Relationship between the two-dimensional $\bar{\psi}\psi\bar{\psi}\psi$ model and Yukawa-type models

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We show to all orders in mean-field perturbation theory that a two-dimensional scalar four-fermion interaction $(\bar{\psi}\psi)^2$ has renormalized Green's functions which are structurally identical to an infinite class of theories having fundamental Yukawa-type couplings with a scalar field σ and additional scalar self-interactions of the form $F(\sigma)$. For a particular choice of renormalized parameters of the Yukawa-type theories which occurs as $Z_3^\sigma \to 0$ the renormalized Green's functions are numerically equal to those of the four-fermion theory.

I. INTRODUCTION

Recently a new systematic method of expanding field theories around a mean field has been fully developed.[1-4] This method is in many (but definitely not all) instances equivalent to the $1/N$ expansion techniques.[5] Even in the cases of equivalence, it is usually possible to identify a small renormalized parameter so that the mean-field method can be applied for the case of only one field.[1,4] The current published applications of this mean-field method are most fully developed for the ϕ^4 scalar theory,[1] but the greatest potential interest of the approach lies in the four-Fermi theories with interactions of the form $(\bar{\psi}o^\alpha\psi)(\bar{\psi}o^\alpha\psi)$. In these theories the mean-field perturbation theory is conveniently obtained by the usual trick of introducing auxiliary fields $\chi_{ij}^\alpha = \bar{\psi}_i o^\alpha \psi_j$, where α is a Lorentz index and i and j are internal-symmetry indices, directly into the action which appears in the path-integral representation of the vacuum functional. Modified in this manner, the action contains the Fermi fields only bilinearly, and hence they can be integrated out of the path integral. The resulting resummed theories bear a topological resemblance to theories of fermions interacting via generalized Yukawa couplings with fundamental fields. They are only missing the kinetic terms for the fields χ_{ij}^α. The lack of these kinetic terms is, of course, entirely nontrivial since this is the mathematical statement of the obvious fact that the Yukawa and pure Fermi theories are canonically entirely different.

Nevertheless, using mean-field expansion techniques in four space-time dimensions it is possible to show that the renormalized Green's functions of the Nambu–Jona-Lasinio model[6] are identical to those of a special σ model[3,7] and that vector-vector four-Fermi theories yield renormalized Green's functions equivalent to gauge-field theories.[1,7,8] Since this equivalence is valid when these theories are expanded in their mean-field approximation, it is necessary and, remarkably, possible to demonstrate that within this particular scheme most four-fermion theories in four dimensions and their related fundamental boson theories are renormalizable.[1,3,9] The ambiguities of the definition of products of operators at a point which are responsible for the high degree of divergence of the normal coupling-constant perturbation expansion of four-fermion theories are handled in a specially defined self-consistent manner in mean-field theories. This makes it easy to level the criticism that equivalence arguments are nothing more than the result of defining what a four-fermion theory should be. To some extent it is, of course, true that any divergent field theory that can be made usable is defined. However, the content of the mean-field method is far deeper than just definition, because the self-consistent nature of the construction indicates that the Fermi theory is only equivalent to a σ model for special values of the parameters of the σ model.[3,7,9] The primary possible advantage of pure fermion models is the reduction of the number of free parameters relative to a more conventional model.

The correspondence of sets of models within mean-field approximations is a common feature in no way unique to scalar-type four-Fermi interactions in four dimensions or theories which are not renormalizable or for that matter even divergent at all. In Ref. 1 it was shown that a ϕ^4 interaction corresponds to a cubic interaction involving two canonical boson fields for a particular limiting choice of the bare coupling of the cubic model. As a result, the usual three independent renormalized parameters of the cubic model were reduced to two in order to establish a correspondence with the quartic model in the mean-field approximation. Similarly, in that paper it was shown that the $j^\mu j_\mu$ four-fermion interaction corresponds exactly to ordinary electrodynamics if α is determined by the Johnson-Baker-Willey eigenvalue condition. In this paper we develop related arguments which show the equivalence of the scalar four-Fermi interaction in two space-time dimensions to a whole class of Yukawa-type models.

In Sec. II we review the expansion technique developed in Ref. 1 and apply it to determine the renormalization parameters of the $(\bar\psi\psi)^2$ model in two dimensions. In Sec. III we show that the lowest-order renormalization parameters are not arbitrary but are related through the self-consistency requirements of the model. In Sec. IV we demonstrate the equivalence of the two-dimensional four-fermion model to a whole class of models with elementary bosons.

II. MEAN-FIELD EXPANSION OF $(\bar\psi\psi)^2$

We shall study the two-space-time-dimensional theory given by the Lagrangian density[10]

$$\mathcal{L}_F' = \bar\psi^{(i)} i\gamma\partial\psi^{(i)} + \tfrac{1}{2}G_0{}^2(\bar\psi^i\psi^i)^2 \tag{2.1}$$

in the mean-field approximation. The index i is summed over from 1 to N and corresponds to the internal symmetry. Henceforth, explicit reference to this summation will be suppressed. We have set the bare Fermi mass m_0 to zero for the sake of simplification, but most of this paper's major conclusions are valid even when $m_0 \neq 0$. In order to do mean-field approximations in the integral formulation (but not the alternative differential formulation[3]) it is necessary to explicitly introduce the auxiliary field σ and write the Lagrangian density in the form

$$\mathcal{L}_F = \bar\psi(i\gamma\partial - G_0\sigma)\psi - \tfrac{1}{2}\sigma^2 + J\sigma + \bar n\psi + \bar\psi n\ . \tag{2.2}$$

Here, $\bar n$, n, and J are external sources. Variation of (2.2) leads to the two independent field equations:

$$(i\gamma\partial - G_0\sigma)\psi + n = 0 \tag{2.3}$$

and

$$\sigma = -G_0\bar\psi\psi + J\ . \tag{2.4}$$

Setting the sources to zero and substituting (2.4) into (2.3) results in

$$(i\gamma\partial + G_0\bar\psi\psi)\psi = 0\ , \tag{2.5}$$

which is the field equation obtained by varying (2.1). Thus all solutions to (2.2) are also solutions to (2.1).

The vacuum functional of (2.2) is given in terms of path integrals by

$$Z(\bar n, n, J) = \langle\ |\ \rangle = \text{const}\times\int[d\sigma][d\bar\psi][d\psi]e^{i\int\mathcal{L}_F d^2x}\ .$$

Because (2.2) is quadratic in the Fermi fields they can be integrated out to yield

$$Z(\bar n, n, J) = \text{const}\times\int[d\sigma]e^{iF(\sigma,\bar n,n,J)}\ , \tag{2.6}$$

where

$$F(\sigma,\bar n,n,J) = \int d^2x\Big(-\tfrac{1}{2}\sigma^2(x) + J(x)\sigma(x) - i\,\mathrm{tr}[\ln S_\sigma^{-1}(x,y)]_{x=y} - \int d^2y\,\bar n(x)S_\sigma(x,y)n(y)\Big)\ . \tag{2.7}$$

Here

$$S_\sigma^{-1}(x,y) \equiv [i\gamma\partial_x - G_0\sigma(x)]\delta^2(x-y)\ . \tag{2.8}$$

Mean-field perturbation theory is defined by introducing by hand a small parameter ϵ and defining a new function of the external sources

$$Z_\epsilon(\bar n, n, J) \equiv \int[d\sigma]e^{(i/\epsilon)F(\sigma,\bar n,n,J)}\ . \tag{2.9}$$

When $\epsilon=1$, (2.9) produces the same connected Green's functions as (2.7). Our approach is to expand (2.9) in Euclidean space by Laplace's method in a power series in ϵ. The order of the approximation of $W_\epsilon \equiv -i\epsilon\ln Z_\epsilon$ is identified by the number of powers of ϵ. The theory is renormalized order by order in ϵ and then ϵ is continued to unity. There is no *a priori* way to justify this technique rigorously or to predict the accuracy of the resulting series. This is also the case for the usual coupling-constant perturbation theory. After the calculation is complete we will see that ϵ always appears multiplied by parameters which can be made small so that the product is small even as ϵ approaches unity. In this model a possible choice for the small parameter is $1/N$ if N is large. There is, however, another possible choice which works even when $N=1$ and which corresponds to a reordering of the renormalized Green's function expansion if N is large. This will be mentioned again later and is used extensively in the four-dimensional version of this model.[9]

The expansion of functional integrals of this type in an asymptotic series in ϵ has been studied in detail in Ref. 1. We shall just review some of the essential features here in order to obtain the graphical expansion of this theory.

