Gauge fixing and canonical quantization

Michael Creutz, I. J. Muzinich, and Thomas N. Tudron
Brookhaven National Laboratory, Upton, New York 11973
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We study the canonical quantization of non-Abelian gauge fields in the temporal gauge $A_0 = 0$. We impose the constraint condition of Gauss's law by performing a point transformation into any of a large class of noncovariant gauges. The Faddeev and Popov operator arises naturally in this procedure; indeed, we prove the equivalence of all gauges in this class. We discuss the nonexistence of some simple gauges and show how topological considerations reduce the theory to quantum mechanics on an infinite-dimensional periodic hypersurface.

I. INTRODUCTION

The quantum mechanics of gauge fields involves physically irrelevant variables. Indeed, to locally formulate a gauge theory one must introduce potentials carrying a gauge ambiguity.[1] Elimination of these extra degrees of freedom normally proceeds in two steps; first some particular gauge is selected and then the time component of the gauge field is eliminated as a nonlocal dependent variable. In such a manner the four-component vector potential of quantum electrodynamics reduces to two independent fields corresponding to the two polarizations of physical photons.

In recent years path integrals have become the primary formalism for the discussion of quantum field theory. This procedure has many advantages, such as in the derivation of Feynman rules, proof of Ward-Takahashi identities, and study of nonperturbative phenomena. In this language, a gauge-fixing procedure eliminates degrees of freedom corresponding to the orbits of the field under gauge transformations. The action is constant over these orbits; consequently, one obtains a factor proportional to the volume of the gauge group at each space-time point. Faddeev and Popov[2] have shown how physical amplitudes are independent of the particular gauge choice. Their ansatz is then justified by demonstrating the equivalence with canonical quantization in some simple gauge.[3]

From the viewpoint of canonical quantization, equivalence of different gauges is more obscure. Indeed, the Hilbert space structure can depend on the gauge choice; for example, an indefinite-metric space is often used with covariant gauges in quantum electrodynamics. In this paper we present a systematic canonical treatment of non-Abelian gauge fields. Our basic starting point is the temporal gauge, where the time component of the vector potential is set to zero.[4] This gauge is clearly peculiar in that eliminating the time component with the gauge constraint precludes its elimination again as a dependent variable. In consequence, this gauge choice is incomplete and leaves unphysical variables associated with the remaining gauge freedom. Imposing Gauss's law as a constraint condition on physical states serves to eliminate these unwanted degrees of freedom. In previous publications we have discussed the mechanics of this gauge for conventional quantum electrodynamics[5] and for the Higgs mechanism of mass generation.[6]

This paper treats non-Abelian gauge fields by eliminating the gauge variables as ignorable coordinates. We accomplish this by a point transformation into any of a large class of noncovariant gauges; indeed, we show that all gauges in this class are equivalent. This procedure complements and extends the classic work of Faddeev[7] on singular Lagrangians. Using a similar procedure, Gervais and Sakita[8] have emphasized the analogy to the collective coordinate method of treating the quantum mechanics of solitons.[9] Some of the new variables introduced are conjugate to the generators of the remaining gauge symmetry of the Hamiltonian, just as the collective "position" of a soliton is conjugate to the generator of translation symmetry.

Using this formalism we discuss certain recently discovered nonperturbative phenomena. In particular, the ambiguities of the Coulomb gauge pointed out by Gribov[10] are related to certain technical difficulties with our canonical transformation. We also discuss gauge-fixing difficulties related to the well-known tunneling phenomena associated with pseudoparticle solutions to the classical Euclidean theory.[11] We resolve these difficulties by formulating the field theory on an infinite-dimensional periodic hypersurface.

Throughout this paper we work with pure non-

Abelian gauge fields. The introduction of further sources should be straightforward. Although we hope that a thorough understanding of the structure of these theories will shed some light on the mechanism of quark confinement, we make no comments on this phenomenon. We also ignore ultraviolet difficulties associated with the definition of a continuum field theory; our assumption is that the global properties of the theory do not depend on short-distance effects.[12] In particular we ignore the necessity of normal ordering of products of fields at the same space-time point, even for the free theory.

The plan of this paper is as follows. In Sec. II we review the basic features of quantization in the temporal gauge. Section III introduces a compact notation that simplifies formulas in later sections. We introduce in Sec. IV the change of coordinates that separates out the ignorable gauge degrees of freedom. In Sec. V we show the nonexistence of a generalized Coulomb gauge and a hypothetical "orthogonal" gauge. The role of topology is discussed in Sec. VI. Finally in Sec. VII we conclude with some unanswered questions.

II. REVIEW OF THE TEMPORAL GAUGE

We will assume that the gauge group is some connected non-Abelian unitary group. The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}F^\alpha_{\mu\nu}F^\alpha_{\mu\nu}, \tag{2.1}$$

where

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - ef^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu. \tag{2.2}$$

The index α is a group index which runs from one to the dimension of the group [N^2-1 for SU(N)], e is the coupling constant of the theory, and $f^{\alpha\beta\gamma}$ are the structure constants,

$$[\lambda^\alpha, \lambda^\beta] = if^{\alpha\beta\gamma}\lambda^\gamma, \tag{2.3}$$

where the Hermitian matrices λ^α generate the fundamental representation of the group, normalized according to

$$\mathrm{Tr}(\lambda^\alpha\lambda^\beta) = \tfrac{1}{2}\delta^{\alpha\beta}. \tag{2.4}$$

[For SU(2), $f^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma}$ and $\lambda^\alpha = \frac{1}{2}\sigma^\alpha$.] It is sometimes useful to work in a matrix notation, so that

$$A_\mu = A^\alpha_\mu\lambda^\alpha, \quad F_{\mu\nu} = F^\alpha_{\mu\nu}\lambda^\alpha,$$
$$A^\alpha_\mu = 2\,\mathrm{Tr}(A_\mu\lambda^\alpha), \quad F^\alpha_{\mu\nu} = 2\,\mathrm{Tr}(F_{\mu\nu}\lambda^\alpha). \tag{2.5}$$

