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# Alternate derivation of vacuum tunneling

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A field-theoretical derivation of the vacuum-tunneling effect is presented which gives results similar to the recent work of Callan and Coleman. An expression for the decay rate is exhibited which takes into account the zero and negative eigenmodes of the linearized Green's function.

## INTRODUCTION

The discovery of Euclidean solutions to classical field equations[1-5] and their interpretation as a tunneling process between vacuum states has led to much speculation and a number of derivations of the corresponding decay rates.[6-14] Most recently Callan and Coleman[14] derived the expression for the decay rate per unit volume for a scalar meson theory,

$$\Gamma/V=Ae^{-B/\hbar}[1+O(\hbar)], \tag{1}$$

where both coefficients $A$ and $B$ were determined. This derivation is cleverly done by classical analogy. However, some difficulty is experienced in handling the zero- and negative-frequency modes associated with the linearized, four-dimensional Green's function.

The purpose of this paper is twofold:

(1) to provide a field-theoretical interpretation for the vacuum decay process,

(2) to discuss the zero- and negative-frequency modes mentioned above.

As a consequence of the analysis an expression for the decay rate is derived which is closely related to the Callan and Coleman result.[14] It differs only in the normalization terms which arise from the zero-frequency modes.

## ANALYSIS

It was established some time ago by several authors[15] that the $U$ matrix in the coherent-state representation is the familiar Feynman path integral

$$\begin{aligned} U(t', t) &= \langle\varphi(t')|e^{-iH(t'-t)}|\varphi(t)\rangle \\ &= \langle\varphi(0)|e^{iH_0t'}e^{-iH(t'-t)}e^{-iH_0t}|\varphi(0)\rangle \\ &= \int\left[\frac{D\varphi}{\sqrt{2\pi}}\right]\exp\left[i\int_t^{t'}\int d^4x\mathcal{L}(\dot\varphi,\varphi)\right], \end{aligned} \tag{2}$$

where

$$|\varphi(t)\rangle = e^{-iH_0t}|\varphi(0)\rangle,$$

and if $\psi(x)$ is a $q$-number configuration field operator

$$\langle\varphi(0)|\psi(x)|\varphi(0)\rangle=\varphi(x). \tag{3}$$

More recently working within the framework of the Lehmann-Symanzik-Zimmermann (LSZ) formalism for cannonical, bilinear quantum field theories we established the same result.[16] It also follows from this work[15,16] that the $S$ matrix in the coherent-state representation is

$$\langle\varphi(0)|S|\varphi(0)\rangle=\lim_{\substack{t'\to\infty\\ t\to-\infty}}\int\left[\frac{D\varphi}{\sqrt{2\pi}}\right]\exp\left[i\int_t^{t'}\int d^4x\mathcal{L}(\dot\varphi,\varphi)\right]. \tag{4}$$

It is also well known by the Riemann-Lebesque lemma that the only states that survive the infinite-time limits are those states in the spectrum of $H_0$. This same result is also often obtained by analytically continuing Minkowski time to Euclidean time, with the corresponding change of limits $t'\to i\infty$, $t\to -i\infty$. In particular

$$\langle\varphi(0)|S|\varphi(0)\rangle=\lim_{\substack{t'\to-i\infty\\ t\to i\infty}}\int\left[\frac{D\varphi}{\sqrt{2\pi}}\right]\exp\left[i\int_t^{t'}\int d^4x\mathcal{L}(\dot\varphi,\varphi)\right], \tag{5}$$

that is, the $S$ matrix is assumed invariant under analytic continuation to the imaginary-time axis. In the following analysis it will prove convenient to take imaginary-time limits as in Eq. (5).

