PHYSICAL REVIEW D VOLUME 19, NUMBER 2 15 JANUARY 1979

Field-strength and dual variable formulations of gauge theory

M. B. Halpern
Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720
(Received 8 September 1978)

Using completely fixed axial-like gauges, I construct the unique inversion $A(F)$ for the potential A in terms of the field strength F. A change of variable to F results in a field-strength formulation of gauge theory. F is constrained to satisfy the Bianchi "identities" $\partial\tilde{F} - eA(F)\tilde{F} = 0$. Dual potentials $\tilde{A}$ may also be introduced as functional Fourier conjugates to the Bianchi forms. For quantum electrodynamics in four dimensions, duality $(F \to \tilde{F},\ A \to \tilde{A})$ is a perfect symmetry. However, residuals of the symmetry persist in all theories: E.g., Gauss's law is an identity; $\tilde{A}$ is canonical to the magnetic fields, and closely related to 't Hooft's disorder operators.

I. INTRODUCTION

Let A be a generic potential for a generic gauge theory, and $F(A)$ its field strength. It is generally believed that the inversion $A(F)$ must be nonunique: For Abelian theories there is the certainty of gauge ambiguity, and for non-Abelian theories there is the problem of field-strength copies[1] (sets of gauge-inequivalent potentials with the same field strength).

However, in a recent publication,[2] I noted that field-strength copy is not a gauge-invariant concept, and I proved that there are no such copies in completely fixed axial gauges.[3] Such gauge choice resolves the Abelian ambiguity as well, of course, and so in these gauges *$A(F)$ is unique for all gauge theories*.

I recently announced the explicit construction of the inversion $A(F)$ in a letter,[4] along with a brief discussion of some immediate applications: unambiguous reformulations of gauge theory in terms of field-strength and/or dual variables. The development is *uniform* for any number of dimensions and arbitrary gauge group. The purpose of this paper is to provide further details and discussion of the reformulations.

The basic idea is as follows: (1) Obtain the unique inversion $A(F)$, with the constraint on allowed F's, $\partial\tilde{F}-eA(F)\tilde{F}=0$ (Bianchi "identities"). (2) Change variables to F. The result is an unambiguous field strength formulation for any gauge theory. F is integrated over the Bianchi identities. (3) Dual potentials $\tilde{A}$ are introduced as functional Fourier conjugates to the Bianchi forms. (4) Integrate out the electric fields to obtain the "dual Hamiltonian," a function of B, the magnetic fields, and $\tilde{A}$, canonical to B.

I offer the following reasons why field strengths and dual variables may be the natural language in which to study current problems in gauge theory.

(1) In ordinary formulations, with residual gauge freedom, physical states must be selected by requiring gauge invariance. In a completely fixed gauge, no further gauge constraint on the state is necessary, and the generators of gauge transformations should vanish identically. With dual variables, this is realized elegantly as an aspect of duality: Gauss's law is an identity—just as the Bianchi identities are identities in the usual formulation.

(2) In the field formulation, Wilson integrals can be expressed as *area* integrals over the field strength. This may be an aid in seeking area effects in the confinement problem.

(3) In a Hamiltonian approach to confinement, one is interested in Wilson integrals over *spatial* (fixed-time) paths. In the field formulation, these are area integrals over the *magnetic* fields B. It may therefore be advantageous to have B as a fundamental variable. Indeed, as I will discuss below, it is easy to construct confining states using the dual variables.

(4) The dual potentials $\tilde{A}$, being canonical to B, generate local disturbances in the magnetic field. $\tilde{A}$ is therefore closely related to the *disorder operators* defined implicitly by 't Hooft.[5] Confinement tends to be found in *disordered states* (spread in B, sharp in $\tilde{A}$).

(5) The dual potentials enter the non-Abelian theories with Higgs-type couplings, and silhouette the question of Meissner-type effects in non-Abelian gauge theory.

(6) In general, the dual variables may be extremely useful in developing the continuum analog (especially for the non-Abelian case) of recent progress in dual-lattice theory.[6]

(7) Formulation of monopole theories in terms of dual potentials appears extremely natural. Dual potentials couple locally to quantum monopole currents, just as ordinary potentials couple locally to ordinary charge.

An outline of the paper is as follows. In Sec. II, I briefly remind the reader of the field-strength copy problem, and its solution in fixed axial-like gauges. I obtain the inversion formulas $A(F)$ for the various theories, as well as the Bianchi-identity constraints on allowed F's. In Sec. III, I briefly apply $A(F)$ to express Wilson integrals in terms of field strengths. In Sec. IV, I make a quantum variable change from A to F, obtaining the field-strength formulation. F is integrated over the Bianchi identities with no further functional measure. Section V is a brief, and I believe quite nice, application of the field formulation to monopole quantization. I introduce dual potentials $\tilde{A}$ in Sec. VI. After interpreting $\tilde{A}$ as those fields which couple locally to monopoles, I discuss saddle-point equations, then a formulation entirely in terms of dual potentials, and, finally, duality. In Sec. VII, I eliminate the electric fields to find the "dual Hamiltonians."

Section VIII is reserved for remarks, including the ease of constructing confining states in the dual formulation. I include also four appendixes dealing with technical matters.

II. THE INVERSION $A(F)$

The simplest case is quantum chromodynamics[7] in one space and one time dimension (QCD_2). Consider the ordinary axial gauge $A_3=0$, $F_{03}(A)=-\partial_3 A_0$ (suppressing obvious color indices). All configurations have field-strength copies of the form

$$A_0'(tz)=A_0(tz)+\Delta(t),\qquad F_{03}(A')=F_{03}(A), \tag{2.1}$$

where $\Delta(t)$ is essentially arbitrary. (The theory is non-Abelian, so A_0, A_0' are not in general gauge-equivalent.) The inversion $A(F)$ cannot be unique in the ordinary axial gauge.

There is, however, a residual gauge freedom in such gauges, which I use here to choose the completely fixed axial gauge[8]

$$A_3(tz)=A_0(tz_0)=0, \tag{2.2}$$

where z_0 is a particular point in z. Now it is trivial to see that

$$A_3=0,\qquad A_0(tz)=-\int_{z_0}^{z} dz' F_{03}(tz') \tag{2.3}$$

is the unique inversion $A(F)$. In such gauges, all field copies have become local action copies.[2] This inversion, Eq. (2.3), is also correct for the Abelian theory (QED_2) in this gauge.

The next simplest example is the case of the Abelian theory in three dimensions (QED_3). The problem here is, of course, only ordinary gauge fixing. I choose the fixed axial gauge

$$0=A_3(xyz)=A_1(xyz_0)=A_2(x_0yz_0),\qquad F_{12}(A)=\partial_1A_2-\partial_2A_1,\ F_{31}(A)=\partial_3A_1,\ F_{23}(A)=-\partial_3A_2. \tag{2.4}$$

Using the forms $F_{31}(A)$, $F_{23}(A)$ and the gauge conditions on A_1, A_2, I have immediately

$$A_1(xyz)=\int_{z_0}^{z} dz' F_{31}(xyz'), \tag{2.5a}$$

$$A_2(xyz)=-\int_{z_0}^{z} dz' F_{23}(xyz')+\Delta(xy), \tag{2.5b}$$

$$\Delta(x_0y)=0. \tag{2.5c}$$

The function Δ must be determined consistently from the equation $F_{12}(A)=F_{12}$:

$$F_{12}=-\int_{z_0}^{z} dz'(\partial_2F_{31}+\partial_1F_{23})(xyz')+\partial_1\Delta. \tag{2.6}$$

I break this equation into its form at $z=z_0$ and the derivative with respect to z:

$$F_{12}(xyz_0)=\partial_1\Delta(xy), \tag{2.7a}$$

$$\partial_3F_{12}+\partial_2F_{31}+\partial_1F_{23}=0. \tag{2.7b}$$

Equation (2.7a) can be uniquely solved for Δ with the boundary condition (2.5c). The final result is

$$A_1(xyz)=\int_{z_0}^{z} dz' F_{31}(xyz'), \tag{2.8a}$$

$$A_2(xyz)=-\int_{z_0}^{z} dz' F_{23}(xyz')+\int_{x_0}^{x} dx' F_{12}(x'yz_0), \tag{2.8b}$$

$$A_3=0, \tag{2.8c}$$

$$I\equiv\partial_\mu\tilde{F}_\mu=0,\quad \tilde{F}_\mu\equiv\tfrac{1}{2}\epsilon_{\mu\nu\rho}F_{\nu\rho}, \tag{2.8d}$$

where $\epsilon_{\mu\nu\rho}$ is completely antisymmetric and $\epsilon_{123}=+1$. Equations (2.8a), (2.8b), and (2.8c) form the unique inversion $A(F)$. Equation (2.8d) is the necessary and sufficient consistency condition on the field strengths. It is recognized as the Bianchi "identities" in three dimensions. In general, for dimension greater than two, the Bianchi identities will emerge as consistency conditions on the fields.

For higher dimensions and/or non-Abelian theories, the algebra is more complicated, but the idea remains the same. I give the details for QCD_4 in Appendix A. Here I will only state the results.

QCD_3 $[A_3(xyz)=A_1(xyz_0)=A_2(x_0yz_0)=0]$.

$$A_1(xyz)=\int_{z_0}^{z} dz' F_{31}(xyz'), \tag{2.9a}$$

$$A_2(xyz) = -\int_{z_0}^{z} dz' F_{23}(xyz') + \int_{x_0}^{x} dx' F_{12}(x'yz_0), \tag{2.9b}$$

$$A_3 = 0, \tag{2.9c}$$

$$I_a = \partial_\mu \tilde{F}^a_\mu - e\epsilon^{abc} A^b_\mu(F) \tilde{F}^c_\mu = 0. \tag{2.9d}$$

Here $A^b_\mu(F)$ is precisely the form $A(F)$ given in Eqs. (2.9a), (2.9b), and (2.9c). Note that the form $A(F)$ continues to be the same for Abelian and non-Abelian theories (in the same dimension). The only difference is the complexity of the consistency conditions (Bianchi identities).