In this notation we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] \tag{2.6}$$

and

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F_{\mu\nu}. \tag{2.7}$$

This Lagrangian density is invariant under an arbitrary gauge transformation

$$A_\mu \to gA_\mu g^{-1} + \frac{i}{e}(\partial_\mu g)g^{-1}, \tag{2.8}$$

where g specifies a mapping of space-time into the fundamental representation of the group. In the vector notation, the potential transforms under the adjoint representation of the group as

$$A^\alpha_\mu \to R^{\alpha\beta}A^\beta_\mu - \frac{1}{eC}f^{\alpha\beta\sigma}(\partial_\mu RR^{-1})^{\beta\sigma}, \tag{2.9}$$

where C is the quadratic Casimir invariant of the adjoint representation, $C\delta^{\alpha\beta} = f^{\alpha\gamma\rho}f^{\beta\gamma\rho}$. If we parameterize the gauge transformation according to

$$g(\vec{x}, t) = \exp[i\omega^\alpha(\vec{x}, t)\lambda^\alpha], \tag{2.10}$$

then the adjoint representation of g is

$$R^{\alpha\beta}(g) = [\exp(i\omega^\sigma T^\sigma)]^{\alpha\beta}, \tag{2.11}$$

where

$$(T^\alpha)^{\beta\gamma} = -if^{\alpha\beta\gamma}. \tag{2.12}$$

The field strength transforms as

$$F_{\mu\nu} \to gF_{\mu\nu}g^{-1} \tag{2.13}$$

in the matrix notation and

$$F^\alpha_{\mu\nu} \to R^{\alpha\beta}F^\beta_{\mu\nu} \tag{2.14}$$

in the vector notation. As usual, if the field ϕ_i carries a given representation of the group, then we can define its covariant derivative

$$(D_\mu\phi)_i = \partial_\mu\phi_i + ieA^\alpha_\mu v^\alpha_{ij}\phi_j, \tag{2.15}$$

where v^α_{ij} generate that representation.

We go to the temporal gauge $A^\alpha_0 = 0$ and consider a fixed time, so that time dependence is suppressed. The dynamical variables are the space components of the gauge potential A^α_i, and their conjugate momenta are the space components of the electric field,

$$\Pi^\alpha_i = F^\alpha_{0i} \equiv E^\alpha_i, \tag{2.16}$$

We can write the Hamiltonian density as

$$\mathcal{H} = \tfrac{1}{2}E^\alpha_i E^\alpha_i + \tfrac{1}{4}F^\alpha_{ij}F^\alpha_{ij}. \tag{2.17}$$

We impose canonical equal-time commutation relations

$$[E^\alpha_i(\vec{x}), A^\beta_j(\vec{y})] = -i\delta_{ij}\delta^{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \tag{2.18}$$

and obtain the equations of motion by commuting the canonical variables with the Hamiltonian, $\int d^3x\mathcal{H}=H$,

$$\partial_0 A_i^\alpha = E_i^\alpha\,,$$
$$\partial_0 E_i^\alpha = \partial_j F_{ji}^\alpha - ef^{\alpha\beta\gamma}A_j^\beta F_{ji}^\gamma = (D_j F_{ji})^\alpha\,, \tag{2.19}$$

where we have used the definition of the covariant derivative in the adjoint representation. Note that since Gauss's law $(D_iE_i)^\alpha = 0$ does not involve time derivatives, it cannot be obtained as one of Hamilton's equations of motion. It does follow that

$$\partial_0 (D_i E_i)^\alpha = 0\,, \tag{2.20}$$

so that we are allowed to diagonalize simultaneously the Gauss's law operator and the Hamiltonian. Thus we would like to impose Gauss's law as a constraint on the physical states

$$(D_i E_i)^\alpha|\psi\rangle = 0\,. \tag{2.21}$$

We are met by the problem that the spectrum of the Gauss's law operator is continuous and hence its eigenstates are not normalizable. This difficulty was resolved in the Abelian case by taking an appropriate limit on states where $\nabla\cdot E$ is smeared about zero.[5,13] For the non-Abelian theory, Goldstone and Jackiw[14] have managed to satisfy the Gauss's law constraints, but their technique obscures any perturbative expansion. We will impose this constraint in a manner consistent with a perturbative treatment by analyzing the remaining gauge invariance of the theory.

Imposition of the temporal gauge condition $A_0^\alpha = 0$ leaves the residual freedom to perform time-independent gauge transformations,

$$A_i(\vec{x}) \to A_i^g(\vec{x}) = g(\vec{x})A_i(\vec{x})g^{-1}(\vec{x}) + \frac{i}{e}(\partial_i g(\vec{x}))g^{-1}(\vec{x})\,. \tag{2.22}$$

This transformation is implemented by a unitary operator U,

$$UA_iU^{-1} = A_i^g\,. \tag{2.23}$$

It is not difficult to show[15] that

$$U = \exp\left[-\frac{i}{e}\int d^3x E_i^\alpha (D_i\omega)^\alpha\right]\,, \tag{2.24}$$

where we have made the formal definition

$$D_i\omega^\alpha = \partial_i\omega^\alpha - e\epsilon^{\alpha\beta\gamma}A_i^\beta\omega^\gamma\,. \tag{2.25}$$

Here ω^α parametrizes g as in Eq. (2.10).

We will assume that we can compactify space so that the gauge potentials are defined on S^3. This implies that $A_i^\alpha(\vec{x})$ obeys the boundary condition that it falls to zero faster than $1/|\vec{x}|$ as $|\vec{x}|\to\infty$. The class of allowed time-independent gauge transformations is thus restricted to those which approach the identity at spatial infinity. Our only justification for this restriction is that microscopic physics should not depend on conditions imposed at infinity. In a previous publication,[15] a discussion was given of gauge transformations which do not satisfy this boundary condition. The symmetry properties of physical states under such transformations were related to the question of confinement. With the above restriction, however, the gauge transformations can be divided into a set of equivalence classes labeled by the integer winding number[11]

$$n = \frac{1}{24\pi^2}\int d^3x\,\epsilon_{ijk}\,\mathrm{Tr}[(\nabla_i g)g^{-1}(\nabla_j g)g^{-1}(\nabla_k g)g^{-1}]\,. \tag{2.26}$$