In Euclidean space Eq. (2.9) takes the form

$$Z_\epsilon(\bar{n}, n, J) = \int [d\sigma] e^{-F(\sigma)/\epsilon} . \tag{2.10}$$

If we require that

$$\left.\frac{\delta F}{\delta \sigma}\right|_{\sigma=\sigma_0} = 0 \tag{2.11}$$

and that

$$A(x,y) \equiv \left.\frac{\delta^2 F}{\delta\sigma(x)\,\delta\sigma(y)}\right|_{\sigma=\sigma_0} > 0 \tag{2.12}$$

and define

$$B(x,y,z) \equiv \left.\frac{\delta^3 F}{\delta\sigma(x)\,\delta\sigma(y)\,\delta\sigma(z)}\right|_{\sigma=\sigma_0} \tag{2.13}$$

and

$$C(x,y,z,w) \equiv \left.\frac{\delta^4 F}{\delta\sigma(x)\,\delta\sigma(y)\,\delta\sigma(z)\,\delta\sigma(w)}\right|_{\sigma=\sigma_0} , \tag{2.14}$$

then Laplace's method applied to functional integrals yields

$$Z_\epsilon \sim e^{-F[\sigma_0]/\epsilon} e^{-(1/2)\operatorname{tr}\ln A} \Big\{ 1 - \frac{\epsilon}{8} \iiiint dx\,dy\,dz\,dw\, C(x,y,z,w) A^{-1}(x,y) A^{-1}(z,w) + \frac{\epsilon}{24} \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int dx\,dy\,dz\,da\,db\,dc\, B(x,y,z) B(a,b,c) \times [2A^{-1}(x,a) A^{-1}(y,b) A^{-1}(z,c) + 3A^{-1}(x,y) A^{-1}(z,a) A^{-1}(b,c)] + O(\epsilon^2) \Big\} \quad (\epsilon \to 0^+) . \tag{2.15}$$

This formula is used to expand the vacuum functional w_ϵ in a mean-field perturbation series. We shall show using it that it is possible to identify the basic vertices of the theory and the corresponding graphical expansion.

Returning to Minkowski space, it follows from condition (2.11) that the extremum value of σ is, from (2.7),

$$\sigma_0(x) = G_0\, i \operatorname{tr} S_{\sigma_0}(x,x) - G_0 \bar{\psi}_c(x) \psi_c(x) + J(x) . \tag{2.16}$$

Here we have introduced the fields

$$\psi_c(x) \equiv \int d^2y\, S_{\sigma_0}(x,y)\, n(y) , \tag{2.17a}$$

$$\bar{\psi}_c(x) \equiv \int d^2y\, \bar{n}(y)\, S_{\sigma_0}(y,x) . \tag{2.17b}$$

From (2.15) it is evident that A^{-1} plays a central role in the expansion. Differentiating (2.7) we find

$$A(x,y) = -\delta(x-y) + G_0^2\, i \operatorname{tr}[S_{\sigma_0}(x,y) S_{\sigma_0}(y,x)] - G_0^2 [\bar{\psi}_c(y) S_{\sigma_0}(y,x) \psi_c(x) + (x \leftrightarrow y)] . \tag{2.18}$$

Thus A is a measure of the correlation strength of pairs of Fermi particles. It is directly confirmed through the use of stationarity conditions (2.11) and (2.7) that the "bound-state propagator"

$$D_\epsilon(x,y) \equiv -\frac{i\delta}{\delta J(x)} - \frac{i\delta}{\delta J(y)} W_\epsilon \tag{2.19}$$

defined in lowest order in ϵ as D_{σ_0} is given by

$$D_{\sigma_0}(x,y) = -\frac{i\delta}{\delta J(x)} - \frac{i\delta}{\delta J(y)} F(\sigma_0, J, n, \bar{n}) = \frac{\delta}{\delta J(x)} \sigma_0(y) , \tag{2.20}$$

which, using (2.16), (2.17), and (2.18), becomes

$$D_{\sigma_0}(x,y) = -A^{-1}(x,y) . \tag{2.21}$$

Thus, (2.12) contains the condition that D_{σ_0} is not tachyonic.

We can, by direct differentiation of (2.7) using (2.8), calculate any term occurring in the expansion of Z_ϵ. For example, we have (in Euclidean space)

$$B(x,y,z) = -G_0^3 \operatorname{tr} S_{\sigma_0}(x,y) S_{\sigma_0}(y,z) S_{\sigma_0}(z,x) - G_0^3 [\bar{\psi}_c(x) S_{\sigma_0}(x,z) S_{\sigma_0}(z,y) \psi_c(y) + \bar{\psi}_c(y) S_{\sigma_0}(y,z) S_{\sigma_0}(z,w) \psi_c(x) + \bar{\psi}_c(x) S_{\sigma_0}(x,y) S_{\sigma_0}(y,z) \psi_c(z) + \bar{\psi}_c(z) S_{\sigma_0}(z,y) S_{\sigma_0}(y,x) \psi_c(x) + \bar{\psi}_c(z) S_{\sigma_0}(z,x) S_{\sigma_0}(x,y) \psi_c(y) + \bar{\psi}_c(y) S_{\sigma_0}(y,x) S_{\sigma_0}(x,z) \psi_c(z)] . \tag{2.22}$$

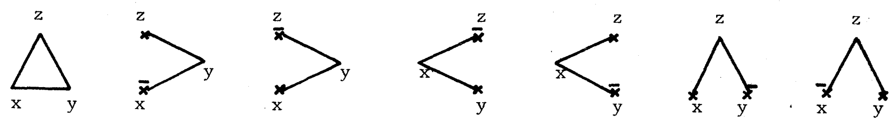

FIG. 1. Graphical representation of $B(x,y,z)$ as given in (2.22). Each solid line represents a factor of $S_{\sigma_0}(x,y)$. $\psi_c(x)$ is represented by a cross at the point x and $\bar{\psi}_c(x)$ is represented by a barred cross at the point x. With each coordinate point we associate a factor G_0.

$B(x,y,z)$ is represented graphically in Fig. 1. It is straightforward but space consuming to calculate $C(x,y,z,w)$. In general it is easy to calculate and to represent graphically all higher variational derivatives of F_n. To represent the nth derivative of F we draw all possible n-sided polygons and all possible $(n-1)$-sided open polygonal paths with a cross and a barred cross at the open ends.