QED$_4$ $[A_3(txyz) = A_1(txyz_0) = A_2(tx_0yz_0) = A_0(tx_0y_0z_0) = 0]$.

$$A_1 = \int_{z_0}^{z} dz' F_{31}(txyz'), \tag{2.10a}$$

$$A_2 = -\int_{z_0}^{z} dz' F_{23}(txyz') + \int_{x_0}^{x} dx' F_{12}(tx'yz_0), \tag{2.10b}$$

$$A_0 = -\int_{z_0}^{z} dz' F_{03}(txyz') - \int_{x_0}^{x} dx' F_{01}(tx'yz_0) - \int_{y_0}^{y} dy' F_{02}(tx_0y'z_0), \tag{2.10c}$$

$$A_3 = 0, \tag{2.10d}$$

$$I_i(txyz) \equiv \partial_\mu \tilde{F}_{\mu i} = 0 \quad (i = 0, 1, 2), \tag{2.10e}$$

$$I_3(tzyz_0) \equiv (\partial_\mu \tilde{F}_{\mu 3})_{z=z_0} = 0. \tag{2.10f}$$

Here $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$, $\epsilon_{0123} = 1$. Notice that the consistency conditions are not the full set of Bianchi identities. This happens as well for QCD$_4$, and I will refer to this as the phenomenon of "3.1 Bianchi identities." It is trivial to show, however, that the the 3.1 Bianchi identities imply the other "0.9,"

$$\partial_3 I_3 = 0. \tag{2.11}$$

QCD$_4$. I make the same gauge choice as in QED$_4$. The resulting $A(F)$ is the same as Eqs. (2.10a), (2.10b), (2.10c), and (2.10d). The consistency conditions are

$$I^a_i(txyz) \equiv \partial_\mu \tilde{F}_{\mu i} - e\epsilon^{abc} A^b_\mu(F) \tilde{F}^c_{\mu i} = 0 \quad (i = 0, 1, 2),$$
$$I^a_3(txyz_0) \equiv [\partial_\mu \tilde{F}^a_{\mu 3} - e\epsilon_{abc} A^b_\mu(F) \tilde{F}^c_{\mu 3}]_{z=z_0} = 0. \tag{2.12}$$

I show in Appendix B that, in analogy to QED$_4$, these 3.1 Bianchi identities imply the last 0.9,

$$\partial_3 I^a_3 = 0. \tag{2.13}$$

I summarize what has been shown thus far. In fixed axial gauges (1) the inversion $A(F)$ is unique for all gauge theories. (2) $A(F)$ is linear in F. Its form is the same for Abelian and non-Abelian theories. (3) The consistency conditions are the Bianchi identities. (4) The Bianchi identities are at most quadratic in F.

III. WILSON INTEGRALS AS FIELD-STRENGTH FUNCTIONALS

By Wilson integrals, I refer to the generic forms

$$W[C] \equiv \exp\left(-ie\oint_C A\cdot dx\right),$$
$$W[C] \equiv \tfrac{1}{2}\mathrm{Tr}\,P\exp\left(-ie\oint_C A\cdot dx\right) \tag{3.1}$$

in the Abelian and non-Abelian cases, respectively. Here Tr and P are trace and path-ordering. In our gauges we can express $W[C]$ directly in terms of the field strengths, simply by replacing $A \to A(F)$. This is well known for the Abelian case, but new for the non-Abelian case. I will mention explicitly two theories which illustrate the general structure for all gauge theories.

QED$_3$. Consider the three paths shown in Fig. 1. Using $A(F)$ in Eq. (2.8), it is trivial to compute for the three cases

$$\oint \vec{A}(F)\cdot d\vec{x} = \int_0^{x_1} dx' \int_0^{z_1} dz' F_{31}(x'yz') \equiv \int_z\int_x d\vec{S}\cdot\vec{\tilde{F}},$$

$$\oint_{yz} \vec{A}(F)\cdot d\vec{x} = \int_0^{y_1} dy' \int_0^{z_1} dz' F_{23}(xy'z') \equiv \int_y\int_z d\vec{S}\cdot\vec{\tilde{F}},$$

$$\oint_{xy} \vec{A}(F)\cdot d\vec{x} = \int_0^{y_1} dx' F_{12}(x'y'z_0) + \int_{z_0}^{z} dz' \left\{ \int_0^{x_1} dx' [F_{31}(x'0z') - F_{31}(x'yz')] - \int_0^{y} dy' [F_{23}(xy'z') - F_{23}(0y'z')] \right\}. \tag{3.2}$$

To simplify the xy-loop form, integrate the Bianchi identity over the area in question:

$$0 = \int_0^{x_1} dx' \int_0^{y_1} dy' \partial_\mu \tilde{F}_\mu(x'y'z). \tag{3.3}$$

This implies immediately that

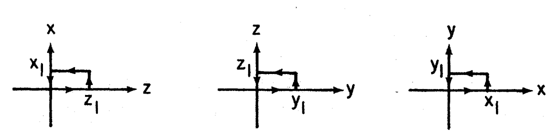

FIG. 1. Three Wilson integral paths.

$$\int_0^{x_1} dx'[F_{31}(x'0z) - F_{31}(x'yz)] - \int_0^{y_1} dy'[F_{23}(xy'z) - F_{23}(0y'z')] = \partial_3 \int_0^{x_1} dx' \int_0^{y_1} dy' F_{12}(x'y'z) \tag{3.4}$$

and hence that

$$\oint_{xy} \vec{A}(F) \cdot d\vec{x} = \int_0^{y_1} dy' \int_0^{x_1} dx' F_{12}(x'y'z) \equiv \iint_{xy} d\vec{S} \cdot \tilde{\vec{F}} \tag{3.5}$$

as expected. The explicit manipulations above will, however, be useful in Sec. V. In general, for Abelian theories, one finds the expected result ($dS_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} dx_\rho dx_\sigma$)

$$\oint_C A(F) \cdot dx = \tfrac{1}{2} \iint_{S(C)} dS_{\mu\nu} \tilde{F}_{\mu\nu}. \tag{3.6}$$

QCD_2. The Wilson integral corresponding to Fig. 2 is

$$W = \tfrac{1}{2} \operatorname{Tr} \left\{ T \exp\left[ie \int_{t_1}^{t_2} dt' \int_{z_0}^{z_2} dz' F_{03}(t'z') \right] T^* \exp\left[ie \int_{t_1}^{t_2} dt' \int_{z_1}^{z_0} dz' F_{03}(t'z') \right] \right\} \tag{3.7}$$

with T, T^* time-ordering and time-reversed-ordering, and I have used the $A(F)$ given in Eq. (2.3). An interesting gauge choice for any loop with one side at $z=z_1$ would be to pick $z_0 = z_1$; then

$$W = \tfrac{1}{2} \operatorname{Tr} \left\{ T \exp\left[ie \int_{t_1}^{t_2} dt' \int_{z_1}^{z_2} dz' F_{03}(t'z') \right] \right\} \tag{3.8}$$

in close analogy to the Abelian form. The reader can easily convince himself that, with appropriate gauge choice, any loop in any non-Abelian theory can be put in this "Abelian" form.

I emphasize also that when the path in the Wilson integral is spatial, then the area integral is over the magnetic fields (not the electric). This is immediate on inspection of, say, Eq. (2.10). It is, of course, just these Wilson integrals that are of interest in a Hamiltonian formulation.

IV. FIELD-STRENGTH FORMULATION OF GAUGE THEORY

I begin with the vacuum generating functional for a generic gauge theory

$$Z \equiv \int \mathfrak{D}A \, \delta[\mathrm{CGF}] \exp\left[-\frac{1}{4} \int F^2(A) \right]. \tag{4.1}$$

Here $\delta[\mathrm{CGF}]$ is a product of functional δ functions for the complete gauge fixing (CGF). The form of $\delta[\mathrm{CGF}]$ for QCD_4 is given in Appendix C.[9] The unique inversion $A(F)$ makes it possible to change variables to the field strengths themselves. The crucial identity is, up to multiplicative constants,

$$\int \mathfrak{D}A \, \delta[\mathrm{CGF}] \delta[G - F(A)] = \delta[I(G)], \tag{4.2}$$

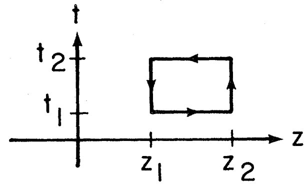

FIG. 2. A Wilson path in a non-Abelian theory.

which is valid for all gauge theories. Here $I(G) = \partial\tilde{G} - eA(G)\tilde{G}$ are the Bianchi forms as a function of G (which is F). The proof of Eq. (4.2) is given in Appendix C for the case of QCD_4. The proof is simpler for other theories. In four dimensions (QED_4 and QCD_4), the right-hand side of Eq. (4.2) actually comes out as δ functionals for the 3.1 Bianchi identities only. However, as the last 0.9 are redundant (Appendix B), I have chosen, for the sake of symmetry, to restore all four Bianchi identities on the right of Eq. (4.2). [Formally this is an extra constant factor $\delta[0]$.]

The field-strength formulation is now immediate. Start with Z, and insert unity in the form

$$1 = \int \mathfrak{D}G \, \delta[G - F(A)]. \tag{4.3}$$

Interchanging integration order and doing the A integration via Eq. (4.2), I obtain

$$Z = \int \mathfrak{D}G \, \delta[I(G)] \exp\left(-\tfrac{1}{4} \int G^2 \right). \tag{4.4}$$

The form Eq. (4.4) provides an unambiguous field-strength formulation[11] for any gauge theory. The field strengths are integrated over the Bianchi identities. There is no further functional measure. I will continue to use G instead of F when it is a fundamental variable.

Any function of A in the original formulation translates immediately to that function of $A \to A(F) \to A(G)$ in the field-strength formulation. In particular, the forms of the Wilson integrals discussed in Secs. III are now appropriate. In general, the formulation is local in time—except in fixed A_0 gauges.

V. A SIMPLE APPLICATION: MONOPOLE QUANTIZATION

Consider QED_3 in the field-strength formulation

$$Z = \int \mathfrak{D}\vec{B} \exp\left(-\tfrac{1}{2} \int d^3x B^2 \right) \delta(\nabla \cdot \vec{B}), \tag{5.1}$$

where I have renamed $\tilde{G}_i = B_i$ ($B_1 = F_{23}, B_2 = F_{31}, B_3 = F_{12}$). The expectation value of a Wilson integral is

$$\langle W[C]\rangle = Z^{-1}\int \mathfrak{D}\vec{B}\exp\left(-\tfrac{1}{2}\int d^3x B^2\right)\delta[\nabla\cdot\vec{B}]W[C]$$
$$= \left\langle \exp\left[-ie\oint_C \vec{A}(B)\cdot d\vec{x}\right]\right\rangle$$
$$= \left\langle \exp\left(-ie\iint_{S(C)} d\vec{S}\cdot\vec{B}\right)\right\rangle. \qquad (5.2)$$

Here $A(B)$ is that $A(F)$ given in Eq. (2.8). I will refer to the first form as the "path form," the second form as the "surface form." The proof that the two forms are equivalent is that given in Sec. III. Thus, the surface form is surface independent. In fact, of course, the surface independence of the surface form can be seen directly [without mentioning $\vec{A}(B)$] from the Bianchi identity.