The equivalence relation is one of homotopy: Two mappings $g_1(\vec{x})$ and $g_2(\vec{x})$ are homotopic if there exists a function $g(\vec{x},s)$ continuous in s such that $g(\vec{x},0)=g_1(\vec{x})$ and $g(\vec{x},1)=g_2(\vec{x})$ and

$$g(\vec{x},s) \xrightarrow[|\vec{x}|\to\infty]{} I \quad \text{for all } s\,.$$

In SU(2) gauge theory any winding number can be reached by taking the appropriate power of the unit-winding-number transformation

$$g(\vec{x}) = -\exp[-\pi i\vec{\sigma}\cdot\vec{x}/(\vec{x}^2+\rho^2)^{1/2}]\,. \tag{2.27}$$

Discussions of the embedding of SU(2) into larger groups and the topology of larger groups have been given in several papers.[16]

If the gauge transformation belongs to the class $n=0$, so that is can be continuously connected to the identity, then a partial integration can be performed in Eq. (2.24) to give

$$U = \exp\left[\frac{i}{e}\int d^3x (D_iE_i)^\alpha\omega^\alpha\right]\,. \tag{2.28}$$

Thus the statement that a physical state obeys Gauss's law is equivalent to saying that the state is invariant under time-independent gauge transformations of the class $n=0$. We will return in Sec. VI to discuss further the nontrivial winding numbers.

III. COMPACT NOTATION

The field $A_i^\alpha(\vec{x})$ is a function of a space index i, an isospin index α, and a position in space $\vec{x}$. Explicitly displaying all these dependences quickly makes the formulas of the next section unmanageable; consequently, we introduce a compact notation reducing $A_i^\alpha(\vec{x})$ to simply A_i, depending on a single index i. One may either regard this index as a shorthand for the full dependence of $A_i^\alpha(\vec{x})$, or one can imagine A_i as a coefficient in an expansion of $A_i^\alpha(\vec{x})$ in some complete set of functions of i, α, and $\vec{x}$. Treating the electric field similarly, we thus begin our formalism with conjugate Hermitian variables A_i and E_i satisfying

$$[E_i, A_j] = -i\delta_{ij}\,. \tag{3.1}$$

A space-dependent group element characterizing a time-independent gauge transformation is denoted simply by

$$g = \exp(i\lambda^\alpha \omega^\alpha)\,. \tag{3.2}$$

Here the index α symbolizes both a group index and a space position. The gauge group generators λ^α satisfy

$$[\lambda^\alpha, \lambda^\beta] = if^{\alpha\beta\gamma}\lambda^\gamma \tag{3.3}$$

and are normalized

$$\mathrm{Tr}(\lambda^\alpha\lambda^\beta) = \tfrac{1}{2}\delta^{\alpha\beta}\,. \tag{3.4}$$

Although these equations look like Eqs. (2.3) and (2.4), they actually differ in that δ functions in the implied space coordinates have been absorbed in the definitions of $f^{\alpha\beta\gamma}$ and $\delta^{\alpha\beta}$. The structure constants $f^{\alpha\beta\gamma}$ remain totally antisymmetric in their indices.

We write the action of a time-independent gauge transformation as

$$A_i \to (A^g)_i = R_{ij}(g)A_j + \Lambda_i(g)\,, \tag{3.5}$$

$$E_i \to (E^g)_i = R_{ij}(g)E_j\,. \tag{3.6}$$

Although one is usually interested in the adjoint representation, we only require that $R_{ij}(g)$ be some nontrivial real unitary irreducible representation of the gauge group; consequently, we have

$$R_{ij}(g)R_{jk}(g') = R_{ik}(gg')\,, \tag{3.7}$$

$$R_{ij}(I) = \delta_{ij}\,, \tag{3.8}$$

where I is the identity element. We let T^α_{ij} denote the generators of this representation

$$R_{ij}(\exp(i\lambda^\alpha\omega^\alpha)) = (\exp(iT^\alpha\omega^\alpha))_{ij}\,. \tag{3.9}$$

The antisymmetric, Hermitian matrices T^α_{ij} satisfy

$$[T^\alpha, T^\beta] = if^{\alpha\beta\gamma}T^\gamma\,, \qquad (T^\alpha T^\alpha)_{ij} = C\delta_{ij}\,, \tag{3.10}$$

where C is the quadratic Casimir operator of the representation. The inhomogeneous term $\Lambda_i(g)$ in Eq. (3.5) has group combination properties determined by requiring

$$[(A^g)^{g'}]_i = (A^{g'g})_i\,. \tag{3.11}$$

This implies

$$\Lambda_i(g'g) = R_{ij}(g')\Lambda_j(g) + \Lambda_i(g')\,. \tag{3.12}$$

Note that setting g' to the identity I gives

$$\Lambda_i(I) = 0\,. \tag{3.13}$$

Although we have not required $R_{ij}(g)$ to be the adjoint representation, we will have need for the latter. Thus we define

$$S^{\alpha\beta}(g) = 2\,\mathrm{Tr}(g^{-1}\lambda^\alpha g\lambda^\beta)\,. \tag{3.14}$$

This real unitary representation of the gauge group is generated by the structure constants

$$S^{\alpha\beta}(\exp(i\lambda^\gamma\omega^\gamma)) = (\exp(f^\gamma\omega^\gamma))^{\alpha\beta}\,, \tag{3.15}$$

$$(f^\gamma)^{\alpha\beta} = f^{\alpha\beta\gamma}\,. \tag{3.16}$$

The derivative operator in our compact notation can be introduced using the inhomogeneous term $\Lambda_i(g)$ for infinitesimal group rotations. We define

$$\Lambda_i(1 + i\lambda^\alpha d\omega^\alpha) = \frac{1}{e}\nabla^\alpha_i d\omega^\alpha\,. \tag{3.17}$$

Here e is the coupling constant of the theory. From Eq. (2.8) this can be shown to correspond to ordinary differentiation. We will also need the concept of covariant derivative which we introduce through an infinitesimal gauge transformation on A_i

$$(A^{(1+i\lambda^\alpha d\omega^\alpha)})_i = A_i + \frac{1}{e}D^\alpha_i(A)d\omega^\alpha\,. \tag{3.18}$$