We adapt the concepts presented by Eqs. (2)–(5) to the case of decaying states by observing that the probability for finding the system in the state $|\varphi(\infty)\rangle_{out}$ after having been initially in the state $|\varphi(0)\rangle$ is

$$\begin{aligned} P &\equiv |N|^{-2}\lim_{t\to\infty}|U(t,0)|^2 \\ &= |N|^{-2}[\lim_{t\to i\infty}U^\dagger(t,0)][\lim_{t\to -i\infty}U(t,0)], \end{aligned} \tag{6}$$

where, by Eq. (2),

$$U(t,0)=\int\left[\frac{D\varphi}{\sqrt{2\pi}}\right]\exp\left[i\int_0^t\int d^4x\mathcal{L}(\dot\varphi,\varphi)\right], \tag{7}$$

and $N$ is a normalization constant which will be discussed later. Similarily the decay rate is given by

$$\Gamma \equiv |N|^{-2} \lim_{t\to\infty} \frac{d}{dt}|U(t,0)|^2$$
$$= |N|^{-2}\left\{\left[\lim_{t\to i\infty} U^\dagger(t,0)\right]\left[\lim_{t\to -i\infty}\frac{d}{dt}U(t,0)\right] + \left[\lim_{t\to i\infty}\frac{d}{dt}U^\dagger(t,0)\right]\left[\lim_{t\to -i\infty}U(t,0)\right]\right\}. \tag{8}$$

The final ingredient needed for the analysis is the observation that for normal-ordered operators,[17] such as the Hamiltonian operator

$$\langle\varphi(0)|H(\psi)|\varphi(0)\rangle = H(\varphi), \tag{9}$$

and for a canonical formalism for which

$$\langle\varphi(0)|[\psi, H]|\varphi(0)\rangle = \langle\varphi(0)|i\mathring{\psi}|\varphi(0)\rangle, \tag{10}$$

$$\langle\varphi(0)|[\pi, H]|\varphi(0)\rangle = \langle\varphi(0)|i\mathring{\pi}|\varphi(0)\rangle, \tag{11}$$

the canonical equations of motion are

$$\frac{\delta H(\varphi)}{\delta\pi} = \dot{\varphi}, \quad \frac{\delta H(\varphi)}{\delta\varphi} = -\mathring{\pi}. \tag{12}$$

Thus an extremum of $H(\varphi)$ is realized if the classical function $\varphi$ is a static solution of the equations of motion. The extremum is surely a minimum if

$$\frac{\delta^2 H(\varphi)}{\delta\varphi^2} > 0, \quad \frac{\delta^2 H(\varphi)}{\delta\pi^2} > 0. \tag{13}$$

It is evident that the ground state, or true vacuum state, is that solution which gives rise to the smallest $H(\varphi)$. This solution we call $\varphi_c$. It is possible $\varphi_c \equiv 0$ is the only solution. In this case the true vacuum state is the Fock vacuum. Otherwise the vacuum is a condensate which has a lower energy than the Fock vacuum state. Clearly the time-dependent solutions of Eq. (12) connect all such stable solutions, or for that matter, quasistable solutions for which Eq. (13) is not satisfied.

The $q$-number interpretation follows from Eq. (3) and Eqs. (9)–(13). For example, if $\psi$ is a Heisenberg field, then in the weak sense

$$\lim_{t\to\infty} \psi = z^{-1/2}\psi_{\text{out}}, \tag{14}$$

where $z$ is a renormalization constant. Corresponding to Eq. (3) we have

$$z^{-1/2}\langle\varphi(0)|\psi_{\text{out}}(x)|\varphi(0)\rangle \equiv \varphi_s(x), \tag{15}$$

and corresponding to Eq. (9)

$$\langle\varphi(0)|H(z^{-1/2}\psi_{\text{out}})|\varphi(0)\rangle = H(\varphi_s), \tag{16}$$