I will now introduce a *monopole* source $\rho_M(\vec{x})$ into the theory by *violating* the Bianchi identity

$$Z[\rho_M] \equiv \int \mathfrak{D}\vec{B}\exp\left(-\tfrac{1}{2}\int d^3x B^2\right)\delta[\nabla\cdot\vec{B} - \rho_M]. \qquad (5.3)$$

I demand that I can still find a "Wilson integral" such that

$$\langle W[C]\rangle_{\rho_M} \equiv Z^{-1}[\rho_M]\int \mathfrak{D}\vec{B}\exp\left(-\tfrac{1}{2}\int d^3x B^2\right) \times\delta[\nabla\cdot\vec{B} - \rho_M]W[C] \qquad (5.4)$$

is gauge invariant and a function of C alone. Which of the two forms in Eq. (5.2) is a viable candidate? I will show that, when $\rho_M \neq 0$, (a) the path form is in general not gauge invariant, while (b) the surface form is in general surface dependent. Moreover, I will show that the following conditions are all equivalent: (1) The condition that the path form is gauge invariant. (2) The condition that the surface form is surface independent. (3) The condition that the path and surface forms are equal. (4) Magnetic monopoles are quantized in the usual manner.

Consider first the path form. Repeating the steps of Sec. III for the three loops considered there, one finds

$$\oint \vec{A}(B)\cdot d\vec{x} = \iint d\vec{S}\cdot\vec{B}$$

for the zx and yz loops (the Bianchi identity is not involved in these computations). The xy loop computation *does* involve the Bianchi identity, however, and now

$$\oint_{xy} \vec{A}(B)\cdot d\vec{x} = \iint_{xy} d\vec{S}\cdot\vec{B} - \int_{z_0}^{z} dz'\int_0^{x_1} d\vec{x}'\int_0^{y_1} dy'\rho_M(\vec{x}'). \qquad (5.5)$$

But z_0 is a gauge parameter, so the path form is not in general gauge invariant.

The surface dependence of the surface form is standard: The difference of two surfaces involves a closed surface integral

$$\iint d\vec{S}\cdot\vec{B} = \int d^3x\,\nabla\cdot\vec{B} = \int d^3x\,\rho_M,$$

according to the (violated) Bianchi identity.

It is immediately clear that the unique cure for either problem solves them both:

$$\exp\left(-ie\int_V d^3x\,\rho_M\right) = 1 \Rightarrow \int_V d^3x\,\rho_M = \frac{2\pi n}{e}, \qquad (5.6)$$

where V is an arbitrary volume and n is an integer. This is the usual monopole quantization, and it is also the condition that the path and surface forms are equal again.

Point magnetic charge at $\vec{u}$ would then be of the form $\rho_M(\vec{x}) = g\delta^3(\vec{x}-\vec{u})$. For $\vec{u}$ on the *boundary* of a volume V, then

$$g = 4\pi n/e. \qquad (5.7)$$

This follows because $\int_{x_1} dx\,\delta(x-x_1) = \frac{1}{2}$ and is the Schwinger quantization.

The reader will notice that I have not mentioned Dirac strings in this discussion. Indeed, in the field-strength formulation, such need not appear, and do not. Nevertheless, I can excavate the strings for pedantic purposes: Starting from a monopole field at the origin $\vec{B} = g\hat{r}/r^2$, I evaluate $\vec{A}(B)$, Eq. (2.8). The result for $A_1(B)$, e.g., is

$$A_1(B) = \frac{gy}{\rho^2}\left[\frac{z}{(\rho^2+z^2)^{1/2}} - \frac{z_0}{(\rho^2+z_0^2)^{1/2}}\right], \quad \rho^2 = x^2+y^2. \qquad (5.8)$$

Near the z axis then

$$A_1(B) \sim \frac{gy}{\rho^2}[\epsilon(z) - \epsilon(z_0)]. \qquad (5.9)$$

So $\vec{A}(B)$ does have the Dirac string: The string is down if $z_0 > 0$ and up if $z_0 < 0$. (One might have anticipated that the strings were all in the z direction from our "trouble" above with Wilson integrals in the xy plane.) Note that if $z_0 = 0$ then near the z axis

$$A_1(B) \sim \frac{gy}{\rho^2}\epsilon(z). \qquad (5.10)$$

This is Schwinger's *double string*. It is not hard to verify that, in general, monopoles placed at z_0 will always carry Schwinger strings, and hence the the Schwinger quantization.

The development for monopoles in QED_4 is parallel. The monopoles are coupled to the theory via $\delta[\partial_\mu\tilde{G}_{\mu\nu}] \to \delta[\partial_\mu\tilde{G}_{\mu\nu} - \tilde{J}_\nu]$. The final results are

$$\langle W[C]\rangle_{\tilde{J}} = \left\langle \exp\left[-ie\oint_C A_\mu(G)dx_\mu\right]\right\rangle_{\tilde{J}}$$
$$= \left\langle \exp\left(-\frac{ie}{2}\iint_{S(C)} dS_{\mu\nu}\tilde{G}_{\mu\nu}\right)\right\rangle_{\tilde{J}}. \qquad (5.11)$$

when

$$\int_V d\sigma_\mu \tilde{J}_\mu = 2n\pi/e. \tag{5.12}$$

VI. DUAL POTENTIALS

By dual potential, I refer to variables such as the magnetic scalar potential, familiar in elementary physics. In QED_4, dual potentials $\tilde{A}_\mu$ would be to $\tilde{F}_{\mu\nu}$ what A_μ is to $F_{\mu\nu}$:

$$\tilde{F}_{\mu\nu}(A) = F_{\mu\nu}(\tilde{A}). \tag{6.1}$$

Such variables may be extremely useful in finding the continuum analog and non-Abelian extension of recent studies in dual-lattice theory.[6] $\tilde{A}$ is also closely related to the local disorder operators 't Hooft[5] defines implicitly.

Having the field-strength formulation, a path to dual variables is clear. In the vacuum functional, Eq. (4.4), write

$$\begin{aligned} \delta(I(G)) &= \int \mathfrak{D}\tilde{A} \exp\left[-i\int \tilde{A} I(G)\right],\\ Z &= \int \mathfrak{D}G\,\mathfrak{D}\tilde{A} \exp\left(-\tfrac{1}{4}\int G^2\right)\\ &\quad\times \exp\left[-i\int \tilde{A} I(G)\right]. \end{aligned} \tag{6.2}$$

The variables $\tilde{A}$ have the indices of the Bianchi forms $I(G)$—so $\tilde{A}_\mu^a$ for QCD_4, $\tilde{\phi}^a$ (magnetic scalar potential) for QCD_3, and so on—and are identifiable as dual potentials. I will discuss various theories individually.

QED_3. The vacuum functional is

$$Z[\rho_M = 0] = \int \mathfrak{D}\vec{B}\,\mathfrak{D}\tilde{\phi} \exp\left(-\tfrac{1}{2}\int d^3x B^2\right) \times \exp\left(-i\int d^3x \nabla\tilde{\phi}\cdot\vec{B}\right). \tag{6.3}$$

The saddle-point equations are

$$\nabla\cdot\vec{B} = 0, \quad \vec{B} + i\nabla\tilde{\phi} = 0. \tag{6.4}$$

As expected in Euclidean space,[12] the saddles are at imaginary $\tilde{\phi}$. Thus, $\bar{\phi} = -i\tilde{\phi}$ is precisely the magnetic scalar potential ($\vec{B} = \nabla\bar{\phi}$).

Consider summing $\tilde{\phi}$ emissions at $\vec{u}$ with strength g. In a simple line of algebra, it is seen that

$$\langle \exp[-ig\tilde{\phi}(\vec{u})]\rangle = Z^{-1}[\rho_M = 0] Z[\rho_M(\vec{x}) = g\delta^{(2)}(\vec{x}-\vec{u})]. \tag{6.5}$$

Such emissions are then precisely equivalent to a monopole of strength g at $\vec{u}$. One might then call the exponential of the dual potential $\Phi(\vec{u}) = \exp[-ig\tilde{\phi}(\vec{u})]$ a "monopole field." More precisely,

$$\left\langle \exp\left(-i\int d^3x\, \rho_M \tilde{\phi}\right)\right\rangle = Z^{-1}[\rho_M = 0] Z[\rho_M] \tag{6.6}$$

in general, therefore: Quantized monopole matter fields couple locally to the dual potentials.

QED_4. The vacuum functional is

$$\begin{aligned} Z &= \int \mathfrak{D}G_{\mu\nu}\mathfrak{D}\tilde{A}_\mu \exp\left(-\tfrac{1}{4}\int d^4x\, G_{\mu\nu}G_{\mu\nu}\right)\\ &\quad\times\exp\left[\tfrac{1}{2}i\int d^4x\, \tilde{F}_{\mu\nu}(\tilde{A})G_{\mu\nu}\right], \end{aligned} \tag{6.7}$$

$$F_{\mu\nu}(\tilde{A}) = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu.$$

The saddle-point equations are

$$\partial_\mu \tilde{G}_{\mu\nu} = 0, \quad \tilde{G}_{\mu\nu} = iF_{\mu\nu}(\tilde{A}). \tag{6.8}$$

Thus, $\bar{A}_\mu \equiv i\tilde{A}_\mu$ are precisely the dual potentials for QED_4. Z is invariant under Abelian gauge transformations on $\tilde{A}_\mu$. This has crept in because I chose to ignore multiplicative constants in Eq. (4.2). $\tilde{A}_\mu$ gauges may be fixed in the usual ways.

A particularly interesting quantity is the gauge-invariant *dual-Wilson integral*

$$\tilde{W}[C'] \equiv \exp\left(-ig\oint_{C'} \tilde{A}_\mu dx_\mu\right). \tag{6.9}$$

As in QED_3, it is simple to see that $\langle\tilde{W}[C']\rangle$ corresponds to a monopole-antimonopole annihilation around the closed path C'. Also, quantized monopole matter fields should couple locally to $\tilde{A}_\mu$.