From Eqs. (3.5), (3.9), and (3.17) one can readily verify

$$D^\alpha_i(A) = \nabla^\alpha_i + ieT^\alpha_{ij}A_j\,. \tag{3.19}$$

In conventional notation this corresponds to Eq. (2.15). In Appendix A we show the simple behavior of D^α_i under an arbitrary gauge transformation

$$D^\alpha_i(A^g) = R_{ij}(g)S^{\alpha\beta}(g)D^\beta_j(A)\,. \tag{3.20}$$

We also prove in Appendix A an identity that will be useful later

$$D^\alpha_i T^\beta_{ij} - D^\beta_i T^\alpha_{ij} - if^{\alpha\beta\gamma}D^\gamma_j = 0\,. \tag{3.21}$$

We close this section by writing the Hamiltonian in this compact notation

$$H = \tfrac{1}{2}E_iE_i + V(A)\,. \tag{3.22}$$

Here the potential energy $V(A)$ is invariant under the gauge transformation in Eq. (3.5).

IV. A POINT TRANSFORMATION

We now wish to change from the variables A_i to a new set of coordinates that explicitly separate out the gauge degrees of freedom. This respresents a generalization of ordinary polar variables for a system with spherical symmetry.

We begin by writing

$$A_i = (\hat{A}^{\hat{g}})_i = R_{ij}(\hat{g})\hat{A}_j + \Lambda_i(\hat{g}). \tag{4.1}$$

The quantities $\hat{A}$ and $\hat{g}$, which represent our new coordinates, need to be constrained in order to avoid having too many degrees of freedom. For now we assume that we have a set of functions $F^\alpha(A)$ vanishing once on the orbit of any A_i under gauge transformations. Later when we discuss the topology of the gauge group, we will see that this assumption must be relaxed slightly. We now fix $\hat{A}_i$ by requiring

$$F^\alpha(\hat{A}) = 0. \tag{4.2}$$

As there is one such constraint for every generator of the gauge group, the variables $\hat{A}_i$ and $\hat{g}$ now carry the same number of degrees of freedom as the original fields A_i. As an example, the use of $F^\alpha(A) = \nabla_i^\alpha A_i$ in the following formalism relates the temporal gauge to the conventional Coulomb gauge.

The concepts of "left" differentiation of $\hat{g}$ and of a projection operator onto the surface $F^\alpha = 0$ will aid the introduction of momentum variables conjugate to $\hat{A}$ and $\hat{g}$. Under an infinitesimal shift $A_i \to A_i + dA_i$, the corresponding change in $\hat{g}$ gives the left differential $d_L\omega^\alpha$ with the formula

$$\hat{g} \to [I + i(d_L\omega)^\alpha \lambda^\alpha]\hat{g}. \tag{4.3}$$

A similarly defined "right" differential would be related to this by an element of the adjoint representation. We now construct a projection operator from the normals to the surface $F^\alpha = 0$,

$$F_i^\alpha = \frac{\partial}{\partial A_i} F^\alpha \bigg|_{\hat{A}}. \tag{4.4}$$

Using the matrix

$$M^{\alpha\beta} = F_i^\alpha F_i^\beta, \tag{4.5}$$

we express the required projection operator as

$$P_{ij} = \delta_{ij} - F_i^\alpha (M^{-1})^{\alpha\beta} F_j^\beta. \tag{4.6}$$

This object has the properties

$$P_{ij} = P_{ji} = P_{ik}P_{kj}. \tag{4.7}$$

For an infinitesimal change $d\hat{A}_i$ lying in the surface $F^\alpha = 0$ we have

$$P_{ij} d\hat{A}_j = d\hat{A}_i. \tag{4.8}$$

The quantity P_{ij} generalizes the transverse δ function used in the conventional canonical treatment of the Coulomb gauge.

Motivated by the chain rule for ordinary differentiation, we define operators Π_i conjugate to $\hat{A}_i$ and l^α conjugate to $\hat{g}$ via the equation

$$E_i = \frac{d\hat{A}_j}{dA_i}\Pi_j + \frac{d_L\omega^\alpha}{dA_i} l^\alpha. \tag{4.9}$$

We will show that this determines Π_i and l^α uniquely if we also impose

$$P_{ij}\Pi_j = \Pi_i. \tag{4.10}$$

The operator Π_i generates shifts in the field $\hat{A}_i$ while maintaining the constraint $F^\alpha(A) = 0$, and the operator l^α generates gauge transformations.

To calculate the coefficients in Eq. (4.9) we use Eqs. (4.1) and (3.18) to relate an infinitesimal change in $\hat{A}_i$ to the changes in A_i and $\hat{g}$,

$$dA_i = R_{ij}(\hat{g}) d\hat{A}_j + \frac{1}{e} D_i^\alpha(A) d_L\omega^\alpha. \tag{4.11}$$

Using Eq. (3.20) this can be rewritten

$$dA_i = R_{ij}(\hat{g})\left(d\hat{A}_j + \frac{1}{e} S^{\alpha\beta}(\hat{g}) D_j^\beta(\hat{A}) d_L\omega^\alpha\right). \tag{4.12}$$

We now use the fact that F_i^α, the normals to the gauge-fixing surface, are orthogonal to the change in $\hat{A}$; that is, $F_i^\alpha d\hat{A}_i = 0$. This gives

$$dA_i R_{ij}(\hat{g}) F_j^\alpha = \frac{1}{e} F_j^\alpha D_j^\beta S^{\gamma\beta}(\hat{g}) d_L\omega^\gamma. \tag{4.13}$$

Defining the Faddeev-Popov matrix

$$N^{\alpha\beta} = F_i^\alpha(\hat{A}) D_i^\beta(\hat{A}), \tag{4.14}$$

we obtain

$$\frac{d_L\omega^\alpha}{dA_i} = eR_{ij}(\hat{g}) S^{\alpha\beta}(\hat{g})(N^{-1})^{\beta\gamma} F_j^\gamma. \tag{4.15}$$