Now using (2.14) we can construct W_ϵ. We find (in Euclidean space)

$$W_\epsilon \sim -F(\sigma_0, \bar{n}, n, J) + \epsilon \operatorname{tr} \ln D_{\sigma_0} + \sum_{n=2} \epsilon^n F_n . \tag{2.23}$$

If we represent $D_{\sigma_0}(x,y)$ by a wiggly line, we find that F_n is composed of only three distinct types of vertices as given in Fig. 2. The vacuum graphs F_n of order n consist of all possible graphs made from the vertices of Fig. 2, so that each graph has no external legs, and, further, has n independent loop integrations involving at least one bound-state propagator D_{σ_0}. Note that in these graphs each bound-state propagator D_{σ_0} always occurs in the combination $G_0^2 D_{\sigma_0}$, with one G_0 coming from each of the space-time points connected by D_{σ_0}. This is a general property of mean-field theory regardless of the explicit form of quartic interaction involved. It essentially guarantees that the bare coupling drops out of the theory except through its effects on mass normalization.

The contribution of order n to a connected Green's function is calculated by differentiating F, D_{σ_0}, or F_n with respect to the appropriate combinations of external sources. In order to do this it is only necessary to know the derivatives of σ_0, ψ_c, $\bar{\psi}_c$, D_{σ_0}, and S_{σ_0} with respect to $\bar{n}$, n, and J. These are messy but extremely straightforward to calculate. All we need to use for the purpose of this paper is that all of these derivatives are expressible in terms of the three vertices of Fig. 2. When the external sources are off, only the Yukawa-type vertex of Fig. 2(a) contributes. We thus conclude that any Green's function calculated in nth order in ϵ is made of the weighted sums of all possible graphs made of vertex (a) of Fig. 2 with the appropriate number of external legs and n independent loop integrations involving at least one bound-state propagator D_{σ_0}. This, along with the asymptotic behavior of the propagator, is enough to study the renormalization properties of this theory.

From Eq. (2.8) we have with the sources off

$$S_{\sigma_0}(p) = \frac{1}{\gamma p - m_0}, \tag{2.24}$$

so, for large momentum,

$$S_{\sigma_0}(p) \underset{p\to\infty}{\sim} \frac{1}{p}. \tag{2.25}$$

From (2.21) and (2.18) it follows that with the sources off

$$D_{\sigma_0}^{-1}(x,y)\big|_{J=\bar{n}=n=0} = \delta(x-y) - G_0^2\, i \operatorname{tr} S_{\sigma_0}(x,y) S_{\sigma_0}(y,x), \tag{2.26}$$

which can be written in momentum space as

$$D_{\sigma_0}^{-1}(q^2) = 1 - G_0^2\, i\Sigma(q^2, m_0), \tag{2.27}$$

with

$$\Sigma(q^2,m_0) \equiv \operatorname{tr}\int \frac{d^2p}{(2\pi)^2}\left(\frac{1}{\gamma(p+q)-m_0+i\epsilon}\right) \times\left(\frac{1}{\gamma p - m_0 + i\epsilon}\right). \tag{2.28}$$

In these equations we have recognized with the sources off (or independent of space-time) that $\sigma_0(x)$ is independent of space-time, so that we can introduce the constant

$$m_0 = g_0\sigma_0 . \tag{2.29}$$

m_0 is, of course, further related to the divergent mean-field self-consistency condition (2.16). Since this is not directly related to our demonstration of renormalizability but rather to the relations among the renormalized parameters, we shall simply

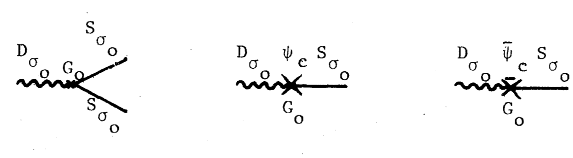

FIG. 2. The basic vertices that occur in the expansion of W_ϵ. When the external Fermi sources are turned off, only the vertex (a) is nonvanishing.

recognize m_0 as the lowest-order renormalized mass of the fermion.[11] The exciting aspects of the theory arising from (2.15) are discussed in Sec. III.

$\Sigma(q^2, m_0)$ as defined by (2.28) is logarithmically divergent, so it must be evaluated through a subtraction procedure. Subtracting at zero momentum, it can be written in the form

$$\Sigma(q^2, m_0) = \Sigma(0, m_0) + \mathrm{sub}_1^0 \Sigma(q^2, m_0)\,. \tag{2.30}$$

$\Sigma(0, m_0)$, if explicitly evaluated, carries the logarithmic divergence, while it is easily shown that

$$\mathrm{sub}_1^0 \Sigma(q^2, m_0) = \frac{iN}{\pi}\left[\frac{1}{2}\left(\frac{4m_0^2 - q^2}{-q^2}\right)^{1/2} \ln\left(\frac{1 + [-q^2/(4m_0^2 - q^2)]^{1/2}}{1 - [-q^2/(4m_0^2 - q^2)]^{1/2}}\right) - 1\right]. \tag{2.31}$$

From (2.27) it follows that the bound-state propagator is

$$D_{\sigma_0}^{-1}(p^2) = 1 - G_0^2\, i[\Sigma(0) + \mathrm{sub}_1^0(q^2, m_0)]\,. \tag{2.32}$$

We shall show in the next section that $D_{\sigma_0}(0)$ is not arbitrary but related to the other parameters of this theory as an expression of the composite nature of D_{σ_0} and condition (2.16).

From Eqs. (2.31) and (2.32) we can study $D_{\sigma_0}(p^2)$ asymptotically for large p^2 to find

$$D_{\sigma_0}(p^2) \underset{p^2 \to \infty}{\sim} \frac{1}{\ln p^2}\,. \tag{2.33}$$

Since we know all graphs are made up from vertex (a) of Fig. 2, which is in turn constructed from (2.24) and (2.32), we can identify the superficially divergent graphs of the theory. If for the moment we ignore the inverse logarithmic behavior of (2.33) and pretend that

$$D_{\sigma_0}(p^2) \underset{p^2 \to \infty}{\sim} 1\,, \tag{2.34}$$

we find that $W(G)$, the degree of superficial divergence of a graph, is given as

$$W(G) = 2 - n_\sigma - \tfrac{1}{2} n_f\,. \tag{2.35}$$

Here n_σ is the number of external σ lines and n_f is the number of external fermion lines on the graph under consideration. Of course, because of the use of (2.34) instead of (2.33), Eq. (2.35) overestimates the degree of superficial divergence. Where necessary we shall take that into account in the enumeration of the graphs with $W(G) \geq 0$ illustrated in Fig. 3. This figure does not include the vacuum graphs, which are divergent but irrelevant for our discussion, or the graphs with one external σ line, which are effectively absorbed into fermion mass renormalization. The graphs of Fig. 3(a) need one subtraction to make them superficially finite. This subtraction is identified with the boson mass renormalization. In addition it is convenient to subtract this graph once again and designate the associated superficially finite parameter as Z_3. The graphs of Fig. 3(b) are slightly more convergent than logarithmic because of (2.33) but they still need two subtractions. The first corresponds to a fermion mass renormalization and the second to the fermion wave-function renormalization Z_2. The graphs of Fig. 3(c) are very slightly divergent because of (2.33) behaving roughly as $\ln(\ln \Lambda)$, where Λ is a cutoff, and hence they still require one subtraction. Associated with this subtraction is the renormalization parameter