Because the Bianchi identities are (even for non-Abelian theories) at most quadratic in G, I can always integrate out the field strengths to obtain a *formulation entirely in terms of dual potentials*. For QED_4, the resulting action density is particularly elegant,

$$\mathfrak{L} = -\tfrac{1}{4}F_{\mu\nu}(\tilde{A})F_{\mu\nu}(\tilde{A}). \tag{6.10}$$

In this sense, QED_4 is perfectly dual.

QCD_3. The vacuum functional is ($\tilde{G}_i^a \equiv B_i^a$)

$$\begin{aligned} Z &= \int \mathfrak{D}B_i^a\,\mathfrak{D}\tilde{\phi}^a \exp\left(-\tfrac{1}{2}\int d^3x\, B_i^a B_i^a\right)\\ &\quad\times\exp\left\{-i\int d^3x\, B_i^a[\partial_i\tilde{\phi}^a - e\epsilon^{abc}A_i^b(B)\tilde{\phi}^c]\right\}, \end{aligned} \tag{6.11}$$

where $A_i^a(B)$ is given in Eq. (2.9). The saddle-point equations are

$$\begin{aligned} &\partial_i B_i^a - e\epsilon^{abc}A_i^b(B)B_i^c = 0,\\ &B_i^a + i[\partial_i\tilde{\phi}^a - e\epsilon^{abc}A_i^b(B)\tilde{\phi}^c] + b_i^a = 0, \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} b_i^a = -ie\epsilon^{abc}&\left[\theta(z-z_0)\int_z^\infty dz' - \theta(z_0 - z)\int_{-\infty}^z dz'\right]\\ &\times(\epsilon_{ij}B_j^b\tilde{\phi}^c)(xyz') \end{aligned} \tag{6.13}$$

for $i=1,2$ (ϵ_{ij} antisymmetric, $\epsilon_{12}=+1$), and

$$b_3^a = ie\epsilon^{abc}\delta(z-z_0)\left[\theta(x-x_0)\int_x^\infty dx' - \theta(x_0-x)\int_{-\infty}^x dx'\right] \times \int_{-\infty}^{+\infty} dz'(B_2^b\tilde{\phi}^c)(x'yz'). \tag{6.14}$$

A number of remarks are relevant here. (1) $\tilde{\phi}^a = -i\tilde{\phi}^a$ are the color-magnetic scalar potentials. (2) The nonlocalities are essentially the nonlocalities of the inversion forms $A(B)$. (3) Notice the singularity in b_3^a at $z=z_0$. This can be traced immediately to the fact that only $F_{12}^a(xyz_0)=B_3^a(xyz_0)$ appears in the inversion $A(B)$, Eq. (2.9). The singularity tells us immediately that, for configurations B_i^a not singular at z_0, the corresponding $\tilde{\phi}^a$ is discontinuous at $z=z_0$. I will mention this again in Sec. VII.

Notice also the Higgs-type coupling of the magnetic scalar potential. This raises the question of a Meissner effect for QCD_3, and focuses attention on regions where $\tilde{\phi}^a$ is a constant. In such regions, the magnetic field is a zero-eigenvalue eigenvector of the operator $M(\tilde{\phi}) \equiv 1+ie\int\tilde{\phi} = 1-e\int\overline{\phi}$ defined by recasting the field equations in the form

$$0 = i\partial_i\tilde{\phi}^a + \left[\left(1+ie\int\tilde{\phi}\right)B\right]_i^a. \tag{6.15}$$

The operator $M(\tilde{\phi})$ appears again in the pure dual potential formulation: Integrating out the field-strengths in Eq. (6.1), I obtain the action density[13]

$$\mathcal{L} = -\tfrac{1}{2}\partial\tilde{\phi}(1+ie\tilde{\phi})^{-1}\partial\tilde{\phi} \tag{6.16}$$

with a measure $[\det(1+ie\int\tilde{\phi})]^{-1/2}$. At large coupling, the zeros of $M(\tilde{\phi})$ approach the real axis and may be enhanced. This subject deserves further investigation.

QCD_4. The vacuum functional is

$$Z = \int \mathfrak{D}G_{\mu\nu}^a \mathfrak{D}\tilde{A}_\mu^a \exp\left(-\tfrac{1}{4}\int d^4x\, G_{\mu\nu}^a G_{\mu\nu}^a\right) \times \exp\left\{i\int d^4x\, \tilde{G}_{\mu\nu}^a[\partial_\mu\tilde{A}_\nu^a - e\epsilon^{abc}A_\mu^b(G)\tilde{A}_\nu^c]\right\}. \tag{6.17}$$

The saddle points are located at

$$\begin{aligned} &\partial_\mu\tilde{G}_{\mu\nu}^a - e\epsilon^{abc}A_\mu^b(G)\tilde{G}_{\mu\nu}^c = 0, \\ &-G_{\mu\nu}^a + i\epsilon_{\mu\nu\rho\sigma}[\partial_\rho\tilde{A}_\sigma^a - e\epsilon^{abc}A_\rho^b(G)\tilde{A}_\sigma^c] + g_{\mu\nu}^a = 0, \end{aligned} \tag{6.18}$$

where

$$\begin{aligned}
g_{01}^a &= -ie\delta(z-z_0)\left[\theta(x-x_0)\int_x^\infty dx' - \theta(x_0-x)\int_{-\infty}^x dx'\right]\int_{-\infty}^{+\infty} dz'\epsilon^{abc}(\tilde{G}_{0\sigma}^b\tilde{A}_\sigma^c)(tx'yz'),\\
g_{02}^a &= -ie\delta(z-z_0)\delta(x-x_0)\left[\theta(y-y_0)\int_y^\infty dy' - \theta(y_0-y)\int_{-\infty}^y dy'\right]\int_{-\infty}^{+\infty} dx'\int_{-\infty}^\infty dz'\epsilon^{abc}(\tilde{G}_{0\sigma}^b A_\sigma^c)(tx'y'z'),\\
g_{03}^a &= -ie\left[\theta(z-z_0)\int_z^\infty dz' - \theta(z_0-z)\int_{-\infty}^z dz'\right]\epsilon^{abc}(\tilde{G}_{0\sigma}^b\tilde{A}_\sigma^c)(txyz'),\\
g_{12}^a &= +ie\delta(z-z_0)\left[\theta(x-x_0)\int_x^\infty dx' - \theta(x_0-x)\int_{-\infty}^x dx'\right]\int_{-\infty}^{+\infty} dz'\epsilon^{abc}(\tilde{G}_{2\sigma}^b\tilde{A}_\sigma^c)(tx'yz'),\\
g_{23}^a &= -ie\left[\theta(z-z_0)\int_z^\infty dz' - \theta(z_0-z)\int_{-\infty}^z dz'\right]\epsilon^{abc}(\tilde{G}_{2\sigma}^b\tilde{A}_\sigma^c)(txyz'),\\
g_{31}^a &= +ie\left[\theta(z-z_0)\int_z^\infty dz' - \theta(z_0-z)\int_{-\infty}^z dz'\right]\epsilon^{abc}(\tilde{G}_{1\sigma}\tilde{A}_\sigma^c)(txyz').
\end{aligned} \tag{6.19}$$

My previous remarks about QCD_3 are compounded here. Notice in particular that *each* component of $\tilde{A}_\mu^a$ has Higgs-type couplings. The field equations may be cast in the form

$$\begin{aligned} &\left[\left(1+ie\int\tilde{A}\right)G\right]_{\mu\nu}^a = i\tilde{F}_{\mu\nu}^{(0)a}(\tilde{A}),\\ &\tilde{F}_{\mu\nu}^{(0)a}(\tilde{A}) = \partial_\mu\tilde{A}_\nu^a - \partial_\nu\tilde{A}_\mu^a. \end{aligned} \tag{6.20}$$

For small coupling then

$$\tilde{G}_{\mu\nu}^a \sim iF_{\mu\nu}^{(0)a}(\tilde{A}), \tag{6.21}$$

as in the Abelian case. In general, however, as in QCD_3, attention is drawn to configurations for which $F_{\mu\nu}^{(0)a}(\tilde{A})=0$ in some region, and hence to the zeros of the operator $(1+ie\int\tilde{A})$. The pure dual potential formulation of QCD_4 has the form

$$\mathcal{L} = -\tfrac{1}{4}\tilde{F}^{(0)}(\tilde{A})\left(1+ie\int\tilde{A}\right)^{-1}\tilde{F}^{(0)}(\tilde{A}) \tag{6.22}$$

with measure $[\det(1+ie\int\tilde{A})]^{-1/2}$.

VII. "DUAL" HAMILTONIAN FORMULATION

In this section, I work in Minkowski space.[14] The generic vacuum functional

$$Z = \int \mathfrak{D}G\,\mathfrak{D}\tilde{A}\exp\left\{-i\int[\tfrac{1}{4}G^2 + \tilde{A}I(G)]\right\} \tag{7.1}$$

contains time derivatives only in the factor

$$\exp\left(i\int d^3x\,\tilde{\phi}\partial_0\tilde{G}^0\right) \tag{7.2a}$$

in three dimensions, or

$$\exp\left(-i\int d^4x\,\tilde{A}_\mu\partial_0\tilde{G}^{0\mu}\right) \tag{7.2b}$$

in four dimensions. Thus in general $\tilde{A}$ is canonically conjugate to the *magnetic* fields. I can pass simply to the canonical form then by integrating out the electric fields $G_{0i}=E_i$, leaving only $\tilde{A}$ and B, the magnetic fields. The resulting "dual" Hamiltonian system has the generic form

$$H=\tfrac{1}{2}\int[B^2+E^2(\tilde{A},B)],\quad [B,\tilde{A}]=i. \tag{7.3}$$

I will briefly discuss each theory in detail, giving the appropriate E in terms of $\tilde{A}$ and B.