Finally we use this to eliminate $d_L\omega^\alpha$ from Eq. (4.12) with the result

$$\begin{aligned} \frac{d\hat{A}_j}{dA_i} &= R_{ik}(\hat{g})[\delta_{kj} - D_j^\alpha(\hat{A})(N^{-1})^{\alpha\beta} F_k^\beta] \\ &= R_{ik}(\hat{g}) Q_{kj}. \end{aligned} \tag{4.16}$$

Here Q is a nonsymmetric projection operator

$$Q_{ij} = \delta_{ij} - D_j^\alpha(\hat{A})(N^{-1})^{\alpha\beta} F_i^\beta(\hat{A}) = Q_{ik}Q_{kj}. \tag{4.17}$$

Products of Q with our earlier projection operator P are particularly simple,

$$\begin{aligned} QP &= Q, \\ PQ &= P. \end{aligned} \tag{4.18}$$

Geometrically, Q projects onto planes perpendicular to the gauge orbits, but it makes this projection along lines perpendicular to the surface $F = 0$. This action is sketched in Fig. 1.

With this bit of tedious algebra out of the way, we can return to Eq. (4.9) which now becomes

$$E_i = R_{ij}(\hat{g})[Q_{jk}\Pi_k + eS^{\alpha\beta}(g)(N^{-1})^{\beta\gamma} F_j^\gamma l^\alpha]. \tag{4.19}$$

Using the fact that Q projects onto planes per-

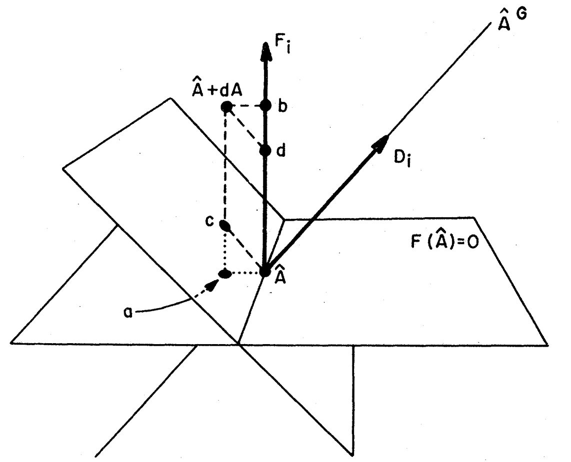

FIG. 1. The action of the projection operators P and Q on the displacement dA from the point $\hat{A}$ on the gauge-fixing surface $F(\hat{A})=0$ is indicated: $a=\hat{A}+PdA$ lies on the surface; $b=\hat{A}+(1-P)dA$; $c=\hat{A}+QdA$ lies on the plane normal to the gauge orbit through $\hat{A}$; $d=\hat{A}+(1-Q)dA$.

pendicular to the orbits, we easily extract l^α as

$$l^\alpha = \frac{1}{e} D_i^\alpha(A) E_i \,, \qquad (4.20)$$

showing explicitly that Gauss's law generates gauge transformations. The projector P can be used to extract Π_i from E; this gives

$$\Pi_i = P_{ij}(\hat{A}) R_{jk}(\hat{g}^{-1}) E_k \,. \qquad (4.21)$$

Using these relations along with the canonical commutator of Eq. (3.1), one can obtain the full set of commutation relations among the new variables

$$[\Pi_i, \hat{A}_j] = -iP_{ij} \,, \qquad (4.22a)$$

$$[l^\alpha, \hat{g}] = \lambda^\alpha \hat{g} \,, \qquad (4.22b)$$

$$[l^\alpha, \hat{A}_i] = [l^\alpha, \Pi_i] = 0 \,, \qquad (4.22c)$$

$$[\hat{A}_i, \hat{g}] = [\Pi_i, \hat{g}] = 0 \,, \qquad (4.22d)$$

$$[l^\alpha, l^\beta] = if^{\alpha\beta\gamma} l^\gamma \,, \qquad (4.22e)$$

$$[\Pi_i, \Pi_j] = (P_{il} P_{jk,l} - P_{jl} P_{ik,l}) \Pi_k \,, \qquad (4.22f)$$

where

$$P_{ij,k} = \frac{\partial}{\partial A_k} P_{ij} \,. \qquad (4.23)$$

Note that the nonvanishing of the right-hand side of Eq. (4.22f) is a measure of the curvature of the surface $F^\alpha = 0$. For a flat surface,[17] such as the Coulomb gauge, this commutator will vanish.

Being defined in terms of objects that do not commute, the operator Π_i is not in general Hermitian. Indeed, from Eq. (4.21) one can readily verify that

$$\Pi_i^\dagger - \Pi_i = -iP_{ij,l} Q_{jl} \,. \qquad (4.24)$$

For a flat gauge, which includes most gauges used in practice, Π_i will be Hermitian. The Hermiticity of l^α is readily verified from Eq. (4.20) using the antisymmetry of the T_{ij}^α occurring in D_i^α.

We now wish to express the Hamiltonian in these new variables. Because of the invariance under gauge transformations, the Hamiltonian is independent of $\hat{g}$. As discussed in Sec. II, we wish to impose Gauss's law as a constraint on physical states. This enables us to eliminate l^α from H in matrix elements between such states. Restricting ourselves to the physical subspace, the Hamiltonian becomes

$$H \to \Pi^\dagger Q^T Q \Pi + V(\hat{A}) \,. \qquad (4.25)$$

For comparison with more standard treatments,[18] we can break $Q\Pi$ into parts "longitudinal" and "transverse" with respect to P. Using Eqs. (4.18) and (4.10) gives

$$Q\Pi = E^T + E^L \,, \qquad (4.26)$$

$$E^T = PQ\Pi = \Pi \,, \qquad (4.27a)$$

$$E^L = (1-P)Q\Pi = (Q-1)\Pi = F_i^\beta (N^{-1})^{\alpha\beta} D_j^\alpha \Pi_j \,. \qquad (4.27b)$$

From Eqs. (4.27a) and (4.27b) it is obvious that E^T and E^L are orthogonal components of the momenta. Therefore, the Hamiltonian of Eq. (4.25) can be rewritten in a more familiar form

$$H = \tfrac{1}{2}(E^T)^\dagger E^T + \tfrac{1}{2}(E^L)^\dagger E^L + V(\hat{A}) \,, \qquad (4.28)$$

where

$$E_i^L = D_j^\alpha (N^{-1})^{\alpha\beta} F_i^\beta \Pi_j \,. \qquad (4.29)$$

The term $(E^L)^\dagger E^L$ generalizes the usual instantaneous interaction familiar from Coulomb gauge where $F_i^\beta = \nabla_i^\beta$ and $N^{\alpha\beta} = \nabla_i^\alpha D_i^\beta$.