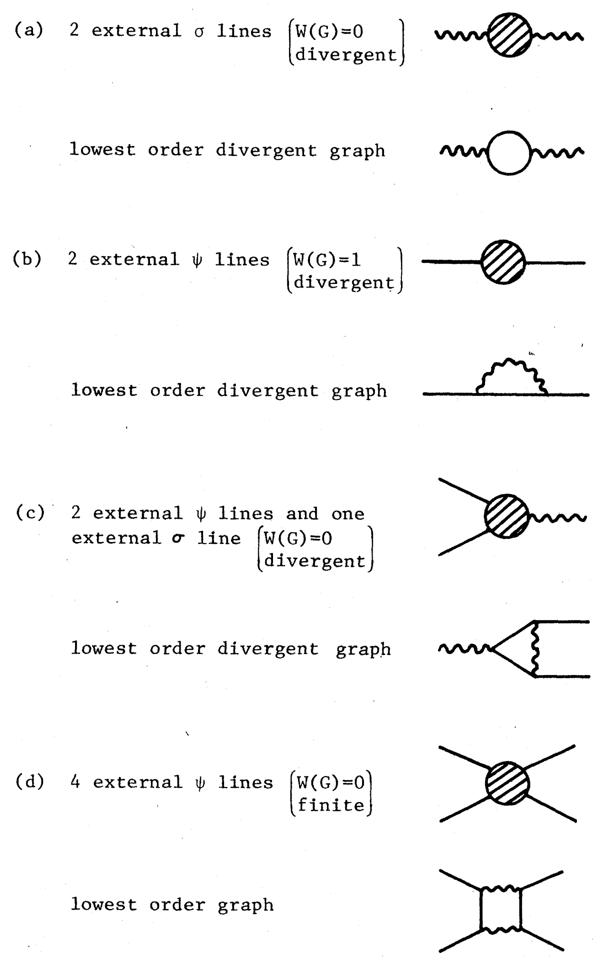

FIG. 3. Classes of graphs which are divergent by power-counting arguments. Because $D_{\sigma 0}$ behaves for large k^2 like $O(1/\ln k^2)$, the class (d) graphs are actually finite.

Z_1. The graphs of Fig. 3(d), although naively divergent are actually superficially finite. This is illustrated by the lowest-order graph, which behaves like

$$\int \frac{d^2k}{k^4(\ln k)^2} \sim \frac{1}{\ln\Lambda}$$

for large cutoff. It is primarily because of this fact that our expansion is interesting since we do not generate any divergent structure of the form of the original interaction in (2.1). Thus, in counterterm language we do not need a counterterm of the four-fermion form added to Lagrangian (2.2). This is a feature which survives in the four-dimensional models and makes it possible to renormalize an interaction that is nonrenormalizable in a coupling-constant expansion.

We conclude that the possible divergences of this theory are contained in the parameters m (the fermion mass), u^2 (the bound-state boson mass), and Z_2 and Z_1. Becuase of the trilinear vertex structure, the Z's only occur in the combination

$$G_0^2 \frac{Z_3 Z_2^2}{Z_1^2} \equiv g_R^2 . \tag{2.36}$$

This is evident, but will be demonstrated elsewhere in our analysis of the Schwinger-Dyson Green's function equations. Thus the parameters of this theory appear to be m, u^2, and g_R^2. We shall show elsewhere[9] that to all orders only one of these is independent. In the next section we demonstrate to lowest order that u^2 and g_R are fixed by m. Finally, we note that, despite the resemblance of (2.2) to a theory with a fundamental field σ, the absence of a boson kinetic term means that this model is not super-renormalizable.

III. THE ZERO-MOMENTUM GREEN'S FUNCTIONS

In the last section we established that $(\bar\psi\psi)^2$ is renormalizable when evaluated in the mean-field approximation in two space-time dimensions and is parametrized by three parameters u^2, m^2, and g_R^2. In this section we demonstrate to lowest order that only one of these parameters is independent and calculate at zero momentum the lowest-order connected one-particle irreducible σ renormalized Green's functions. These results will be generalized to all orders elsewhere.

The key to this analysis is the gap equation (2.16) in the absence of fermion sources[12]

$$\sigma_0(x) = G_0 i \,\mathrm{tr} S_{\sigma_0}(x,x) + J(x) \equiv \sigma_0(x; J(x)) . \tag{3.1}$$

In order to derive equations relating all the lowest-order meson Green's functions to S_{σ_0}, we differentiate (3.1) with respect to J. It is convenient to use the chain rule to write

$$\begin{aligned}\frac{\delta}{\delta J(y)} &= \int d^2z \,\frac{\delta\sigma_0(z)}{\delta J(x)} \frac{\delta}{\delta\sigma_0(z)} \\ &= \int d^2z\, D_{\sigma_0}(y,z) \frac{\delta}{\delta\sigma_0(z)} .\end{aligned} \tag{3.2}$$

Next observe that given $F(\sigma_0(x; J(x)))$ an implicit function of the source $J(x)$, we may compare this function for two different source dependences $J_b(x)$ and $J_a(x)$ through a Taylor expansion

$$\begin{aligned}F(\sigma_0(x;J_b(x))) &= F(\sigma_0(x;J_a(x))) + \int d^2x_1[\sigma_0(x_1;J_b(x_1)) - \sigma_0(x_1;J_a(x_1))]\left[\frac{\delta F(\sigma_0(x;J(x)))}{\delta\sigma_0(x_1;J(x_1))}\right]_{J=J_a} \\ &+ \tfrac{1}{2}\int d^2x_1 \int d^2x_2 [\sigma_0(x_1;J_b(x_1)) - \sigma_0(x_1;J_a(x_1))][\sigma_0(x_2;J_b(x_2)) - \sigma_0(x_2,J_a(x_2))] \\ &\times\left[\frac{\delta^2}{\delta\sigma_0(x_1;J(x_1))\,\delta\sigma_0(x_2;J(x_2))} F(\sigma_0(x;J(x)))\right]_{J=J_a} + \cdots .\end{aligned} \tag{3.3}$$

If we choose both J_a and J_b to be independent of coordinate, this equation simplifies considerably to become

$$\begin{aligned}F(\sigma_0(J_b)) &= F(\sigma_0(J_a)) + [\sigma_0(J_b) - \sigma_0(J_a)] \int d^2x_1 \left[\frac{\delta F(\sigma_0(x;J(x)))}{\delta\sigma_0(x_1;J(x_1))}\right]_{\sigma_0(x;J(x))=\sigma_0(J_a)} \\ &+ \tfrac{1}{2}[\sigma_0(J_b) - \sigma_0(J_a)]^2 \int d^2x_1 \int d^2x_2 \left[\frac{\delta^2 F(\sigma_0(x;J(x)))}{\delta\sigma_0(x_1;J(x_1))\,\delta\sigma_0(x_2,J(x))}\right]_{\sigma_0(x;J(x))=\sigma_0(J_a)} + \cdots .\end{aligned} \tag{3.4}$$

Here for space-time constant sources J we have introduced the notation

$$\sigma_0(J) \equiv \sigma_0(0; J(0)) .$$

Equation (3.4) gives us the relation between functional differentiation and normal differentiation for space-time independent sources

$$\frac{dF(\sigma_0(J))}{d\sigma_0(J)} = \int d^2x_1 \left[\frac{\delta F(\sigma_0(x;J(x)))}{\delta\sigma_0(x_1;J(x_1))}\right]_{\sigma_0(x;J(x))=\sigma_0(J)} \tag{3.5a}$$

$$\frac{d^2F(\sigma_0(J))}{d^2\sigma_0(J)}=\int d^2x_1\int d^2x_2\left[\frac{\delta^2F(\sigma_0(x;J(x)))}{\delta\sigma_0(x_1;J(x_1))\delta\sigma_0(x_2;J(x_2))}\right]_{\sigma_0(x;J(x))=\sigma_0(J)} \tag{3.5b}$$

or in general

$$\frac{d^nF(\sigma_0(J))}{d^n\sigma_0(J)}=\int d^2x_1\cdots\int d^2x_n\left[\frac{\delta^nF(\sigma_0(x;J(x)))}{\prod_i\delta\sigma_0(x_i;J(x_i))}\right]_{\sigma_0(x;J(x))=\sigma_0(J)}. \tag{3.5c}$$