QED_3. In the previous sections, I worked in a manifestly Euclidean form for three dimensions. If I identify $3\to 0$ (time), I am in a fixed A_0 gauge, and there is $z\to t$ (time) nonlocality. I do not know how to achieve a Hamiltonian formulation for these gauges. I convert trivally to an axial gauge with the reidentification $1\to 1$, $2\to 0$, $3\to 2$. This is now a fixed $A_2=0$ gauge:

$$\begin{aligned} A_1&=-\int_{y_0}^{y}dy'B(xy'),\\ A_0&=-\int_{y_0}^{y}dy'E_2(xy')-\int_{x_0}^{x}dx'E_1(x'y_0), \end{aligned} \tag{7.4}$$

where I have used $E_i=G_{0i}$, $B=G_{12}$, $\tilde{G}_\mu=\frac{1}{2}\epsilon_{\mu\nu\rho}G^{\nu\rho}$, $\epsilon_{012}=1$. Notice that the spatial potential (A_1) is a function of B alone; hence fixed-time Wilson integrals are area integrals over the magnetic field. This persists generically.

The action density is

$$\mathcal{L}=-\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}+\tilde{\phi}\partial_\mu\tilde{G}^\mu. \tag{7.5}$$

Integrating G_{0i}, I obtain the dual Hamiltonian, Eq. (7.3) with

$$\begin{aligned} &E_1=\partial_2\tilde{\phi},\quad E_2=-\partial_1\tilde{\phi},\\ &[B(\vec{x}),\tilde{\phi}(\vec{y})]=i\delta^{(2)}(\vec{x}-\vec{y}). \end{aligned} \tag{7.6}$$

Notice that Gauss's law $\nabla\cdot\vec{E}=0$ is an identity in the the dual formulation, just as the Bianchi identities are identities in the usual formulation. This also persists generically.

Consider the local operator

$$\Phi(x)\equiv\exp[-ig\tilde{\phi}(\vec{x})]. \tag{7.7}$$

Since $\tilde{\phi}$ and B are canonical, Φ generates local disturbances in the magnetic field,

$$\Phi(\vec{x})B(\vec{y})\Phi^\dagger(x)=B(\vec{y})-g\delta^{(2)}(\vec{y}-\vec{x}). \tag{7.8}$$

The disturbance is a string with magnetic flux g. The Wilson integrals at fixed time are area integrals over the magnetic field, and hence

$$\Phi(\vec{x})W[C]=W[C]\Phi(x)\exp\left[ieg\iint_{S(C)}\delta^{(2)}(\vec{y}-\vec{x})d^2y\right]. \tag{7.9}$$

If I choose $\frac{1}{2}$ Dirac unit of magnetic flux,

$$\Phi_{1/2}(\vec{x})=\exp\left[-i\frac{\pi}{e}\tilde{\phi}(\vec{x})\right], \tag{7.10}$$

then $\Phi_{1/2}(\vec{x})$ commutes with $W[C]$ when $\vec{x}$ is outside C, and anticommutes when $\vec{x}$ is inside C. Evidently, $\Phi_{1/2}(\vec{x})$ is the Abelian analog of the [SU(2) in three dimensions] disorder operator defined implicitly by 't Hooft.[5]

QED_4. In four dimensions, my notation is $E_i=G_{0i}$, $B_i=\frac{1}{2}\epsilon_{ijk}G_{jk}$, $\tilde{G}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$, $\epsilon_{0123}=+1$. The inversion Eq. (2.10) is then

$$\begin{aligned} A_1&=\int_{z_0}^{z}dz'B_2(xyz'),\\ A_2&=-\int_{z_0}^{z}dz'B_1(xyz')+\int_{x_0}^{x}dx'B_3(x'yz_0),\\ A_0&=-\int_{z_0}^{z}dz'E_3(xyz')-\int_{x_0}^{x}dx'E_1(x'yz_0)\\ &\quad-\int_{y_0}^{y}dy'E_2(x_0y'z_0). \end{aligned} \tag{7.11}$$

Again the spatial components of A_μ are functions of B alone, and hence fixed-time Wilson integrals are area integrals over B. The action density is

$$\mathcal{L}=-\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}-\tilde{A}_\nu\partial_\mu\tilde{G}^{\mu\nu}. \tag{7.12}$$

But the momentum conjugate to $\tilde{A}_0$ is zero. I choose therefore not to introduce $\tilde{A}_0$, leaving $\delta[\partial_\mu\tilde{G}^{\mu 0}]=\delta[\nabla\cdot\vec{B}]=\delta[I_0(G)]$ as a constraint. Integrating out G_{0i}, I obtain the dual Hamiltonian Eq. (7.3) with

$$\begin{aligned} &\vec{E}=\nabla\times\vec{\tilde{A}},\\ &[B_i(\vec{x}),\tilde{A}_j(\vec{y})]=i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}). \end{aligned} \tag{7.13}$$

This Hamiltonian must be taken with the constraint $I_0=\nabla\cdot\vec{B}=0$. Note that $I_0=\nabla\cdot\vec{B}$ is the generator of $\tilde{A}_i$ gauge transformations, so duality

$$\vec{B}\to\vec{E},\quad \vec{A}\to\vec{\tilde{A}},\quad \nabla\cdot\vec{E}\to\nabla\cdot\vec{B} \tag{7.14}$$

is quite perfect. Gauss's law $\nabla\cdot\vec{E}=0$ is again an identity.

Consider the dual Wilson integral as an operator, with C' a fixed-time path:

$$\tilde{W}[C']=\exp\left(-ig\oint_{C'}\vec{\tilde{A}}\cdot d\vec{x}\right). \tag{7.15}$$

The fixed-time Wilson integrals are area integrals over the magnetic fields, and so

$$\tilde{W}[C']W[C]=W[C]\tilde{W}[C']\exp(iegN),$$
$$N=\int_{S[C]} d\vec{S}(\vec{x})\cdot\oint_{C'} d\vec{x}\,\delta^{(3)}(\vec{x}-\vec{x}'). \tag{7.16}$$

N is the number of directed crossings of C' through C. With $\frac{1}{2}$ Dirac unit of flux, $\tilde{W}[C']$ is the Abelian analog of the disorder operator [SU(2) in four dimensions] 't Hooft[5] defines implicitly.

There is a final comment that I wish to make here about $\tilde{A}$ gauge fixing. At the level of QED_4, the remark is essentially pedantic—but it will be helpful in our discussion of QCD_4.

The $\tilde{A}$ gauge may be fixed in the usual Faddeev-Popov fashion, but a certain bizarre gauge choice has particular significance. Remember that in four dimensions I chose to complete the identity Eq. (4.2) to all four Bianchi identities. If I had left only the 3.1 Bianchi identities, I could have exponentiated the third Bianchi identity with an $\tilde{A}_3(txy)$ only. I leave it as an exercise for the reader to show that, in our more symmetric formulation, this corresponds to the bizarre $\tilde{A}$ gauge choice

$$\delta[\tilde{A}_3(txyz)-\tilde{A}_3'(txy)\delta(z-z_0)], \tag{7.17}$$

that is, gauging away the z dependence of $\tilde{A}_3$, except at z_0. The most direct way to use this gauge is to solve the Bianchi identity

$$B_3(xyz)=B_3(xyz_0)-\int_{z_0}^{z}dz'(\partial_1B_1+\partial_2B_2)(xyz'), \tag{7.18}$$

keeping only $\tilde{A}_3'(xy)$, $B_3(xyz_0)$ as canonical variables.

QCD_3. I use the same fixed $A_2=0$ gauge as for QED_3. The inversion is still Eq. (7.4), and the action density is

$$\mathcal{L}=-\tfrac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a+\tilde{\phi}^a[\partial_\mu\tilde{G}^\mu_a-e\epsilon^{abc}A^b_\mu(G)\tilde{G}^\mu_c]. \tag{7.19}$$

Integrating G^a_{0i}, I obtain the Hamiltonian with

$$E^a_1=\partial_2\tilde{\phi}^a-e\delta(y-y_0)\times\left[\theta(x-x_0)\int_x^\infty dx'-\theta(x_0-x)\int_{-\infty}^x dx'\right]\int_{-\infty}^{+\infty}dy'\mathcal{J}^a(x'y'),$$
$$E^a_2=-[\partial_1\tilde{\phi}^a-e\epsilon^{abc}A^b_1(B)\tilde{\phi}^c]-e\left[\theta(y-y_0)\int_y^\infty dy'-\theta(y_0-y)\int_{-\infty}^y dy'\right]\mathcal{J}^a(xy'), \tag{7.20}$$
$$[B^a(\vec{x}),\tilde{\phi}^b(\vec{y})]=i\delta^{ab}\delta^{(2)}(\vec{x}-\vec{y}).$$

Here $A_1(B)$ is given in Eq. (7.4) and $\mathcal{J}^a\equiv\epsilon^{abc}B^b\tilde{\phi}^c$ is a rotation operator for $\tilde{\phi}^a$ and B^a. Observe the persistence of the Higgs coupling of the dual potential. The singular term $\delta(y-y_0)$, seen in the equations of motion, also persists.[15]

Note that

$$Q^a\equiv\int_{-\infty}^{+\infty}dx'\int_{-\infty}^{+\infty}dy'\mathcal{J}^a(x'y') \tag{7.21}$$

is a constant of the motion. Q^a generates a global gauge transformation, and this freedom reflects the fact that my gauges are not fixed up to such a trivial transformation. Physical states can easily be chosen to satisfy $Q^a=0$.

From the definition of E^a_i, it is easy to compute that

$$\nabla\cdot\vec{E}^a-e\epsilon^{abc}A^b_i(B)E^c_i=-e\delta(y-y_0)\delta(x-x_0)Q^a, \tag{7.22}$$

so on physical states ($Q^a=0$), Gauss's law will be an identity.

I also repeat that Wilson integrals over spatial paths are functionals of B only [because $A_1=A_1(B)$]. Having B as a fundamental variable may then be advantageous in studying the confinement problem.

I have not yet found the exact expression of 't Hooft's operators for QCD_3. Although operators such as $\exp(-ig\tilde{\phi}^a)$, $\mathrm{Tr}\exp(-ig\tilde{\phi}^a\frac{1}{2}\tau_a)$ have the correct effect on the magnetic field, they generate nonlocalized changes in $\vec{E}$. This deserves further investigation.