where we take

$$H(z^{-1/2}\psi_{\text{out}}) \equiv H_0. \tag{17}$$

From the form of Eqs. (15)–(17) compared to Eqs. (9)–(13), it is seen that $\varphi_s$ must satisfy the same field equations as does $\varphi$. Note that with this choice for $H_0$, Eq. (17), we can identify $\varphi_s(\vec{x},0) = \varphi(\vec{x},0)$ and from Eqs. (3), (14), and (15), $\varphi_s(\vec{x},\infty) = \varphi(\vec{x},\infty) = \varphi_c$ so that $\varphi_s$ represents a classical interpolating field that connects $\varphi(\vec{x},0)$ to $\varphi_c$. If we further can choose $\varphi(\vec{x},0)$ to be a second static solution of Eq. (12), we have the interpretation that $U(\infty,0)$ is the evolution, backwards through time of the true vacuum state $|\varphi_c\rangle$ to another vacuum state $|\varphi(0)\rangle$, which is also an extremum of $\langle\varphi(0)|H|\varphi(0)\rangle$. In particular if $\varphi(\vec{x},0)\equiv 0$, $|\varphi(0)\rangle = |\Omega\rangle$, the Fock vacuum state. If both $|\varphi_c\rangle$ and $|\varphi(0)\rangle$ satisfy Eq. (13), we interpret the effect as vacuum tunneling since the two minima of $H(\varphi)$ must be separated by at least one maximum. If only $\varphi_c$ satisfies Eq. (13) the process is the decay of a quasistable state.

In evaluating the functional integrals of Eq. (2) we will write

$$\varphi(x) = \varphi_s(x) + \xi(x) \tag{18}$$

and assume invariance of the measure of the functional integration,

$$[D\varphi] = [D\xi],$$

so that the interpolating field remains fixed. This requires

$$\xi(\vec{x},0) = \xi(\vec{x},\infty) = 0.$$

In a parallel fashion we define

$$\mathcal{L}(\dot{\varphi},\varphi) = \mathcal{L}(\dot{\varphi}_s,\varphi_s) + \mathcal{L}'(\dot{\varphi}_s,\varphi_s,\dot{\xi},\xi), \tag{19}$$

so that

$$U(t,0) = \exp\left[i\int_0^t\int d^4x\,\mathcal{L}(\dot{\varphi}_s,\varphi_s)\right] \times \int\left[\frac{D\xi}{\sqrt{2\pi}}\right]\exp\left(i\int_0^t\int d^4x\,\mathcal{L}'\right). \tag{20}$$

Since $\varphi_s$ satisfies the same field equations as does $\varphi$, it is clear that $\varphi_s$ minimizes the action

$$S = \int d^4x\,\mathcal{L}(\dot{\varphi}_s,\varphi_s),$$

providing that

$$\frac{\partial^2 S}{\partial\varphi_s^{\,2}} > 0. \tag{21}$$

It will be seen later that it is advantageous if a $\varphi_s$ can be found which satisfies Eq. (21).

To obtain the normalization factor $N$, we also need an interpolating field $\varphi_{s'}$ which carries a solution into itself,

$$\varphi = \varphi_{s'} + \xi', \tag{22}$$

with the boundary conditions

$$\varphi_{s'}(\vec{x},0) = \varphi(\vec{x},0), \quad \xi'(\vec{x},0) = 0$$
$$\varphi_{s'}(\vec{x},\infty) = \varphi(\vec{x},0), \quad \xi'(\vec{x},\infty) = 0.$$

With these modifications

$$N=\lim_{t\to -i\infty}\exp\left[i\int_0^t\int d^4x\mathcal{L}(\dot\varphi_{s'},\varphi_{s'})\right]$$
$$\times\int\left[\frac{D\xi'}{\sqrt{2\pi}}\right]\exp\left[i\int_0^t\int d^4x\mathcal{L}'(\dot\varphi_{s'},\varphi_{s'},\dot\xi',\xi')\right],\tag{23}$$

and Eq. (6) becomes

$$P\sim\exp\left(\int_{-t}^t\int d^4x\mathcal{L}\right)\left\{\int\left[\frac{D\xi}{\sqrt{2\pi}}\right]\exp\left(\int_0^t\int d^4x\mathcal{L}'\right)\right.$$
$$\left.\times\int\left[\frac{D\xi}{\sqrt{2\pi}}\right]\exp\left(\int_{-t}^0\int d^4x\mathcal{L}'\right)\right\},$$

where, except for the differential $d^4x$, the arguments of the functional integrals are evaluated at the Euclidean time points $ix_0$. If it is assumed that $\mathcal{L}'$ is analytic in the neighborhood of $x_0=0$, the definition of functional integration allows us to write