Before we analyze the σ Green's functions in terms of these constant-source relations, it is of interest to examine (3.1) for a constant source J. In that case, using (2.29) we have for the stable solution[10] that

$$m_0=m_0\left[\frac{G_0^2\,i\,\mathrm{tr}S_{\sigma_0}(x,x)}{m_0}\right]+G_0J\,. \tag{3.6}$$

Introducing

$$f(m_0)=\frac{G_0^2\,i\,\mathrm{tr}S_{\sigma_0}(x,x)}{m_0}, \tag{3.7}$$

(3.6) becomes

$$m_0=m_0f(m_0)+G_0J \tag{3.8a}$$

or

$$1=f(m_0)+\frac{G_0J}{m_0}. \tag{3.8b}$$

Explicit evaluation of $f(m_0)$ in the context of its propagator definition is, of course, not meaningful since we then see

$$f(m_0)=\frac{iNG_0^2}{2\pi^2}\int\frac{d^2p}{p^2-m_0^2+i\epsilon}$$
$$=\frac{N}{2\pi}G_0^2\ln\left(\frac{1+\Lambda^2}{m_0^2}\right). \tag{3.9a}$$

Here Λ is introduced as a symmetric Euclidean cutoff.[13] That (3.9a) is divergent should be no surprise, since as mentioned in the preceding section and as is apparent there, this divergence is associated with the fermion mass renormalization. Accordingly, $f(m_0)$ is not defined except through a subtraction procedure. The remaining finite part of $f(m_0)$ which is all that of interest is then determined by Eq. (3.8). The cutoff form of (3.9a) is, however, useful in that we can evaluate derivatives of $f(m_0)$ all of which are finite by differentiating the right-hand side. The results that we obtain are, of course, cutoff independent and are identical to those obtained by just evaluating finite integrals involving multiple fermion propagators. Consequently, from (3.9a) and (3.7) we find the following useful catalog of results:

$$\frac{d^nf(m_0)}{dm_0^n}=(-1)^n(n-1)!\,\frac{NG_0^2}{\pi}\,\frac{1}{m_0^n}, \tag{3.9b}$$

so, for $n\geq 2$

$$\left[\frac{d}{d\sigma_0(J)}\right]^nG_0\,i\,\mathrm{tr}S_{\sigma_0}(x,x)=\left(\frac{d}{dm_0}\right)^n[G_0^{n-1}m_0f(m_0)]$$
$$=\frac{NG_0^{1+n}}{\pi m_0^{n-1}}(-1)^{(n-1)}(n-2)!\,, \tag{3.10}$$

while for the special case $n=1$ using (3.8) we have

$$\frac{d}{d\sigma_0(J)}G_0\,i\,\mathrm{tr}S_{\sigma_0}(x,x)=\frac{d}{dm_0}m_0f(m_0)$$
$$=1-\frac{N}{\pi}G_0^2-\frac{G_0J}{m_0}. \tag{3.11}$$

Now we may begin the actual evaluation of the lowest-order σ-field Green's functions at zero momentum. We use (3.5c) to find

$$\frac{dJ}{d\sigma_0(J)}=\int d^2x_1\frac{\delta J(x)}{\delta\sigma_0(x_1;J(x_1))}$$
$$=\int d^2x_1D_{\sigma_0}^{-1}(x,x_1)$$
$$=D_{\sigma_0}^{-1}(0), \tag{3.12}$$

where it is understood that the last equality is for the propagator in momentum space. We use (3.12) and (3.5c) again to get the general relation

$$\frac{d^nJ}{d\sigma_0^n(J)}=\int d^2x_1\cdots\int d^2x_n\frac{\delta}{\delta\sigma_0(x_2;J(x_2))}\cdots\frac{\delta}{\delta\sigma_0(x_n;J(x_n))}D_{\sigma_0}^{-1}(x,x_1)$$
$$\equiv\int d^2x_1\cdots\int d^2x_n\Gamma^{n+1}(x,x_1,\ldots,x_n)=\Gamma^{n+1}(0,0,\ldots,0_n). \tag{3.13}$$

Here we have introduced the usual amputated single-particle irreducible vertices Γ. It is understood that the last term in this relation is written in momentum space.[14] We now generate a whole string of zero-momentum constant-source identities by differentiating (3.8a) with respect to $\sigma(J)$ to obtain, using (3.10),

$$\Gamma^{n+1}(0,\ldots,0_n)=\frac{(-1)^nNG_0^{n+1}}{\pi m_0^{n-1}}(n-2)! \tag{3.14}$$

for $n \geq 2$. In the special case $n=1$, using (3.11),

$$D_{\sigma_0}{}^{-1}(0) = \frac{NG_0{}^2}{\pi} + \frac{G_0 J}{m_0}, \tag{3.15}$$

which, for the case of final interest $J=0$, becomes

$$D_{\sigma_0}{}^{-1}(0) = \frac{NG_0{}^2}{\pi}. \tag{3.16}$$

We thus see that the unrenormalized propagator at zero momentum is not arbitrary, as might be anticipated from the necessity of bound-state mass renormalization pointed out in the preceding section. Instead, we have found, because of the composite-field nature of σ as expressed through the gap equation (3.1), that the bound-state mass renormalization is determined in lowest order through the mass renormalization of the fundamental Fermi field ψ.[15] Inserting (3.16) into (2.32) and using (2.31) it follows that

$$D_{\sigma_0}{}^{-1}(p^2) = \frac{G_0{}^2 N}{2\pi}\left(\frac{4m_0{}^2 - q^2}{-q^2}\right)^{1/2} \times \ln\left(\frac{1+[-q^2/(4m_0{}^2-q^2)]^{1/2}}{1-[-q^2/(4m_0{}^2-q^2)]^{1/2}}\right). \tag{3.17a}$$

In the region $0 \leq q^2 \leq 4m_0{}^2$, this is more familiarly

$$D_{\sigma_0}{}^{-1}(p^2) = \frac{G_0{}^2 N}{\pi}(4m_0{}^2 - p^2)[p^2(4m_0{}^2-p^2)]^{-1/2} \times \tan^{-1}\left(\frac{p^2}{4m_0{}^2-p^2}\right)^{1/2}, \tag{3.17b}$$

which demonstrates that $D_{\sigma 0}(p^2)$ develops a bound-state pole $m_\sigma{}^2$ at threshold $m_\sigma{}^2 = 4m_0{}^2$. As pointed out in Sec. II the relevant quantity in the mean-field expansion scheme is $G_0{}^2 D_{\sigma_0}(p^2)$, which is now seen from (3.18) to be independent of G_0 except through the implicit dependence of the lowest-order renormalized Fermi mass m_0. From (3.18) we can directly calculate the bound-state wave-function renormalization at zero momentum to find

$$Z_3{}^{-1} \equiv -\left[\frac{dD_{\sigma_0}{}^{-1}(p^2)}{dp^2}\right]_{p^2=0} = \frac{G_0{}^2 N}{12\pi m_0{}^2}. \tag{3.18}$$

It is convenient to use this definition to write yet another form of the propagator

$$\bar{D}_{\sigma_0}{}^{-1}(k^2) = -k^2 + (G_0{}^2 Z_3)[N/\pi - i\,\mathrm{sub}_2^0\Sigma(k^2, m_0)]. \tag{3.19}$$

Since to this lowest order the fermion propagator is given by (2.24) it follows that $Z_2 = 1$. Similarly, the vertex function to this order has no structure, and hence $Z_1 = 1$. It follows, using (2.36) and (3.18), that the lowest-order and dimensional coupling constant is

$$g_R{}^2 = \frac{12\pi m_0{}^2}{N}. \tag{3.20}$$

Thus we see that the lowest-order renormalized bound-state propagator as given by (3.19) depends only on m_0. Note that as anticipated the coupling decreases as $1/N$, so is small as N becomes large. However, even if N is 1, we can obtain a small expansion parameter by letting $m_0{}^2 = \xi u^2$, where u is a fixed massive parameter and ξ is a variable parameter which can be taken to be arbitrarily small. This identification of parameter suggests a re-expansion of our renormalized mean-field results.