QCD_4. The inversion is given, as for QED_4, in Eq. (7.11). The action density is

$$\mathcal{L}=-\tfrac{1}{4}G^aG^{\mu\nu}_a-\tilde{A}^a_\nu[\partial_\mu\tilde{G}^{\mu\nu}_a-\epsilon^{abc}A^b_\mu(G)\tilde{G}^{\mu\nu}_c]. \tag{7.23}$$

However, as in QED_4, I choose to leave $\delta(I^a_0(B))$ as a constraint, keeping only $\tilde{A}^a_i$. Integrating G^a_{0i}, I obtain the dual Hamiltonian Eq. (7.3) with

$$E^a_i=\mathcal{C}^a_i+J^a_i,\qquad \mathcal{C}^a_i\equiv\epsilon_{ijk}[\partial_j\tilde{A}^a_k-e\epsilon^{abc}A^b_j(B)\tilde{A}^c_k], \tag{7.24}$$

$$J^a_1=-e\delta(z-z_0)\left[\theta(x-x_0)\int_x^\infty dx'-\theta(x_0-x)\int_{-\infty}^x dx'\right]\int_{-\infty}^{+\infty}dz'\mathcal{J}^a(x'yz'),$$
$$J^a_2=-e\delta(z-z_0)\delta(x-x_0)\left[\theta(y-y_0)\int_y^\infty dy'-\theta(y_0-y)\int_{-\infty}^y dy'\right]\int_{-\infty}^{+\infty}dx'\int_{-\infty}^{+\infty}dz'\mathcal{J}^a(x'y'z'),$$
$$J^a_3=-e\left[\theta(z-z_0)\int_z^\infty dz'-\theta(z_0-z)\int_{-\infty}^z dz'\right]\mathcal{J}^a(xyz'), \tag{7.25}$$
$$[B^a_i(\vec{x}),\tilde{A}^b_j(\vec{y})]=i\delta^{ab}\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}),\qquad \mathcal{J}^a\equiv\epsilon^{abc}B^b_i\tilde{A}^c_i.$$

This Hamiltonian must be taken with the constraint

$$I_0^a = \partial_i B_i^a - e\epsilon^{abc} A_i^b(B) B_i^c = 0. \qquad (7.26)$$

As with QCD_3, note the persistence of the Higgs-type couplings of the dual potentials. $Q^a \equiv \int d^3x\, \mathcal{G}^a$ is a constant of the motion, and can be set equal to zero on physical states. Wilson integrals with spatial contours are all functions of B_i^a. Forms such as

$$\mathrm{Tr}P\exp\left(-ig\oint_{C'} \tilde{A}_i^a \tfrac{1}{2}\tau_a\, dx_i\right)$$

are candidates for 't Hooft operators.

I wish to discuss in some detail the structure of this constrained Hamiltonian system and its parallels with QED_4. In QED_4, the constraint $I_0 = \nabla\cdot\vec{B}$ generates Abelian gauge transformations on $\tilde{A}_i$ and a symmetry of the Hamiltonian. In Appendix D, I show a similar property for the QCD_4 constraints $I_0^a(B)$: Modulo ordering problems

$$\frac{d}{dt} I_0^a = e\epsilon^{abc}\left[E_3^b \int_{z_0}^{z} dz' I_0^c(xyz') + A_0^b I_0^c\right]. \qquad (7.27)$$

Thus, if one starts with a state at time t_0 with $I_0^a = 0$, it will stay zero. Put another way, $I_0^a(B)$ generates a change in the Hamiltonian which vanishes in the presence of the constraint. Therefore, if desired, gauges may be fixed in the manner of Faddeev and Popov.

What are the transformations generated by $I_0^a(B)$? They are *Abelian, and act only on* $\tilde{A}_i^a$, just as in QED_4. A simple computation results in

$$[I_0^a(xyz), \tilde{A}_i^b(x'y'z')] = i\delta^{ab}\partial_i\delta^{(3)}(\vec{x}-\vec{x}') + ie\epsilon^{abc}[\delta_{i1}A_1^c(B) + \delta_{i2}A_2^c(B)]\,\delta^{(3)}(\vec{x}-\vec{x}')$$
$$+ ie\epsilon^{abc}\delta(x-x')\delta(y-y')\theta(zz'z_0)(\delta_{i1}B_2^c - \delta_{i2}B_1^c) - ie\epsilon^{abc}\delta_{i3}\delta(y-y')\delta(z_0-z')\theta(xx'x_0)B_2^c(xyz), \qquad (7.28)$$

where

$$\theta(zz'z_0) \equiv \theta(z-z')\theta(z'-z_0) - \theta(z_0-z')\theta(z'-z)$$
$$= \theta(z'-z_0) - \theta(z'-z). \qquad (7.29)$$

The feature of note in Eq. (7.28) is that

$$[I_0^a(xyz), \tilde{A}_3^b(x'y'z')]_{z'\neq z_0} = i\delta^{ab}\partial_3\delta^{(3)}(\vec{x}-\vec{x}'). \qquad (7.30)$$

Thus, as long as one stays away from the (gauge) point $z = z_0$, the gauge transformation on $\tilde{A}_3^a$ is the usual Abelian form. As a result, the gauge choice Eq. (7.17) can be made here for QCD_4, with constant measure.

This brings us full circle for QCD_4; as explained in QED_4, this is the form I would have obtained if I had not completed all four Bianchi identities in Eq. (4.2). The Abelian invariance in QCD_4 is a direct result of completing the Bianchi identities. As in QED_4, the most direct way to use this gauge is to solve $I_0 = 0$:

$$B_3^a(xyz) = B_3^a(xyz_0) + \int_{z_0}^{z} dz[-\partial_1 B_1^a - \partial_2 B_2^a + e\epsilon^{abc}A_i^b(B)B_i^c](xyz'), \qquad (7.31)$$

thus eliminating B_3^a in favor of $B_3^a(xyz_0)$, which is canonical to $\tilde{A}_3^{a\prime}(xy)$.

Finally, I remark on Gauss's law. It is straightforward to compute that[16]

$$\partial_i E_i^a - e\epsilon^{abc}A_i^b(B)E_i^c = -e\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)Q^a + e\epsilon^{abc}[B_3^b - F_{12}^b(A(B))]\tilde{A}_3^c. \qquad (7.32)$$

When the solution (7.31) is used, $F_{12}^b(A(B)) = B_3^b$, so, on states with $Q^a = 0$, Gauss's law is again an identity.

VIII. REMARKS

There are a number of remarks that properly belong with this piece of work. Perhaps the most important is what I promised in the Introduction: *Confining states are easy to construct in the dual formulation.* Just as an example, consider the states

$$|\alpha, \tilde{\phi}_0\rangle \equiv \int \mathfrak{D}B' \exp\left[\int d^2x(-\tfrac{1}{2}\alpha B'^2 + iB'\tilde{\phi}_0)\right]|B'\rangle \qquad (8.1)$$

in QED_3. Here $|B'\rangle$ is an eigenstate of B; $\tilde{\phi}_0$ is the center of the Gaussian in $\tilde{\phi}'$, eigenvalues of the dual potential. Because the Wilson integral is an area integral of B, one trivially computes

$$\frac{\langle\alpha, \tilde{\phi}_0|W[C]|\alpha, \tilde{\phi}\rangle}{\langle\alpha, \tilde{\phi}_0|\alpha, \tilde{\phi}_0\rangle} = \exp\left(-\frac{e^2 A}{4\alpha}\right), \qquad (8.2)$$

where A is the area of the Wilson loop. Note that confinement is stronger for small α: This is the "*disordered*" state, characterized by a broad spread in B' (or $\tilde{\phi}'$ peaked fairly sharply at $\tilde{\phi}_0$). Similar states can be constructed in the non-Abelian case as well. The problem, of course, is finding a meaningful minimum of the Hamiltonian in such confining states.

My second remark concerns the manifestation of the old field-strength copies, now in the field-strength formulation. As discussed in Ref. 2, the field copies from *every* gauge have become local-action copies in the fixed gauges. In the field-strength description, local-action copies are sets of field strengths, all satisfying $I(G) = 0$, related to each other by local rotations. I will call such rotations pseudogauge transformations, as they are *not* related to ordinary gauge transformations. (My gauges are fixed, and the local action copies are *not* gauge equivalent.)

In QCD_2, for example,

$$Z=\int \mathfrak{D}G^a_{03}\exp\left(-\tfrac{1}{2}\int d^2x\, G^a_{03}G^a_{03}\right), \tag{8.3}$$

and *every* configuration is a member of a continuous family of local-action copies. In higher dimension, the sets of local-action copies can be characterized as little "pockets" of pseudogauge equivalence. This is, of course, the possible "enhancement" I discussed in Ref. 2, and deserves further investigation.

My third remark is on the possibility of "other" dual potentials besides my choice. In particular, it would be interesting to find an $\tilde{A}^a_\mu$ for QCD_4 that transforms as a gauge field (my $\tilde{A}^a_\mu$ essentially only rotates). Such an $\tilde{A}^a_\mu$ might be one satisfying

$$\tilde{F}^a_{\mu\nu}(A)=F^a_{\mu\nu}(\tilde{A}). \tag{8.4}$$

Unfortunately, it is easy to show that this $\tilde{A}$ cannot exist for all A: The form $F^a_{\mu\nu}(\tilde{A})$ implies the Bianchi identity $\partial\tilde{F}(\tilde{A})-e\tilde{A}\tilde{F}(\tilde{A})=0$, and hence, using Eq. (8.4), $\partial F(A)-e\tilde{A}F(A)=0$. This is to be compared with the saddle-point equations $\partial F(A)-eAF(A)=0$. For configurations with $\det F(A)\neq 0$,[12] one easily shows that $A=\tilde{A}$ and hence, from Eq. (8.4) again, that F is self-dual. This is a contradiction for configurations such as the anti-instanton.

I mention two other candidates for $\tilde{A}^a_\mu$ whose merit is existence for all A. (1) $\tilde{A}$ defined by $\partial F(A)-e\tilde{A}F(A)=0$. $\tilde{A}$ is dual to A in the sense that its defining equation is dual to the Bianchi identity. For $\det F\neq 0$,[12] $\tilde{A}=e^{-1}F^{-1}\partial F$. At saddle points with $\det F\neq 0$, $A=\tilde{A}$. This object was discussed in Ref. 12. (2) In the main development of the text, I computed $A(F)$. One can define $\tilde{A}\equiv A(\tilde{F})$. The utility of either of these two definitions remains to be studied.

I remark finally that the path-dependent formalism[17] may be quite useful in studying general features of my reformulations.

ACKNOWLEDGMENTS

I wish to thank K. Bardakci, D. Friedan, M. Hillery, and S. Mandelstam for helpful discussions. This work was supported by the U.S. Department of Energy and the National Science Foundation under Grant No. PHY 77-23512.