$$P(t)=\exp\left[\int_{-t}^t\int d^4x\mathcal{L}(\dot\varphi_s,\varphi_s)-\int d^4x\mathcal{L}(\dot\varphi_{s'},\varphi_{s'})\right]$$
$$\times\int\left[\frac{D\xi}{\sqrt{2\pi}}\right]\exp\left[\int_{-t}^t\int d^4x\mathcal{L}'(\dot\varphi_s,\varphi_s,\dot\xi,\xi)\right]$$
$$\times\int\left[\frac{D\xi'}{\sqrt{2\pi}}\right]\exp\left[-\int d^4x\mathcal{L}'(\dot\varphi_{s'},\varphi_{s'},\xi',\xi')\right].\tag{24}$$

It is now a simple exercise to show

$$\Gamma\sim\exp\left\{\int d^4x\left[\mathcal{L}\left(-i\frac{d}{dx_0}\varphi_s(\vec{x},ix_0),\varphi_s(\vec{x},ix_0)\right)\right.\right.$$
$$\left.\left.-\mathcal{L}\left(-i\frac{d}{dx_0}\varphi_{s'}(\vec{x},ix_0),\varphi_{s'}(\vec{x},ix_0)\right)\right]\right\},$$

which is the result to be expected.[14]

## SCALAR FIELD EXAMPLE

As an illustration of the above analysis we consider the scalar field Lagrangian[18]

$$\mathcal{L}=\tfrac{1}{2}[\dot\varphi^2-(\vec\nabla\varphi)^2]-U(\varphi).\tag{25}$$

The corresponding Hamiltonian is

$$H=\int d^3x\{\tfrac{1}{2}[\pi^2+(\vec\nabla\varphi)^2]+U(\varphi)\},\tag{26}$$

with $\pi=\dot\varphi$. This gives for Eqs. (12) and (13)

$$\frac{\delta H}{\delta\pi}=\dot\varphi=\pi,$$
$$\frac{\delta H}{\delta\varphi}=-\dot\pi=-\vec\nabla^2\varphi+U'(\varphi),$$

which summarizes to the field equation

$$(-\vec\nabla^2+\partial^2/\partial t^2)\varphi+U'(\varphi)=0.\tag{27}$$

The stability conditions become

$$\frac{\delta^2H}{\delta\pi(y)\delta\pi(x)}=\delta(\vec{x}-\vec{y}),$$
$$\frac{\delta^2H}{\delta\varphi(y)\delta\varphi(x)}=[-\vec\nabla^2+U''(\varphi)]\delta(\vec{x}-\vec{y}).\tag{28}$$

To make $|\varphi_c\rangle$ the true vacuum state we must require the normalizable eigenfunctions $\hat\eta_j$, which satisfy

$$[-\vec\nabla^2+U''(\varphi_c)]\hat\eta_j=\lambda_j\hat\eta_j,$$

to have non-negative eigenvalues

$$\lambda_j\geq 0,\tag{29}$$

with at least one positive eigenvalue, and

$$-\vec\nabla^2\varphi_c+U'(\varphi_c)=0.\tag{30}$$

Note that $\vec\nabla\varphi_c$, properly normalized, is a member of the set $\{\hat\eta_j\}$ with eigenvalue zero. The minimum energy becomes

$$E_0\equiv H(\varphi_c)=\int d^3x[\tfrac{1}{2}(\vec\nabla\varphi_c)^2+U(\varphi_c)].\tag{31}$$

One possible solution of Eq. (30) is $\varphi_c\equiv\text{const}$ which mimimizes $U(\varphi_c)$, that is, $U'(\varphi_c)=0$, $U''(\varphi_c)>0$. Then, if a coordinate frame is chosen so that $U(\varphi_c)=0$, $E_0=0$. The eigenvalues of Eq. (29) for this case would be

$$\lambda_p=U''(\varphi_c)-p^2,$$

where $p$ is the eigenvalue of the $\vec\nabla$ operator. To keep $\lambda_p\geq 0$, so that we have a minimum in the energy, the possible values of $p$ are real if $p^2<U''(\varphi_c)$. Otherwise they are pure imaginary. It is convenient to choose phases as shown in Fig. 1,

$$\lambda_p=e^{i\pi}[p^2-U''(\varphi_c)],$$
$$\lambda_p^{1/2}=e^{i\pi/2}\{p-[U''(\varphi_c)]^{1/2}\}^{1/2}\{p+[U''(\varphi_c)]^{1/2}\}^{1/2}.\tag{32}$$