The vertex functions at zero momentum can now be written in terms of renormalized quantities. Defining in the usual way

$$\bar{\Gamma}_n = Z_3{}^{n/2}\Gamma_n \quad (n > 3), \tag{3.21}$$

we see that (3.14) with (3.19) and (3.20) yields

$$\bar{\Gamma}_{n+1} = g_R{}^{n+1}\frac{(-1)^n N(n-2)!}{\pi m_0{}^{n-1}} = \left(\frac{12\pi}{N}\right)^{(n+1)/2}\frac{m_0{}^2}{\pi}(-1)^n N(n-2)!\,. \tag{3.22}$$

Note that this formula leads to the relation

$$\bar{\Gamma}_4 = -12 g_R{}^2. \tag{3.23}$$

Relations of exactly this type hold between the renormalized couplings of the four-dimensional analog of this model despite its vastly more complicated renormalization structure. This is because these relations follow from constant-source identities.[9]

We can now easily derive a formula for the vertex functions at arbitrary momentum. The derivation is similar to the constant-momentum derivation, but for completeness we provide the details. First recall that

$$\Gamma^{n+1}(x, x_1 \cdots x_n) = \frac{\delta^n J(x)}{\delta\sigma_0(x_1)\cdots\delta\sigma_0(x_n)}, \tag{3.24}$$

so using (3.1) we find that

$$\Gamma^{n+1}(x, x_1\cdots x_n) = -G_0 i \frac{\delta^n}{\delta\sigma_0(x_1)\cdots\delta\sigma_0(x_n)}\mathrm{tr}S_{\sigma_0}(x,x). \tag{3.25}$$

Transforming this to momentum space, subtracting once, and using (3.14), it follows that

$$\Gamma^{n+1}(p_1 - p_n) = \frac{(-1)^n N G_0{}^{n+1}}{\pi m_0{}^{n-1}}(n-2)! - G_0 i\,\mathrm{sub}_1^0\,\mathrm{F.T.}\frac{\delta^n}{\sigma_0(x_1)\cdots\delta\sigma_0(x_n)}\mathrm{tr}S_{\sigma_0}(x,x). \tag{3.26}$$

Here, F.T. means Fourier transform. Using (3.21) this may be rewritten in the renormalized form

$$\bar\Gamma^{n+1}=\frac{(-1)^n N g_R^{\ n+1}}{\pi m_0^{\ n-1}}(n-2)!-g_R^{\ n+1}\,i\,\mathrm{sub}_1^0\,\mathrm{F.T.}\,\frac{\delta^n}{\delta G_0\sigma_0(x_1)\cdots\delta G_0\sigma_0(x_n)}\,\mathrm{tr}S_{\sigma_0}(x,x)\,. \tag{3.27}$$

IV. EQUIVALENT MODELS WITH ELEMENTARY BOSONS

In this section we will demonstrate that there is an infinite class of models with elementary bosons which, with the appropriate choice of parameters, result in renormalized Green's functions identical to those of the $(\bar\psi\psi)^2$ model. This is an every-order result in the mean-field expansion.

We begin this demonstration by considering the two-space-time-dimensional Lagrangian density

$$\mathcal{L}_\sigma=\bar\psi(i\gamma\partial-M-g\sigma)\psi+\tfrac{1}{2}(\partial_\mu\sigma)^2+H(\sigma)+J\sigma+\bar n\psi+\bar\psi n\,. \tag{4.1}$$

It is understood that, as before, ψ has N components and that H has a power-series expansion of the form

$$H(\sigma)=\sum_{n=2}^{\infty}h_n\sigma^n\,. \tag{4.2}$$

Note that Lagrangian (4.1) differs markedly from (2.2) in that explicit kinetic terms are present for the σ field, which is consequently a true canonical field. It is easy to see that (4.1) is a super-renormalizable theory in a conventional coupling-constant perturbation expansion. This Lagrangian is also super-renormalizable in our mean-field expansion.

The vacuum functional, as usual, is given by

$$Z^\sigma(\bar n,n,J)=\mathrm{const}\times\int[d\sigma][d\bar\psi][d\psi]e^{i\int\mathcal{L}_\sigma d^2x}\,.$$

Performing the integration over Fermi variables we find

$$Z^\sigma(\bar n,n,J)=\mathrm{const}\times\int[d\sigma]e^{iF_\sigma(\sigma,\bar n,n,J)}\,, \tag{4.3}$$

where

$$F_\sigma(\sigma,\bar n,n,J)=\int d^2x\Big(\tfrac{1}{2}[\partial_\mu\sigma(x)]^2+H(\sigma(x))+J(x)\sigma(x)-i\,\mathrm{tr}[\ln{}_gS_\sigma^{-1}(x,y)]_{x=y}-\int d^2y\bar n(x)\,{}_gS_\sigma(x,y)n(y)\Big)\,. \tag{4.4}$$

The definition

$${}_gS_\sigma^{-1}(x,y)\equiv[i\gamma\partial_\lambda-M-g\sigma(x)]\delta^2(x-y) \tag{4.5}$$

has been made.

A mean-field expansion is performed as before by studying the expansion of

$$Z_\epsilon^\sigma\equiv\int[d\sigma]e^{(i/\epsilon)F_\sigma(\sigma,\bar n,n,J)}$$

term by term in an asymptotic series in powers of ϵ, renormalizing, identifying a new small parameter, and then letting $\epsilon\to1$. The requirement (2.11) that the expansion is made about an extremum for this theory is

$$\partial^2\sigma_0(x)-\sum_{n=2}^{\infty}nh_n\sigma_0^{\ n-1}(x)=J(x)-g\bar\psi_c^g\psi_c^g+ig\,\mathrm{tr}\,{}_gS_{\sigma_0}(x,x)\,. \tag{4.6}$$

Here,

$$\psi_c^g(x)\equiv\int d^2y\,{}_gS_{\sigma_0}(x,y)n(y) \tag{4.7a}$$

and

$$\bar\psi_c^g(x)\equiv\int d^2y\bar n(y)\,{}_gS_{\sigma_0}(y,x)\,. \tag{4.7b}$$

When the sources are off, it is useful to introduce the notation

$$M_T=M+g\sigma_0\,. \tag{4.8}$$

From (4.6) with the Fermi sources off, J constant, and using (3.7), we obtain the gap equation for this generalized Yukawa model,

$$J=-\sum_{n=2}^{\infty}nh_n\left(\frac{M_T-M}{g}\right)^{n-1}-\frac{M_T}{g}f_g(M_T)\,. \tag{4.9}$$

Here in complete analogy with (3.9a) we have

$$f_g(M_T)=\frac{N}{(2\pi)}g^2\ln\left(1+\frac{\Lambda^2}{M_T^{\ 2}}\right)=\frac{g^2i\,\mathrm{tr}\,{}_gS_\sigma(x,x)}{M_T}\,. \tag{4.10}$$