To summarize this section, we have reexpressed $H(E_i, A_i)$ as a function of our new variables $\hat{A}_i$, $\hat{g}_i$, l^α, and Π_i. Because of gauge invariance and Gauss's law, we are left with a Hamiltonian only depending on the conjugate coordinates $\hat{A}_i$ and Π_i. This reproduces the canonical treatment in the gauge $F^\alpha(A)=0$. As a by-product the Faddeev-Popov operator $N^{\alpha\beta}$ emerges in a canonical way for this large class of noncovariant gauges. Note, however, that we have not included covariant gauges. This would be more complicated and would involve use of an unphysical space with an indefinite metric; we leave this problem open for the present time. Throughout this section we have assumed that an appropriate surface

$F^\alpha(\hat{A})=0$ exists; in the remaining sections of this paper we discuss certain limitations on this assumption.

V. NONEXISTENCE OF SOME SIMPLE GAUGES

Since the shape of the orbits in field space is determined by the structure of the theory, one might think that it is possible to define a "natural" gauge by demanding that the orbits intersect the surface $F^\alpha(A)=0$ orthogonally everywhere. In this case, the gauge structure of the theory would uniquely define the gauge-fixing condition. Indeed one can show that this orthogonal set of coordinates cannot be achieved even in some small neighborhood of the classical vacuum field if the gauge group is non-Abelian. Assume, to the contrary, that such a choice were possible. Then in this neighborhood the projection operators P and Q of Eqs. (4.6) and (4.17) of the previous section would be identical. In order for this to be true there would have to exist a linear relation between $D_i^\alpha(A)$ and $F_i^\beta(A)$,

$$D_i^\alpha(A)=M^{\alpha\beta}F_i^\beta+L_i^{\alpha\beta}F^\beta, \tag{5.1}$$

where M and L in general depend on A. Since this equality holds within a finite region, we can differentiate this relationship with respect to A_j, and project onto the surface to obtain

$$ie(PT^\alpha P)_{ij}=P_{ik}M^{\alpha\beta}F_{kl}^\beta P_{lj}, \tag{5.2}$$

where

$$F_{kl}^\beta=\frac{\partial}{\partial A_k}\frac{\partial}{\partial A_l}F^\beta .$$

Now the left-hand side of this equation is antisymmetric under $i \leftrightarrow j$ while the right-hand side is symmetric. Thus

$$(PT^\alpha P)_{ij}=0 \tag{5.3}$$

for all points on the surface. Consider the point $\hat{A}$ gauge equivalent to the classical vacuum $A=0$. Here Eq. (5.3) implies

$$R_{ij}(\hat{g})R_{nm}(\hat{g})S^{\alpha\beta}\delta_{jk}^{\text{Tr}}T_{kl}^\beta\delta_{lm}^{\text{Tr}}=0, \tag{5.4}$$

where $\hat{g}$ is the gauge transformation that takes $\hat{A}$ to $A=0$ and the transverse δ function is

$$\delta_{ij}^{\text{Tr}}=\delta_{ij}-\nabla_i^\alpha\frac{1}{(\nabla_k\nabla_k)^{\alpha\beta}}\nabla_j^\beta . \tag{5.5}$$

In Eq. (5.4) we understand that

$$\nabla_i^\alpha \to \delta^{\alpha\beta}\partial_i\delta^3(\vec{x}-\vec{y})$$

and

$$T_{kl}^\alpha \to \delta^3(\vec{x}-\vec{y})\delta^3(\vec{x}-\vec{z})\delta_{kl}f^{\alpha\beta\gamma},$$

when full indices are restored. Thus Eq. (5.4) implies that $f^{\alpha\beta\gamma}=0$, i.e., the group is Abelian. Indeed, in the Abelian gauge theory, the usual Coulomb condition $\vec{\nabla}\cdot\vec{A}=0$ is orthogonal to the orbits under gauge transformation.

In a non-Abelian theory the Coulomb condition specifies a flat surface which at $A_i=0$ is orthogonal to the orbit through that point. The above result says that, away from $A_i=0$, gauge orbits must intersect this surface in a non-orthogonal manner. Indeed, Gribov has shown that sufficiently far from the classical vacuum orbit, there exist orbits tangent to the Coulomb surface. We now prove an extension of this result due to Singer,[19] that there are tangent orbits to any generalized Coulomb surface. By a generalized Coulomb surface we mean a flat surface through some point $\bar{A}_i$ and orthogonal to the gauge orbit through that point, i.e., orthogonal to $D_i^\alpha(\bar{A})$. A gauge function giving this surface is $F^\alpha(A)=D_i^\alpha(\bar{A})(A_i-\bar{A}_i)$. The point $\bar{A}_i+\lambda\tau_i$, where λ is a real number, will lie in this surface if $D_i^\alpha(\bar{A})\tau_i=0$ for all α. Since $D_i^\alpha(\bar{A}+\lambda\tau)$ is a vector in the direction of the orbit through $\bar{A}+\lambda\tau$, what we want to show is that

$$M^{\alpha\beta}=D_i^\alpha(\bar{A})D_i^\beta(\bar{A}+\lambda\tau) \tag{5.6}$$

has a zero eigenvalue for some λ. Using Eq. (3.19) we expand $M^{\alpha\beta}$,

$$M^{\alpha\beta}=D_i^\alpha(\bar{A})D_i^\beta(\bar{A})+ie\lambda D_i^\alpha(\bar{A})T_{ij}^\beta\tau_j . \tag{5.7}$$

We now show that M is a Hermitian operator. Indeed, the anti-Hermitian part of M is

$$(M-M^\dagger)^{\alpha\beta}=i(D_i^\alpha T_{ij}^\beta\tau_j-D_i^\beta T_{ij}^\alpha\tau_j). \tag{5.8}$$

Using Eq. (3.21) and $D_i\tau_i=0$, we see that this vanishes and $M^{\alpha\beta}$ is Hermitian. Therefore its eigenvalues are real. At $\lambda=0$, $M^{\alpha\beta}$ is positive definite while for λ sufficiently far from zero, it cannot remain so if $D_i^\alpha T_{ij}^\beta\tau_j$ does not vanish. By continuity there must exist an intermediate λ where $M^{\alpha\beta}$ has a zero eigenvalue. At this point the gauge orbit is tangent to the gauge-fixing plane.