With this choice of phase the contour $\mathcal{C}_+$, also shown in Fig. 1, is used to sum[19] over states of different $p$ for negative $\lambda_p^{1/2}$ eigensolutions. For $\text{Im}p>0$ these states correspond to outgoing spherical waves in four-dimensional Minkowski space, whereas for $\text{Im}p<0$ they are incoming spherical waves. For $p$ real the eigensolutions are irregular at the origin and therefore must be discarded. The contour $\mathcal{C}_-$ which sums over antiparticle solutions is on another sheet.

Similarily if the initial field $\varphi(\vec{x},0)$ describes a secondary minimum,

$$-\vec\nabla^2\varphi(\vec{x},0)+U'(\varphi(\vec{x},0))=0,$$
$$[-\vec\nabla^2+U''(\varphi(\vec{x},0))]\hat\eta_j'=\lambda_j'\eta_j',\quad\lambda_j\geq 0$$
$$E_1=\int d^3x\{\tfrac{1}{2}[\vec\nabla\varphi(\vec{x},0)]^2+U(\varphi(\vec{x},0))\}>E_0.$$

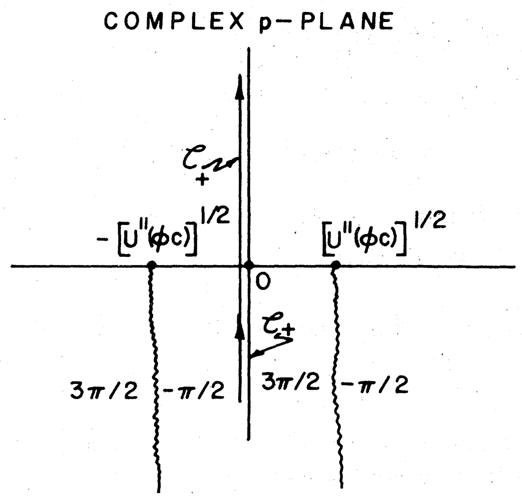


FIG. 1. Complex $p$ plane showing the branches and choice of phases for $p-[U''(\varphi_c)]^{1/2}$ and $p+[U''(\varphi_c)]^{1/2}$. The contour $C_1$ is used to sum over the states of different $p$ for negative $\lambda_p^{1/2}$.

From Eqs. (18) and (19) we have

$$\mathcal{L}(\dot\varphi_s, \varphi_s) = \tfrac{1}{2}[(\dot\varphi_s)^2 - (\vec\nabla\varphi_s)^2] - U(\varphi_s),$$
$$(\dot\varphi_s, \varphi_s, \dot\xi, \xi) = \tfrac{1}{2}\{\xi\vec\nabla^2\xi - \xi(\partial^2\xi/\partial t^2) - U''(\varphi_s)\xi^2 - 2[U(\varphi_s+\xi) - U(\varphi_s) - U'(\varphi_s)\xi - \tfrac{1}{2}U''(\varphi_s)\xi^2]\}, \tag{33}$$

where we have used the field equations for $\varphi_s$ and discarded certain total time derivatives since they do not contribute to the action. In the usual discussion for vacuum decay processes the method of steepest descent is used to determine the quantum fluctuating part of the path integral. Thus if $U(\varphi_s+\xi)$ is expanded as

$$U(\varphi_s+\xi) - U(\varphi_s) - U'(\varphi_s)\xi - \tfrac{1}{2}U''(\varphi_s)\xi^2 \simeq \frac{1}{3!}U^{(3)}\xi^3 + \frac{1}{4!}U^{(4)}\xi^4, \tag{34}$$