As emphasized in Sec. III, equations of the form (4.9) are meaningful only within the framework of a subtraction procedure, so this equation in no way sets the cutoff, but only determines the finite part of $f_g(M_T)$. We display the equation in this form because its mass derivatives are finite and cutoff independent, and serve to determine the various zero-momentum Green's functions of the theory as in the preceding section. In particular, Eqs. (3.9b) and (3.10) are valid here if everywhere m_0 and G_0 occur we substitute M_T and g. The analog of (3.11) is

$$\frac{d}{d\sigma_0(J)}\, gi\, \mathrm{tr}\, {}_gS_{\sigma_0}(x,x) = \frac{d}{dM_T} M_T f_g(M_T) = -\sum_{n=2}^{\infty} \frac{nh_n g}{M_T}\left(\frac{M_T-M}{g}\right)^{n-1} - \frac{gJ}{M_T} - \frac{Ng^2}{\pi}. \tag{4.11}$$

Using Eq. (3.12) with (4.9), it follows that if as before the boson propagator is defined by Eq. (2.19) then to lowest order we have

$${}_gD_{\sigma_0}^{-1}(0) = \frac{gJ}{M_T} + \frac{Ng^2}{\pi} + \sum_{n=2}^{\infty}\left[2n - n^2 - \left(\frac{M}{M_T}\right)n\right] h_n \left(\frac{M_T-M}{g}\right)^{n-2}. \tag{4.12}$$

Similarly, we can calculate the rest of the vertex functions at zero momentum using (4.9) and (3.13) to find

$$\Gamma_{n+1}(0,\ldots,0) = \frac{(-1)^n (n-2)!\, Ng^{n+1}}{\pi M_T^{\,n-1}} - \sum_{m=n+1}^{\infty} \frac{m!}{(m-n-1)!} h_m \left(\frac{M_T-M}{g}\right)^{m-n-1} \tag{4.13}$$

for $n \geq 2$.

Using (2.19) again and proceeding as before, it follows that the full lowest propagator is

$${}_gD_{\sigma_0}^{-1}(x,y) = \frac{\delta J(x)}{\delta\sigma_0(y)}. \tag{4.14}$$

With the Fermi sources off, this is constant

$${}_gD_{\sigma_0}^{-1}(x,y) = \left(\partial^2 - \sum_{n=2}^{\infty} n(n-1)\, h_n \sigma_0^{\,n-2}(x)\right)\delta(x-y) - ig^2\, \mathrm{tr}\left[{}_gS_{\sigma_0}(x,y)\, {}_gS_{\sigma_0}(y,x)\right]. \tag{4.15}$$

Note that this equation is structurally nearly identical to (2.26). As we shall show, the differences will become irrelevant in the appropriate limit. In momentum space, (4.15) with $J(x)$ constant is

$${}_gD_{\sigma_0}^{-1}(k^2) = \left(-k^2 - \sum_{n=2}^{\infty} (n)(n-1)\, h_n \sigma_0^{\,n-2}\right) - g^2 i\Sigma(k^2, M_T). \tag{4.16}$$

Here $\Sigma(k^2, M_T)$ is defined by Eq. (2.28) and given explicitly by (2.30) and (2.31). Using (4.12) it follows that

$${}_gD_{\sigma_0}^{-1}(k^2) = -k^2 + \sum_{n=2}^{\infty} n\left(2 - \frac{M}{M_T} - n\right) h_n \left(\frac{M_T-M}{g}\right)^{n-2} + g^2\left[\frac{N}{\pi} - i\, \mathrm{sub}_1^0 \Sigma(k^2, M_T)\right] + \frac{gJ}{M_T}. \tag{4.17}$$

Using

$${}_gZ_3^{-1} = -\left[\frac{dD^{-1}(k^2)}{dk^2}\right]_{k^2=0}$$

and (3.19), we now find for this model that

$${}_gZ_3^{-1} = 1 + \frac{g^2 N}{12\pi M_T^{\,2}}. \tag{4.18}$$

With this the meson propagator (4.17) may be rewritten as

$${}_gD_{\sigma_0}^{-1}(k^2) = {}_gZ_3^{-1}\, {}_g\bar{D}_{\sigma_0}^{-1}(k^2), \tag{4.19}$$

where

$${}_g\bar{D}_{\sigma_0}^{-1}(k^2) = -k^2 + u^2\, {}_gZ_3 + g^2\, {}_gZ_3\left[\frac{N}{\pi} - i\, \mathrm{sub}_2^0(k^2)\right] \tag{4.20}$$

and we have introduced the convenient definition

$$u^2 = \sum_{n=2}^{\infty} n\left(2 - \frac{M}{M_T} - n\right) h_n \left(\frac{M_T-M}{g}\right)^{n-2}. \tag{4.21}$$

Similarly, the meson vertex functions may be examined with the sources off using

$${}_g\Gamma^{n+1}(x, x_1 \cdots x_n) = \frac{\delta^n}{\delta\sigma_0(x_1)\cdots\delta\sigma_0(x_n)} J(x) = -\sum_{m=2}^{\infty} \frac{m!}{(m-n-2)!} h_m \left(\frac{M_T-M}{g}\right)^{m-n-1} - ig\frac{\delta^n}{\delta\sigma_0(x_1)\cdots\delta\sigma_0(x_n)}\, \mathrm{tr}\, {}_gS_{\sigma_0}(x,x), \quad n \geq 2. \tag{4.22}$$

In momentum space with one subtraction and using (4.13) we then find

$${}_g\Gamma^{n+1}(p_1,\ldots,p_n) = \frac{(-1)^n (n-2)!\, Ng^{n+1}}{\pi M_T^{\,n-1}} - \sum_{m=2}^{\infty} \frac{m!\, h_m}{(m-n-1)!}\left(\frac{M_T-M}{g}\right)^{m-n-1} - ig^{n+1}\, \mathrm{sub}_1^0\, \mathrm{F.T.} \frac{\delta^n}{\delta g\sigma_0(x_1)\cdots\delta g\sigma_0(x_n)}\, \mathrm{tr}\, {}_gS_{\sigma_0}(x,x). \tag{4.23}$$

Now we are going to demonstrate the fact that to this lowest order the renormalized Green's functions of the two dimensional $(\bar{\psi}\psi)^2$ model and the generalized Yukawa model are identical for an appropriate choice of the renormalization parameters. The first requirement for this to be so is that

$$ {}_gS_{\sigma_0}{}^{-1}(x,y)|_{J=\bar{\eta}=\eta=0} = S_{\sigma_0}{}^{-1}(x,y)|_{J=\bar{\eta}=\eta=0} \,. $$

From Eqs. (4.8) and (2.29) it follows that

$$ M_T = m_0 \tag{4.24} $$

fulfills the requirement. This just says that we choose the undetermined (because of renormalization) lowest-order Fermi masses to be equal. The requirement that the renormalized boson propagators, as given by (4.20) and (3.19) be equal is more restrictive. Using (3.20) it results in the conditions

$$ g^2\,{}_gZ_3 = G_0{}^2 Z_3 = g_R{}^2 = \frac{12\pi m_0{}^2}{N} \tag{4.25} $$

and

$$ u^2\,{}_gZ_3 = 0\,. \tag{4.26} $$

Using the definitions of ${}_gZ_3$ as given by (4.18), Eq. (4.25) becomes

$$ \frac{g^2}{1+g^2N/12\pi m_0{}^2} = \frac{12 m_0{}^2}{N}\,. \tag{4.27} $$

It is clear that this condition can hold only in the limit

$$ g^2 \to \infty\,, \tag{4.28} $$

in which case it also follows[16] that

$$ {}_gZ_3 \approx O\left(\frac{1}{g^2}\right) \to 0\,. \tag{4.29} $$

Equation (4.26) is automatically satisfied if (4.25) is satisfied through (4.28) and if h_n is picked so that

$$ h_n < g^n\,. \tag{4.30} $$

Of course (4.26) can also be satisfied through complicated cancellations among various terms. We have now matched the renormalized fermion and boson propagators in lowest approximation. The renormalized vertex functions are formed by using the analog of (3.21) on (4.23). If (4.30) is valid, then we find that the extra terms in (4.23) do not contribute and the renormalized vertices in this Yukawa-type model are identical to those given for the four-fermion model by Eq. (3.27). It is straightforward to confirm, moreover, that to this order all the renormalized Green's functions are identical for both models.