APPENDIX A: $A(F)$ AND 3.1 BIANCHI IDENTITIES FOR QCD_4

I work in the fixed axial gauge $0=A_3(txyz)=A_1(txyz_0)=A_2(tx_0yz_0)=A_0(tx_0y_0z_0)$, and I list for reference in this gauge

$$\begin{aligned}
&F^a_{01}(A)=\partial_0A^a_1-\partial_1A^a_0-e\epsilon^{abc}A^b_0A^c_1,\\
&F^a_{02}(A)=\partial_0A^a_2-\partial_2A^a_0-e\epsilon^{abc}A^b_0A^c_2,\\
&F^a_{03}(A)=-\partial_3A^a_0,\quad F^a_{31}(A)=\partial_3A^a_1,\quad F^a_{23}(A)=-\partial_3A^a_2,\\
&F^a_{12}(A)=\partial_1A^a_2-\partial_2A^a_1-e\epsilon^{abc}A^b_1A^c_2.
\end{aligned} \tag{A1}$$

From the form $F_{31}(A)$ and the boundary condition on A_1, I have immediately

$$A_1(F)=\int_{z_0}^{z}dz'F_{31}(txyz'). \tag{A2}$$

From the form $F_{23}(A)$ and the boundary condition on A_2, I have

$$A_2=-\int_{z_0}^{z}dz'F_{23}(txyz')+\Delta_2(txy),\quad \Delta_2(tx_0y)=0. \tag{A3}$$

Substituting $A_1(F)$ and this form for A_2 into the form $F_{12}(A)$ at $z=z_0$, I obtain

$$\begin{aligned}
&F_{12}(txyz_0)=\partial_1\Delta_2(txy)\\
&\Rightarrow\Delta_2=\int_{x_0}^{x}dx'F_{12}(tx'yz_0)\\
&\Rightarrow A_2(F)=-\int_{z_0}^{z}dz'F_{23}(txyz')+\int_{x_0}^{x}dx'F_{12}(tx'yz_0).
\end{aligned} \tag{A4}$$

$A_0(F)$ is the most difficult. From $F_{03}(A)$ and the boundary condition on A_0, I have

$$A_0=-\int_{z_0}^{z}dz'F_{03}(txyz')+\Delta_0(txy),\quad \Delta_0(tx_0y_0)=0. \tag{A5}$$

Using this and $A_1(F)$ in $F_{01}(A)$ at $z=z_0$ results in

$$\begin{aligned}
&F_{01}(txyz_0)=-\partial_1\Delta_0(txy)\\
&\Rightarrow\Delta_0(txy)=-\int_{x_0}^{x}dx'F_{01}(tx'yz_0)+\bar{\Delta}_0(ty),\quad \bar{\Delta}_0(ty_0)=0.
\end{aligned} \tag{A6}$$

This result and $A_2(F)$ substituted into the form $F_{02}(A)$ at $z=z_0$, $x=x_0$ allows the completion of $A(F)$,

$$\begin{aligned}
&F_{02}(tx_0yz_0)=-\partial_2\bar{\Delta}_0(ty)\\
&\Rightarrow\bar{\Delta}_0=-\int_{y_0}^{y}dy'F_{02}(tx_0yz_0),\\
&A_0(F)=-\int_{z_0}^{z}dz'F_{03}(txyz')-\int_{x_0}^{x}dx'F_{01}(tx'yz_0)\\
&\qquad-\int_{y_0}^{y}dy'F_{02}(tx_0y'z_0).
\end{aligned} \tag{A7}$$

I must next explore the consistency conditions $F\{A(F)\}=F$; we systematically substitute each $A(F)$ into the form $F(A)$ and require it to **be** F. There is no difficulty verifying $F_{03}\{A(F)\}=F_{03}$, $F_{31}\{A(F)\}=F_{31}$, $F_{23}\{A(F)\}=F_{23}$. Already for F_{12}, however, we obtain the restriction on the allowed field strengths,

$$F^a_{12}-F^a_{12}(txyz_0)=-\int_{z_0}^{z}dz'[\partial_1 F^a_{23}(txyz')+\partial_2 F^a_{31}(txyz')]$$
$$+e\epsilon^{abc}\int_{z_0}^{z}dz'F^b_{31}(txyz')\int_{z_0}^{z'}dz''F^c_{23}(txyz'')-e\epsilon^{abc}\int_{z_0}^{z}dz'F_{31}(txyz')\int_{x_0}^{x}dx'F^c_{12}(tx'yz_0). \tag{A8}$$

This equation is manifestly true at $z=z_0$ [I required it at that point in the derivation of $A(F)$], so I lose no information by differentiating with respect to z. With simple algebra, the differentiated form may be written

$$I^a_0(F)=\partial_\mu \tilde{F}^a_{\mu 0}-e\epsilon^{abc}A^b_\mu(F)\tilde{F}^c_{\mu 0}=0,\quad \tilde{F}_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma},\quad \epsilon_{0123}=1, \tag{A9}$$

which is recognized as the temporal Bianchi identity. Here $A^a_\mu(F)$ is just our inversion $A(F)$ [and $A_3(F)=0$]. I continue to use the notation $I^a_\nu(F)=\partial_\mu\tilde{F}^a_{\mu\nu}-e\epsilon^{abc}A^b_\mu F^c_{\mu\nu}$ for the Bianchi forms.

Turning now to F_{01}, one obtains a similar condition manifestly true at $z=z_0$. Differentiating with respect to z results this time in the second Bianchi identity

$$I^a_2(F)=0. \tag{A10}$$

Finally, substituting into F_{02}, I obtain

$$F^a_{02}=\int_{z_0}^{z}dz'[-\partial_0 F^a_{23}(txyz')+\partial_2 F^a_{03}(txyz')]+\int_{x_0}^{x}dx'[\partial_0 F^a_{12}(tx'yz_0)+\partial_2 F^a_{01}(tx'yz_0)]+F^a_{02}(tx_0yz_0)$$
$$+e\epsilon^{abc}\left[\int_{z_0}^{z}dz'F^b_{03}(txyz')+\int_{x_0}^{x}dx'F^b_{01}(tx'yz_0)+\int_{y_0}^{y}dy'F^b_{02}(tx_0y'z_0)\right]$$
$$\times\left[-\int_{z_0}^{z}dz''F^c_{23}(txyz'')+\int_{x_0}^{x}dx''F^c_{12}(tx''yz_0)\right]. \tag{A11}$$

As above, the derivative with respect to z of Eq. (A11) results in another Bianchi identity,

$$I^a_1(F)=0. \tag{A12}$$

However, Eq. (A11) is not manifestly true at $z=z_0$. I must therefore require it explicitly. Substituting the restriction Eq. (A12) into Eq. (A11) at $z=z_0$, and doing some simple algebra (all terms involving z integrations cancel) results in

$$0=F^a_{02}(tx_0yz_0)-F^a_{02}(txyz_0)+\int_{x_0}^{x}dx'[\partial_0 F^a_{12}(tx'yz_0)+\partial_2 F^a_{01}(tx'yz_0)]$$
$$+e\epsilon^{abc}\left[\int_{x_0}^{x}dx'F^b_{01}(tx'yz_0)+\int_{y_0}^{y}dy'F^b_{02}(tx_0yz_0)\right]\int_{x_0}^{x}dx'F^c_{12}(tx'yz_0). \tag{A13}$$

This is manifestly true at $x=x_0$, so I can differentiate with respect to x. This results in the last 0.1 Bianchi identity

$$I^a_3(txyz_0)=0. \tag{A14}$$

I have demonstrated that the form $A(F)$ [Eq. (2.10) in the text] together with the 3.1 Bianchi identities (as restrictions on allowed field strengths) constitute the completely unique inversion for QCD_4.

APPENDIX B: REDUNDANCY OF LAST 0.9 BIANCHI IDENTITY FOR QCD_4

In four space-time dimensions, the consistency conditions on the field strengths are only 3.1 Bianchi identities—the Bianchi identity in the gauge direction μ_G (for our case $\mu_G=3$) being required only at one point in the μ_G direction (for our case $z=z_0$). In this appendix, I want to demonstrate that these 3.1 Bianchi identities imply the full set of four. The result is gauge independent, so for notational simplicity I will work as if I had performed the inversion in the $A_0=0$ (temporal gauge).

I begin with $A(F)$ in the temporal gauge, and the 3.1 Bianchi identities $I^a_i=0$ ($i=1,2,3$), $I^a_0(t_0xyz)=0$. I will not need the explicit form of $A(F)$—two properties will suffice: (1) $A_0(F)=0$ and (2) $F\{A(F)\}=F$ (because of the 3.1 Bianchi identities). I want to show that then $\partial_0 I^a_0=0$ follows.

Consider $\partial_0 I^a_0=\partial_0[\partial_i\tilde{F}^a_{i0}-e\epsilon^{abc}A^b_i(F)\tilde{F}^c_{i0}]$ where i is summed from 1 to 3. Noting that $\partial_0 A^b_i(F)=F^b_{0i}$ and using $I^a_i=0$ ($i=1,2,3$) to eliminate all other time derivatives, I compute

$$\partial_0 I^a_0=\partial_i\partial_j\tilde{F}^a_{ij}-e\epsilon^{abc}\partial_i A^b_j\tilde{F}^c_{ji}$$
$$-e\epsilon^{abc}F^b_{0i}\tilde{F}^c_{i0}+e^2\epsilon^{abc}\epsilon^{cde}A^b_i A^d_j\tilde{F}^e_{ji}. \tag{B1}$$

Of course $\partial_i\partial_j\tilde{F}^a_{ij}=0$ (this would complete the proof trivially for the Abelian case). Because $F_{ij}(A(F)) = F_{ij}$, the other terms are easily rearranged to

$$\partial_0 I_0^a = -\tfrac{1}{2} e\epsilon^{abc} F_{ij}^b \tilde{F}_{ij}^c - e\epsilon^{abc} F_{0i}^b \tilde{F}_{i0}^c = 0. \qquad \text{(B2)}$$

This completes the proof that the last 0.9 Bianchi identity is redundant in four space-time dimensions.