we can write

$$\int\left[\frac{D\xi}{\sqrt{2\pi}}\right]\exp\left(\int d^4x\mathcal{L}'\right) = \int\left[\frac{D\xi}{\sqrt{\pi}}\right]\exp\left(-\int d^4x\,\xi G^{-1}\xi\right) + \text{h.o}$$
$$= \text{Det}^{-1/2}(G)^{-1} + \text{h.o.}, \tag{35}$$

where h.o. means higher-order terms,

$$G^{-1} = -\vec\nabla^2 - (\partial^2/\partial x_0^2) + U''(\varphi_s(\vec{x}, ix_0)),$$

and, since $\varphi_s$ also satisfies Eq. (27), the minimization condition, Eq. (21), becomes

$$[-\vec\nabla^2 - \partial^2/\partial x_0^2 + U''(\varphi_s(\vec{x}, ix_0))]\delta^4(x-y) > 0.$$

If this equation is satisfied, $G^{-1}$ has no negative eigenvalues. Now we see a basic difficulty. Since $\varphi_s$ satisfies Eq. (27), the inverse Green's function $G^{-1}$ must have zero eigenvalues[20] corresponding to the eigensolutions $\vec\nabla\varphi_s$, $\dot\varphi_s$. The path integral of Eq. (35) therefore diverges. If $G^{-1}$ has negative eigenvalues, that is Eq. (21) is not satisfied, the Gaussian integrations are undamped so that Eq. (35) becomes even more divergent. It is clear these divergences are caused by the drastic truncation of the series for the potential function, Eq. (34), because if terms through $\xi^4$ are retained and $U^{(4)}(\varphi_s) > 0$, the path integral would surely converge regardless of the value of the eigenvalues of $G^{-1}$. Thus these terms can be ignored *only* for the *positive* eigenvalues of $G^{-1}$.

The path integral then becomes

$$\int\left[\frac{D\xi}{\sqrt{2\pi}}\right]\exp\left(\int d^4x\mathcal{L}'\right) = [\text{Det}'^{-1/2}(G)^{-1}]\int\left[\frac{D\xi}{\sqrt{\pi}}\right]\exp\left\{-\int d^4x[\bar\xi G^{-1}\bar\xi + (\sqrt{2}/3)U^{(3)}\bar\xi^3 + \tfrac{1}{6}U^{(4)}\bar\xi^4]\right\} + \text{h.o.}, \tag{36}$$

where the prime on Det means ignore the zero and negative eigenvalues, and $\bar\xi$ is assumed expandable in terms of only those eigenmodes for which the eigenvalues of $G^{-1}$ are zero or negative. Cross terms between $\bar\xi$ and eigenmodes corresponding to positive eigenvalues are assumed negligable when compared to $\text{Det}'^{-1/2}$ and presumed to be included in h.o.

For the purpose of simplicity and for comparison to other work[14] we assume in the following analysis that $G^{-1}$ has no negative eigenvalues, that is, Eq. (21) is satisfied perhaps in a manner similar to that above for the case of $\varphi_c = \text{const}$. With these further assumptions,

$$\int\left[\frac{D\xi}{\sqrt{2\pi}}\right]\exp\left(\int d^4x\mathcal{L}'\right) = \frac{B_{\varphi_s}^2 A_{\varphi_s}}{4\pi^2}\text{Det}'^{-1/2}(G)^{-1},$$

where

$$A_{\varphi_s} = \int d^3C_i dC_0 \exp\left\{-\int d^4x[(1/3!)U^{(3)}\xi_0^3 + (1/4!)U^{(4)}\xi_0^4]\right\} \tag{37}$$

with

$$\xi_0 = \sum_{i=1}^{3} C_i(\partial\varphi_s/\partial x_i) + C_0(\partial\varphi_s/\partial x_0), \tag{38}$$

and where $B_{\varphi_s}$ is a factor which arises from the normalization of the eigensolutions $\partial_\mu \varphi_s$. Note that if the exponent is considered negligible, $A_{\varphi_s}$ is the four-dimensional volume $VT$ and we obtain

$$(P/VT) \equiv (\Gamma/V) = \frac{B_{\varphi_s}{}^2 |N|^{-2}}{4\pi^2} \mathrm{Det}'^{-1/2}(G)^{-1} \exp\left[\int d^4x \mathcal{L}(\dot\varphi_s, \varphi_s)\right]. \tag{39}$$