This observation is enough to demonstrate that the Green's functions to any order expressed in terms of the lowest-order renormalized Green's functions (with the lowest-order renormalization factors removed from external lines) are identical in both models. This is because these any-order Green's functions are constructed in terms of the mean-field approximation using the results of Sec. II of this paper. We have shown that the basic vertex which occurs for any G_0 or as $g\to\infty$ is only the trilinear one and in either model it occurs in such a way that it always has the weight g_R as given by Eq. (3.20).

We emphasize that these Green's functions in general need further renormalization according to the prescriptions of this paper. They have been renormalized only to lowest order since this is adequate for the establishment of the identity of the fully renormalized Green's functions of the two theories. Similarly, we have argued that, when $g\to\infty$, ${}_gZ_3\to 0$, where ${}_gZ_3$ is the lowest-order wave-function renormalization. Because of the additive nature of the contributions to ${}_gZ_3{}^{-1}$, it should be clear that in fact ${}_gZ_3$ vanishes as $g\to\infty$ in any order. This and related points having to do with the exact Green's-function equations of the theory will be discussed elsewhere.

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[11]We do not use explicit Lagrangian counterterm renormalization procedures in our discussion but instead regard divergent Green's functions as to be defined by subtracting in momentum space until they are finite and then adding back in the requisite number of powers of momentum multiplied by finite (but undefined except through identities of the theory) constants. This method is obviously fully equivalent to the Lagrangian counterterm method but to our own taste is somewhat more elegant for dealing with mean-field problems.

[12]The analysis that follows is easily generalized to nonvanishing constant Fermi sources, but we do not display the details since the simpler case is adequate for our purposes.

[13]We emphasize that any alternative explicit regularization scheme could have been chosen. In fact, no regularization scheme at all need have been introduced since our results, defined through a subtraction procedure, make no reference to a regularion scheme.

[14]Most readers are probably more familiar with arguments of this type being written in terms of the effective action. It is easily demonstrated to this order that the effective action, defined as

$$\Gamma = W(J,n,\bar{n}) - \frac{\delta W}{\delta n} n - \bar{n}\frac{\delta W}{\delta \bar{n}} - J\frac{\delta W}{\delta J},$$

is given by

$$\Gamma = \iint d^2x\, d^2y\, \bar{\psi}_c(x) S_{\sigma_0}^{-1}(x,y)\psi_c(y) - \int d^2x \left\{ \frac{\sigma_0^2(x)}{2} + iN\mathrm{tr}\,[\ln S_{\sigma_0}^{-1}(x,y)]_{x=y} \right\}.$$

Differentiation of this expression generates all of the vertex functions in the usual manner. The formulation of the text and the effective-action formulation are clearly entirely equivalent. We find the formalism used in the text somewhat simple for higher-order analysis.

[15]This observation and indeed many of the observations made so far in this section are not new but were known in the earliest works on mean-field approximations in four dimensions such as Ref. 2, 6, and 8, as well as in the original work explicitly on quartic scalar fermion couplings in two dimensions (Ref. 10). We have repeated the analysis in detail for the sake of completeness of this work and because the methods used here are a simple variant of methods that work very elegantly for relating renormalized parameters to all orders in this theory as well as the more complicated case of the related theories in four dimensions, as is explained in detail in Ref. 9.

[16]That a relation exists between composite particles of the kind we are discussing and $Z_3 = 0$ has been known for some time. An early discussion of the relation to the Nambu model and many other references is contained in D. Lurie and A. J. Macfarlane, Phys. Rev. 136, B816 (1964). Note that Z_3 in our model is superficially finite and thus naive application of the arguments used in the 1960's which exploited the divergent nature of the higher-dimensional field theories can lead to incorrect results. Detailed discussions of the relationships among the renormalized parameters of mean-field theories equivalent through this mechanism are contained in Ref. 1 and 9. A recent discussion of $Z_3 = 0$ in the context of models of the type we are examining is also contained in Ken-ichi Shizuya, University of Tokyo report (unpublished).

[17]If care is not taken, it is very easy to fall into an interesting trap by insisting that the renormalized propagators of both models coincide. To arrange for this trap we write Eq. (3.17) in the form

$$D_{\sigma_0}^{-1}(p^2) = -G_0^2 iA(p^2).$$

We may expand this expression about an arbitrary subtraction mass u_0^2 with $0 \leq u_0^2 \leq 4m_0^2$ to find

$$\begin{aligned} D_{\sigma_0}^{-1}(p^2) \equiv Z_3^{u_0^2} D_{\sigma_0}^{-1}(p^2) &= \frac{1}{G_0^2 B(u_0^2)} D_{\sigma_0}^{-1}(p^2) \\ &= \frac{-iA(u_0^2)}{B(u_0^2)} + (-p^2 + u_0^2) \\ &\quad - \frac{i}{B(u_0^2)} \mathrm{sub}_2^{u_0^2} A(p^2). \end{aligned}$$

Similarly, for the Yukawa-type model it is easy to find

$$\begin{aligned} {}_g\bar{D}_{\sigma_0}^{-1}(p^2) &\equiv {}_g Z_3^{u_0^2}\, {}_g D_{\sigma_0}^{-1}(p^2) \\ &= \frac{1}{1+g^2 B(u_0^2)}\, {}_g D_{\sigma_0}^{-1}(p^2) \\ &= \frac{u^2 - u_0^2 - g^2 iA(u_0^2)}{1+g^2 B(u_0^2)} + (-p^2 + u_0^2) \\ &\quad - \frac{g^2}{1+g^2 B(u_0^2)} \mathrm{sub}_2^{\mu_0^2} A(p^2). \end{aligned}$$

In the above,

$$B(u_0^2) \equiv i\left[\frac{\partial}{\partial p^2} A(p^2)\right]_{p^2 = u_0^2}.$$

It is of course natural to choose the subtraction point to be at the "bound-state" mass $4m_0^2$, in which case $A(4m_0^2) = 0$, and the condition that the two renormalized propagators have the same mass "pole" is

$$\frac{u_0^2 - u^2}{1+g^2 B(u^2)} = 0.$$

Obviously if $u^2=u_0{}^2$, this condition is met. If we examine a σ-type model, where in Eq. (4.1) we insert

$$H(\sigma)=-\tfrac{1}{4}(\sigma^2-f^2)^2+\tfrac{1}{4}f^4,$$

the condition $u_0{}^2=u^2$ requires that $\lambda=2g^2$. Thus the bare Yukawa parameters appear to be related by the condition that the σ propagators have a pole at the same location as the four-fermion model. However, this condition is excessively restrictive since in fact the total equality of the propagators requires that

$$\frac{g^2}{1+g^2B(4m_0{}^2)}=\frac{1}{B(4m_0{}^2)},$$

which can be satisfied only for $g^2\to\infty$ or ${}_gZ_3^{4m_0{}^2}\to 0$, as was equivalently shown in the text with the subtraction point at zero. $g^2\to\infty$ automatically guarantees that the "poles" of the renormalized propagators in the two different models coincide without any further restrictions on the bare couplings. The condition $\lambda=2g^2$ was originally derived for comparison of the σ model to four-fermion interactions evaluated in the mean-field approximation in four dimensions. In four dimensions the $Z_3\to 0$ type argument given as above is very heuristic since $B(4m^2)$ is divergent, but even in this case any restriction on bare parameters is incorrect. This will be discussed in detail elsewhere.