To complete the proof we show in Appendix B that there always exists a τ_i such that $D_i^\alpha\tau_i$ vanishes while $D_i^\alpha T_{ij}^\beta\tau_j$ does not.

VI. TOPOLOGY AND THE TEMPORAL GAUGE

We now show how the nontrivial topology of the gauge orbits forces an interesting structure upon the gauge-fixing surface $F^\alpha=0$. As discussed in Sec. II we assume boundary conditions restricting allowed gauge transformations to those going to the identity at spatial infinity. This discussion does not apply to gauge conditions such as the axial gauge where surface variables are important.[20] We thus have a set of gauge transformations that is not connected; they divide into homotopy classes labeled by integers such as the winding number in Eq. (2.26). In consequence, the orbit of any particular gauge field is actually made up of a countably infinite number of disconnected pieces.[19]

Assume that a surface $F^\alpha = 0$ has been chosen such that $\hat{A}_i$ and $\hat{g}$ are continuous nonsingular functions of A_i. Since the space of allowed gauge fields is simply connected, $\hat{g}$ must always lie in the same homotopy class which, for simplicity, we may take to be the trivial class. This immediately implies that the gauge fixing surface must intersect all the separate disconnected pieces of every gauge orbit. In particular, for every $\hat{A}_i$ lying on this surface, there must exist a gauge transformation g_1, with unit winding number as defined in Eq. (2.26) that leaves one on this surface. In equations we have

$$F^\alpha(\hat{A}^{g_1}) = F^\alpha(\hat{A}) = 0 . \tag{6.1}$$

This topological result, as stressed by Singer,[19] implies that no gauge-fixing function F^α can only pick a single $\hat{A}$ on every orbit.

That any gauge surface uniquely determines a g_1 for every $\hat{A}$ is a rather strong requirement. In particular, it precludes any gauge fixing which is invariant under ordinary space translation because no topologically nontrivial g_1 can be translationally invariant.[21] This is another way of seeing that difficulties must occur with the Coulomb gauge.

Although g_1 is in general a function of $\hat{A}$, by deformation of the surface it should be possible to have g_1 be some fixed unit-winding-number transformation. In this case we can iterate Eq. (6.1) to obtain for every $\hat{A}$ on the surface,

$$F^\alpha(\hat{A}^{g_1{}^n}) = 0 , \tag{6.2}$$

where g_1 is taken to an integer power n. Remembering that the Hamiltonian is gauge invariant, we see that the theory possesses a periodicity on the surface $F^\alpha = 0$. The discrete symmetry associated with this periodicity is

$$Q\Pi \to UQ\Pi U^{-1} = R(g_1)Q\Pi , \tag{6.3}$$

$$\hat{A} \to U\hat{A}U^{-1} = R(g_1)\hat{A} + \Lambda(g_1) , \tag{6.4}$$

where U is the operator of Eq. (2.24) with the appropriate ω^α. This unitary operator can be simultaneously diagonalized with the Hamiltonian, giving eigenvalues which are phases

$$U|\psi\rangle = e^{i\theta}|\psi\rangle . \tag{6.5}$$

As has been extensively discussed elsewhere,[11] our Hilbert space breaks up into sectors labeled by the parameter θ, while gauge-invariant operators have vanishing matrix elements between different sectors.

The potential $V(\hat{A})$ in the Hamiltonian of Eq. (4.25) is minimum wherever the gauge surface intersects the orbit of fields gauge equivalent to $A_i = 0$. This countable infinity of intersections represents the set of classical vacuum states. In a semiclassical picture the quantum system will tunnel between these classical vacuums. As discussed in Ref. 22 the most probable tunneling path is given by the pseudoparticle solutions of Ref. 11.

VII. REMAINING QUESTIONS

Working via the temporal gauge $A_0 = 0$ we have related a large class of canonical gauges. These gauges are determined by the vanishing of a function $F^\alpha(\hat{A})$ which depends only on the space components of the gauge field. It would be interesting to extend this formalism to include gauge conditions depending as well on the time component of $\hat{A}$ defined by

$$\hat{A}_0 = \frac{i}{e}\partial_0 \hat{g}\hat{g}^{-1} . \tag{7.1}$$

Such a condition involves time derivatives of the coordinates $\hat{g}$ and thus the corresponding canonical transformation is not a simple change of coordinates. The Lorentz gauge $\partial_\mu \hat{A}_\mu = 0$ would be even more complicated as the gauge condition involves second time derivatives of the coordinates $\hat{g}$.

Goldstone and Jackiw[14] have investigated a transformation among the momenta by writing

$$E_i = \hat{E}_i^{\hat{g}} \tag{7.2}$$

This approach also allows the elimination of ignorable coordinates corresponding to the gauge symmetry of the theory. Since the transformation is on the momentum variables, the complexity of the problem is shifted to the potential-energy term in Eq. (3.22). Also the lack of an inhomogeneous term in the gauge transformation of the electric fields makes constraints on $\hat{E}$ more complicated to handle.

It would be interesting to have a construction of a gauge surface with the properties discussed in Sec. VI. Indeed, without such an explicit example one might worry that such a surface may not exist for reasons that we have missed.