Correspondingly we would have

$$|N|^2 = \mathrm{Det}^{-1/2}[G(\varphi_{s'})]^{-1} \exp\left[\int d^4x \mathcal{L}(\dot\varphi_{s'}, \varphi_{s'})\right],$$

where $G^{-1}(\varphi_{s'})$ is $G^{-1}$ with $\varphi_s$ replaced by $\varphi_{s'}$. If $\varphi_{s'}$ is taken to be $\varphi(0) \equiv \varphi_+ = \mathrm{const}$, the secondary minimum assumed by Callan and Coleman,[14] Eq. (39) is in agreement with their result.[14] However, in general $\varphi_{s'} \neq \mathrm{const}$ so $G(\varphi_{s'})^{-1}$ may also have zero eigenvalues. These zero modes must also be removed. To allow for this generalization we continue the integration indicated in Eq. (37) for the coefficient $A_{\varphi_s}$.

In the calculation of the decay rate, using Eq. (8), it is convenient to reexpress Eq. (37) as

$$\int \left[\frac{D\xi}{\sqrt{2\pi}}\right] \exp\left(\int_{-t}^{t} dx_0 \int d^3x \mathcal{L}'\right) = \frac{B_{\varphi_s}{}^2 A_{\varphi_s}(t)}{4\pi^2} \exp\left[-\tfrac{1}{2}\int_{-t}^{t} dx_0 \int d^3x \ln(G)^{-1}\right] + O(t^{-1}),$$

where

$$A_{\varphi_s}(t) = \int d^3C_i dC_0 \exp\left\{-\int_{-t}^{t} dx_0 \int d^3x [(1/3!)U^{(3)}\xi_0{}^3 + (1/4!)U^{(4)}\xi_0{}^4]\right\}. \tag{40}$$

To evaluate $A_{\varphi_s}(t)$ we restrict $\varphi_s$ and $\varphi_{s'}$ to be isotropic in $\vec{x}$ and $t$. For example,[14]

$$\varphi_s(\vec{x}, it) = \varphi_s(\rho), \quad \rho = (\vec{x}^2 + t^2)^{1/2} \tag{41}$$

with

$$B_{\varphi_s} \equiv \tfrac{1}{4} \int d^4x \left[\sum_i (\partial\varphi_s/\partial x_i)^2 + (\partial\varphi_s/\partial x_0)^2\right],$$

as is customary.[14] Our boundary condition

$$i\dot\varphi_s(\vec{x}, -i\infty) = \lim_{x_0 \to \infty} (d\varphi_s/d\rho)(x_0/\rho) = 0$$

then implies

$$\lim_{|x_0| \to \infty} (d\varphi_s/d\rho) = 0 \tag{42}$$

for all $\vec{x}$. This assumption is sufficient to show[21]

$$A_{\varphi_s}(\infty) = 16\left[\int_0^\infty \rho^3 d\rho U^{(4)}(\varphi_s)(d\varphi_s/d\rho)^4\right]^{-1}. \tag{43}$$

and $\lim_{|x_0| \to \infty} \dot A_{\varphi_s}(x_0) \to 0$ so that it does not contribute to $\Gamma$. It is now a simple matter to show

$$\Gamma = \frac{B_{\varphi_s}^2 A_{\varphi_s}(\infty)}{B_{\varphi_{s'}}{}^2 A_{\varphi_{s'}}} \frac{\mathrm{Det}'^{1/2}[-\vec\nabla^2 - (\partial^2/\partial t^2) + U''(\varphi_{s'})]}{\mathrm{Det}'^{1/2}[-\vec\nabla^2 - (\partial^2/\partial t^2) + U''(\varphi_s)]} \times \exp\left\{\int d^4x [\mathcal{L}(\dot\varphi_s, \varphi_s) - \mathcal{L}(\dot\varphi_{s'}, \varphi_{s'})]\right\} \times (2\,\mathrm{Im}\{\mathcal{L}(0, \varphi_c) + \ln \mathrm{Det}^{1/2}[-\vec\nabla^2 + U''(\varphi_c)]\}). \tag{44}$$

