\tilde{r}|=(-\tilde{r}^2)^{1/2}\,, \tag{3.5}$$

and the subsidiary equation is

$$K\cdot(\partial/\partial r)\psi=0\,. \tag{3.6}$$

It is easy to check that, insofar as (3.4) is assumed, one gets the same result starting from (2.5) and (2.6) with $m'=m$, $W=0$, and

$$-2V=2mU+U^2\,. \tag{3.7}$$

From the independence of U with respect to the relative time, our condition (2.4) is manifestly satisfied.

Finally, Eq. (2.5) becomes

$$[\Delta_K+\tfrac{1}{4}K^2-(m+U)^2]\psi(r^\alpha)=0\,, \tag{3.8}$$

where

$$\Delta_K=\frac{1}{K^2}\left(K\cdot\frac{\partial}{\partial r}\right)^2-\Box_r\,. \tag{3.9}$$

But Δ_K reduces to the usual Laplacian in coordinates adapted to the center of mass ($\vec{K}=0$).

Equation (3.8) has been solved by Gunion and Li for

$$U=k|\tilde{r}|\,, \tag{3.10}$$

giving rise to a quite reasonable charmonium spectrum. This spectrum can be considered as obtained by quantization from the classical system defined by (1.3) in which $W=0$ and V is related by Eq. (3.7) to a classical phase-space function,

$$U_{\text{class}}=k|\tilde{z}|\,.$$

The study of the classical motion can be performed according to the method given in Ref. 12.

It goes without saying that we have just exhibited one example among many. In order to fit some more recent experimental data, the Gunion-Li model should be in fact modified or replaced in a way which has been discussed by Crater.[20] But it is clear that our method of interpretation can still be applied in a similar way to a large class of models, since, to speak only of the simple case $W=0$, $m=m'$, the essential thing is that the interaction term in the Klein-Gordon equation must be independent of the relative time.

Naturally, a more realistic treatment should take spin into account. In that case, the underlying classical system should be, in fact, semiclassical, i.e., must involve degrees of freedom represented by Grassmann numbers. But, at the price of standard and obvious modifications (replacing Klein-Gordon operators by Dirac operators, etc.) our present point of view can be generalized.

---

[1]R. P. Feynman, M. Kislinger, and F. Ravndal, Phys. Rev. D 3, 2706 (1971). See also I. Montvay, I. T. P. Budapest, Report No. 291, 1971 (unpublished).

[2]Y. S. Kim and M. E. Noz, Phys. Rev. D 15, 335 (1977).

[3]J. F. Gunion and L. F. Li, Phys. Rev. D 12, 3583 (1975).

[4]H. Leutwyler and J. Stern, Nucl. Phys. B133, 115 (1978).

[5]G. Preparata and N. S. Craigie, Nucl. Phys. B102, 478 (1976).

[6]See also the treatment of relativistic wave equations in C. Fronsdal and L. E. Lundberg, Phys. Rev. D 1, 3247 (1970); C. Fronsdal, *ibid.* 4, 1689 (1971). These papers are rather devoted to weak, electromagnetic, or gravitational interactions and take field theory into account. Owing to the abundance of the literature we admit that the list of Refs. 1–6 is by no means exhaustive.

[7]This version of action-at-a-distance is sometimes called *finitely predictive* (referring to a phase space

of finite dimension), or *instantaneous*, or even *Newtonian*, because it makes use of force laws in instantaneous form. There have been many approaches to this subject. A lot of them are quoted in Refs. 6, 9, 11, and 12. We do not display here all the technical details. They can be found in Refs. 11 and 12.

[8]At least insofaras weak, electromagnetic, or gravitational interactions are concerned, predictive action-at-a-distance theory is related with classical field theory *in the perturbative way*: F. J. Kennedy, J. Math. Phys. 10, 1349 (1969); L. Bel, A. Salas, and J. M. Sanchez, Phys. Rev. D 7, 1099 (1973); L. Bel and J. Martin, *ibid.* 8, 4347 (1973); J. Ibanez and J. L. Sanz, Report No. FTUAM/77-12, 1977 (unpublished).

[9]Ph. Droz-Vincent, Lett. Nuovo Cimento 1, 839 (1969); Phys. Scr. 2, 129 (1970).

[10]Ph. Droz-Vincent, Lett. Nuovo Cimento 7, 206 (1973).

[11]Ph. Droz-Vincent, Rep. Math. Phys. 8, 79 (1975).

[12]Ph. Droz-Vincent, Ann. Inst. Henri Poincaré 27, 407 (1977).

[13]P. A. M. Dirac, Commun. Dublin Inst. Adv. Studies A No. 2, 1943. In particular, this paper contains an equation similar to (1.5). Unfortunately the famous "no-interaction theorem" was not yet discovered at this time, so the positions were inconsistently assumed to have canonical Poisson brackets, and this attempt broke down.

[14]D. G. Currie, J. Math. Phys. 4, 1470 (1963); Phys. Rev. 142, 817 (1966); D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963).

[15]The relationship between the positions and the canonical variables is studied with a lot of details in Refs. 11 and 12. The most appealing point of view consists in requiring that q, q' coincide with x, x' *at equal times*. This determines the parametrization of the world lines.

[16]Ph. Droz-Vincent, Lett. Nuovo Cimento 19, 800 (1971).

[17]$\{q, p\} = \{q', p'\} = \eta$; other brackets vanish.

[18]The calculation is the same as for proving that $[X, X'] = 0$ in Ph. Droz-Vincent, Ann. Inst. Henri Poincaré 20, 269 (1974).

[19]One could perhaps interpret the two-surfaces as the phase-space image of a pair of interacting strings, but this would be quite another story. Enlargement of the geometrical basic concepts is another possible way to save the point-particle interpretation, or alternatively, incorporation of extra degrees of freedom. We do not wish to follow these lines.

[20]H. Crater, Phys. Rev. D 16, 1580 (1977).