APPENDIX A

Here we derive the simple behavior of $D_i^\alpha(A)$ under gauge transformations. From Eqs. (3.19) and (3.5) we have

$$D_i^\alpha(A^g)=\nabla_i^\alpha+ieT_{ij}^\alpha(R_{jk}A_k+\Lambda_j)\,. \tag{A1}$$

Properties of the adjoint representation imply

$$T_{ij}^\alpha R_{jk}=R_{ij}S^{\alpha\beta}T_{jk}^\beta\,. \tag{A2}$$

Inserting an infinitesimal g' in Eq. (3.12) gives

$$\begin{aligned}\Lambda_i[(1+i\lambda^\alpha\delta^\alpha)g]&=\Lambda_i(g)\\&\quad+\left(iT_{ij}^\alpha\Lambda_j(g)+\frac{1}{e}\nabla_i^\alpha\right)\delta^\alpha\\&=\Lambda_i\{g[1+i\delta^\alpha S^{\alpha\beta}(g)\lambda^\beta]\}\\&=\Lambda_i(g)+\frac{1}{e}R_{ij}(g)\nabla_j^\beta S^{\alpha\beta}(g)\delta^\alpha\,.\end{aligned} \tag{A3}$$

Thus we have

$$ieT_{ij}^\alpha\Lambda_j=-\nabla_i^\alpha+R_{ij}S^{\alpha\beta}\nabla_j^\beta\,. \tag{A4}$$

Inserting Eqs. (A4) and (A2) into (A1) gives the desired result [Eq. (3.20)]

$$D_i^\alpha(A^g)=R_{ij}(g)S^{\alpha\beta}(g)D_j^\beta(A)\,.$$

If we now consider infinitesimal g in Eq. (A4) we obtain

$$\nabla_i^\alpha T_{ij}^\beta-\nabla_i^\beta T_{ij}^\alpha-if^{\alpha\beta\gamma}\nabla_j^\gamma=0\,. \tag{A5}$$

To obtain Eq. (3.21) from this, just add Eq. (3.10) multiplied by a factor of A.

APPENDIX B

Theorem: There does not exist an $\bar{A}_i$ such that $D_i^\alpha(\bar{A})\tau_i=0$ for all α implies $D_i^\alpha(\bar{A})T_{ij}^\beta\tau_j=0$ for all α and β.

Proof: Assuming that the theorem is false, we iterate to obtain

$$D_i^\alpha(\bar{A})R_{ij}(g^{-1})\tau_j=0 \tag{B1}$$

for arbitrary g and τ satisfying $D_i^\alpha(\bar{A})\tau_i=0$. Using the gauge transformation properties of D_i^α, we obtain

$$D_i^\alpha(\bar{A}^g)\tau_i=0\,. \tag{B2}$$

Thus every τ orthogonal to the orbit at $\bar{A}$ is orthogonal to the orbit everywhere. We conclude that the orbit of $\bar{A}$ must be straight, i.e., all gauge fields of the form

$$A_i=\bar{A}_i+h^\alpha D_i^\alpha(\bar{A}) \tag{B3}$$

lie on the gauge orbit of $\bar{A}_i$. However, the invariance of $D_i^\alpha D_i^\alpha$ over the orbit implies

$$\begin{aligned}0&=\frac{d}{dh^\alpha}\frac{d}{dh^\beta}D_i^\gamma(A)D_i^\gamma(A)\\&=2e^2D_i^\alpha(\bar{A})T_{ij}^\gamma T_{jk}^\gamma D_k^\beta(\bar{A})\\&=2e^2CD_i^\alpha(\bar{A})D_i^\beta(\bar{A})\,,\end{aligned} \tag{B4}$$

where C is the quadratic Casimir operator of Eq. (3.10). Since the right-hand side of Eq. (B4) does not vanish, we have proven the theorem by contradiction.

[1] S. Weinberg, Phys. Rev. 138, B988 (1965).
[2] L. Faddeev and V. Popov, Phys. Lett. 25B, 29 (1967).
[3] S. Coleman, in *Laws of Hadronic Matter*, edited by A. Zichichi (Academic, New York, 1975).
[4] J. Kogut and L. Susskind, Phys. Rev. D 10, 3468 (1974); 11, 3594 (1975); S. S. Chang, *ibid.* 17, 2611 (1978).
[5] M. Creutz, Ann. Phys. (N.Y.) (to be published).
[6] M. Creutz and T. N. Tudron, Phys. Rev. D 17, 2619 (1978).
[7] L. D. Faddeev, Teor. Mat. Fiz. 1, 3 (1969) [Theor. Math. Phys. 1, 1 (1969)].
[8] J. L. Gervais and B. Sakita, Phys. Rev. D 18, 453 (1978).
[9] J. L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975).
[10] V. N. Gribov, Nucl. Phys. B139, 1 (1978); R. Jackiw, I. Muzinich, and C. Rebbi, Phys. Rev. D 17, 1576 (1978); M. Ademollo, E. Napolitano, and S. Sciuto, Nucl. Phys. B134, 477 (1978).
[11] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Y. S. Tyupkin, Phys. Lett. 59B, 85 (1975); C. G. Callan, R. F. Dashen, and D. J. Gross, *ibid.* 63B, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).
[12] For a different point of view see M. Peskin, Cornell Univ. Report Nos. CLNS-395 and CLNS-396, 1978 (unpublished).
[13] J. F. Willemsen, Phys. Rev. D 17, 574 (1978).
[14] J. Goldstone and R. Jackiw, Phys. Lett. 74B, 81 (1978).
[15] M. Creutz and T. N. Tudron, Phys. Rev. D 16, 2978 (1977).
[16] K. M. Bitar, Fermilab Report No. 77/15, 1977 (unpublished); C. Meyers, M. De Roo, and P. Sorba, Nucl. Phys. B140, 533 (1978).
[17] By flat we mean that if A and A' both lie in the surface, so does $\alpha\hat{A}+(1-\alpha)\hat{A}'$ for every α.
[18] R. N. Mohapatra, Phys. Rev. D 4, 378 (1971); 4, 1007 (1971); E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).
[19] I. M. Singer, Commun. Math. Phys. (to be published).
[20] S. Coleman, Harvard Report No. HUTP-78/A004, 1978 (unpublished); I. Bars and F. Green, Nucl. Phys. B142, 157 (1978).
[21] For SU(2), $g_1(\bar{x})$ must pass through the particular element (-1) at a definite space point.
[22] K. M. Bitar and S.-Y. Chang, Phys. Rev. D 17, 486 (1978).