In general we expect $\mathcal{L}(0, \varphi_c)$ to be real so this term will not contribute. The logrithmic term will vanish except for "continuum" eigenvalues such as those of Eq. (32) for $p$ imaginary. We take the $\varphi_c = \mathrm{const}$ example as a guide and assume the principal contribution to the sum over all eigenvalues, $p$, arises only from large $p$. For the phases chosen in Fig. 1,

$$2\,\mathrm{Im} \ln \mathrm{Det}^{1/2}[-\vec\nabla^2 + U''(\varphi_c)] = 2\,\mathrm{Im}\left\{\sum_{\mathfrak{C}_+} [\ln|p^2 - U''(\varphi_c)|^{1/2} + i\pi]\right\}, = V(2\pi)^{-3}(2\pi) = V(2\pi)^{-2},$$

where $V$ is the three-dimensional space volume. Assuming this is typical of the large-$p$ behavior for the more general case $U''(\varphi_c) \neq \mathrm{const}$, we obtain finally

$$(\Gamma/V) = (4\pi)^{-2} \frac{B_{\varphi_s}{}^2 A_{\varphi_s}(\infty)}{B_{\varphi_{s'}}{}^2 A_{\varphi_{s'}}(\infty)} \frac{\mathrm{Det}'^{1/2}[-\vec\nabla^2 - (\partial^2/\partial t^2) + U''(\varphi_{s'})]}{\mathrm{Det}'^{1/2}[-\vec\nabla^2 - (\partial^2/\partial t^2) + U''(\varphi_s)]} \exp\left\{\int d^4x [\mathcal{L}(\dot\varphi_s, \varphi_s) - \mathcal{L}(\dot\varphi_{s'}, \varphi_{s'})]\right\}. \tag{45}$$

Of course this equation will not be valid if $G^{-1}$ has negative eigenvalues. In that event, although more algebraically complicated, the same method as used to extract the zero eigenmodes can also be used to extract the negative eigenmodes.

## CONCLUSION

We have given a field-theoretical interpretation for the vacuum-tunneling process which is consistent with the previous classical treatment.[14] In fact our Eq. (45), as shown by the steps leading to Eq. (39), reduces to the result of Callan and Coleman[14] [Eq. (39)] if we assume, as they do, that

(1) the interpolating solution $\varphi_{s'}$ [Eq. (22)] is a constant so that $G^{-1}(\varphi_{s'})$ has no nontrivial zero eigenmode solutions, and

(2) the quantum fluctuations may be taken to lowest order thereby effectively setting the exponent

of Eq. (37) equal to zero.

Thus if a secondary constant interpolating solution exists the difference between our result and that of Callan and Coleman[14] depends only upon higher-order quantum fluctuations and may therefore be negligible.

While we used the formalism to describe the vacuum-tunneling process, it could equally well be used to describe the decay of quasistable vacuum state as described just prior to Eq. (18) or the decay of excitations of the vacuum state $|\varphi_c\rangle$. In this latter case we need only to describe the initial condition $\varphi(\vec{x}, 0)$ as

$$\varphi(\vec{x}, 0)= \sum_k C_k \hat{\eta}_k \qquad (46)$$

where the eigenfunctions $\hat{\eta}_i$ are given by Eq. (29).

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[21]Note this condition enforces the boundary condition $\xi(x,\infty)=0$ for the zero eigenmode. See the definition for $\xi_0$ below, Eq. (37). If the isotropic assumption were not true it would be possible for $\lim_{t\to\infty}\vec{\nabla}\varphi_s = \vec{\nabla}\varphi_c \neq 0$. The boundary condition would then require $\xi_0 = C_0\dot{\varphi}_s$ be the only zero eigenmode.