APPENDIX C: THE CRUCIAL IDENTITY IN THE QUANTUM VARIABLE CHANGE $A \to F$

I will sketch the computation for QCD_4. I begin with (I suppress products Π_a over color indices)

$$\begin{aligned} M[G] &\equiv \int \mathfrak{D}A\,\delta[\text{CGF}]\,\delta[G-F(A)] \\ &= \int \mathfrak{D}A_1^a \mathfrak{D}A_2^a \mathfrak{D}A_0^a\, \delta[A_1^a(txyz_0)]\,\delta[A_2^a(tx_0yz_0)]\,\delta[A_0^a(tx_0y_0z_0)]\,\delta[G_{03}^a+\partial_3 A_0^a]\,\delta[G_{31}^a-\partial_3 A_1^a] \\ &\quad \times \delta[G_{23}^a+\partial_3 A_2^a]\,\delta[G_{01}^a-(\partial_0 A_1^a-\partial_1 A_0^a - e\epsilon^{abc}A_0^b A_1^c)]\,\delta[G_{02}^a-(\partial_0 A_2^a-\partial_2 A_0^a-e\epsilon^{abc}A_0^b A_2^c] \\ &\quad \times \delta[G_{12}^a-(\partial_1 A_2^a-\partial_2 A_1^a - e\epsilon^{abc}A_1^b A_2^c)]\,. \end{aligned} \qquad \text{(C1)}$$

The integration strategy is a slavish imitation of the steps I followed during the inversion (see Appendix A).

The A_1 integration is easily done, using the gauge-fixing $A_1\delta$ functional and $\delta[G_{31}^a-\partial_3 A_1^a]$. The value of A_1 selected is

$$A_1 = A_1(G) = \int_{z_0}^{z} dz' G_{31}^a(txyz').$$

Ignoring multiplicative constants one can read $M[G]$ as the rest of the integrand with $A_1 \to A_1(G)$.

To do the A_2 integration, first change variables to Δ_2^a via the shift

$$A_2^a = -\int_{z_0}^{z} dz' G_{23}^a(txyz') + \Delta_2^a(txyz).$$

In terms of the new integration variables, I have

$$\begin{aligned} M[G] &= \int \mathfrak{D}\Delta_2^a \mathfrak{D}A_0^a\, \delta[\Delta_2^a(tx_0yz_0)]\,\delta[A_0^a(tx_0y_0z_0)]\,\delta[G_{03}+\partial_3 A_0^a] \\ &\quad \times \delta[\partial_3 \Delta_2^a]\,\delta[G_{01}^a-[\partial_0 A_1(G)-\partial_1 A_0^a - e\epsilon^{abc}A_0^b A_1^c(G)]]\,\delta[G_{02}^a+\cdots] \\ &\quad \times \delta\left[G_{12}^a + \int_{z_0}^{z} dz'(\partial_1 G_{23}^a+\partial_2 G_{31}^a)(txyz') - \partial_1\Delta_2^a + e\epsilon^{abc}A_1^b(G)\left(-\int_{z_0}^{z} dz' G_{23}^c(txyz') + \Delta_2^c\right)\right]. \end{aligned} \qquad \text{(C2)}$$

Split the last δ at $z=z_0$ (into the constraint at z_0 times the z-derivative constraint):

$$\delta[G_{12}^a+\cdots] = \delta[(G_{12}^a+\cdots)_{z_0}]\,\delta[\partial_3(G_{12}^a+\cdots)] \qquad \text{(C3a)}$$

$$\begin{aligned} &= \delta[G_{12}^a(txyz_0) - \partial_1 \Delta_2^a(txyz_0)] \\ &\quad \times \delta\left[\partial_3 G_{12}^a + \partial_1 G_{23}^a + \partial_2 G_{31} + e\epsilon^{abc} G_{31}^b\left(-\int_{z_0}^{z} dz' G_{23}^c(txyz') + \Delta_2^c\right)\right. \\ &\quad \left. - e\epsilon^{abc}\int_{z_0}^{z} dz' G_{31}^b(txyz') G_{23}^c\right]. \end{aligned} \qquad \text{(C3b)}$$

The Δ_2^a integrations can now be done, using the first and fourth δ functional in Eq. (C2) and the first factor of Eq. (C3b). The value selected is $\Delta_2^a = \int_{x_0}^{x} dx' G_{12}^a(tx'yz_0)$. Thus

$$M[G] = \delta[I_0^a(G)] \int \mathfrak{D}A_0{}^2\,\delta[A_0^a(tx_0y_0z_0)]\,\delta[G_{03}^a+\partial_3 A_0^a]\,\delta[G_{01}^a+\cdots]\,\delta[G_{02}^a+\cdots], \qquad \text{(C4)}$$

where $I_0^a(G)$ is the temporal Bianchi form.

To do the A_0^a integrations, the most economical shift is

$$\begin{aligned} A_0^a &= -\int_{z_0}^{z} dz' G_{03}^a(txyz') - \int_{x_0}^{x} dx' G_{01}^a(tx'yz) \\ &\quad - \int_{y_0}^{y} dy' G_{02}^a(tx_0y'z_0) + \Delta_0^a, \end{aligned}$$

with Δ_0^a the new variables of integration. One must split $\delta[G_{01}^a+\cdots]$ at z_0, and $\delta[G_{02}^a+\cdots]$ at z_0. Then *resplit* $\delta[\{G_{02}^a+\cdots\}_{z_0}]$ at x_0. I leave the algebra as an exercise for the interested reader. The value of Δ_0^a selected is $\Delta_0^a=0$, and the final result is

$$\begin{aligned} M[G] &= \delta[I_0^a(G)]\,\delta[I_1^a(G)]\,\delta[I_2^a(G)]\,\delta[\{I_3^a(G)\}_{z_0}] \\ &\equiv \delta[\text{“3.1”}] \end{aligned} \qquad \text{(C5)}$$

with no further functional measure. As might be anticipated from previous discussions, only 3.1 Bianchi identities emerge in four dimensions. (The same is true for QED_4.) By Appendix B, however, the last 0.9 Bianchi identity is redundant, so

$$\delta[I(G)] \equiv \prod_{\mu} \delta[I_{\mu}^{a}(G)] = \delta[\text{“3.1”}]\delta[\partial_3 I_3^a] = \delta[\text{“3.1”}]\delta[0]. \tag{C6}$$

Up to multiplicative constants then, I have shown that

$$\int \mathfrak{D}A\, \delta[\text{CGF}]\delta[G-F(A)] = \delta[I(G)] \tag{C7}$$

as stated in the text. The form Eq. (C7) is uniform for all gauge theories in any number of dimensions.

APPENDIX D: TIME DEPENDENCE OF $I_0^a(B)$

I need to compute

$$\frac{d}{dt} I_0^a = i[H, I_0^a] = \frac{d}{dt}[\partial_i B_i^a - e\epsilon^{abc} A_i^b(B) B_i^c]. \tag{D1}$$

I shall ignore ordering problems here, though this may need further investigation. I begin by noting that the other three Bianchi identities follow from the Hamiltonian:

$$\frac{d}{dt} B_i^a = i[H, B_i^a] = \epsilon_{ijk}[\partial_j E_k^a - e\epsilon^{abc} A_j^b(B) E_k^c] + e\epsilon^{abc} A_0^b(E) B_i^c. \tag{D2}$$

I can use these to do the time derivatives of B_i^a in Eq. (D1). After some algebra, I arrive at

$$\frac{d}{dt} I_0^a = e\epsilon^{abc}\{F_{12}^a[A(B)] - B_3^b\} E_3^c + e\epsilon^{abc} A_0^b(E) I_0^c. \tag{D3}$$

Without the fourth Bianchi identity, I must evaluate $F_{12}^a[A(B)]$ directly from $A(B)$, Eq. (7.11). The final result is

$$\frac{d}{dt} I_0^a = e\epsilon^{abc}\left[E_3^b \int_{z_0}^{z} dz' I_0^c(xyz') + A_0^b(E) I_0^c\right] \tag{D4}$$

as quoted in the text.

---

[1]T. T. Wu and C. N. Yang, Phys. Rev. D 12, 3843 (1975).

[2]M. B. Halpern, Nucl. Phys. B139, 477 (1978).

[3]Fixed axial gauges have been studied independently and for different purposes by A. Chodos, Phys. Rev. D 17, 2624 (1978).

[4]M. B. Halpern, Berkeley report, 1978 (unpublished).

[5]G. 't Hooft, Nucl. Phys. B138, 1 (1978).

[6]R. Balian, J. M. Drouffe, and C. Itzykson, Phys. Rev. D 10, 3376 (1974); 11, 2098 (1975); 11, 2104 (1975); E. Fradkin and L. Susskind, *ibid.* 17, 2637 (1978).

[7]My notation for the non-Abelian theories is

$$F_{\mu\nu}^a(A) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - e\epsilon^{abc} A_\mu^b A_\nu^c,$$

with the obvious simplification for the Abelian cases. I work in Euclidean space until I discuss canonical quantization in Sec. VII.

[8]Strictly speaking, I am leaving a global (constant) gauge freedom. This too can be fixed, but it is not relevant for the uniqueness of $A(F)$.

[9]For completely fixed axial-like gauges, the Faddeev-Popov determinant is a finite constant.

[10]This is true modulo ordering problems. I have not investigated this point.

[11]The present field-strength formulation should not be confused with my development in Phys. Rev. D 16, 1798 (1977). In that work, the field-strength-like variables transform as field strengths, but they equal field strengths only at saddle points. Otherwise, they are functionally Fourier conjugate to the field strengths.

[12]See, e.g., the footnote in Ref. 11.

[13]An attempt to integrate out the field strengths in Minkowski space runs head-on into the singularities of $M^{-1}(\bar{\phi})$. Presumably, such must be defined by continuation from Euclidean space.

[14]My Minkowski metric is $A\cdot B = A_0 B_0 - \vec{A}\cdot\vec{B}$, $A^0 = A_0$, $A^i = -A_i$.

[15]This may be a divergence, perhaps related to that of Chodos, Ref. 3, and S. Mandelstam, invited talk at the Washington meeting of the APS, 1977 (unpublished).

[16]Useful identities in this computation are $A_1^a(B)J_1^b = A_2^a(B)J_2^b = A_3^a J_3^b = 0$.

[17]S. Mandelstam, Ann. Phys. (N.Y.) 19, 1 (1962); 19, 25 (1962); Phys. Rev. 175, 1580 (1968); I. Bialynicki-Birula, Bull. Acad. Pol. Sci. 11, 135 (1963); N. H. Christ, Phys. Rev. Lett. 34, 355 (1975).