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Finite field equation for asymptotically free ϕ^4 theory

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We consider the finite local field equation $-(\Box+m^2)\phi(x)=\lim_{\xi\to 0}[(1/6)gZ(\xi^2):\phi(x-\xi)\phi(x)\phi(x+\xi):-\Delta(\xi^2)\phi(x)+\sigma(\xi^2)(\xi\cdot\partial_x)^2\phi(x)]$, which rigorously describes $g\phi^4$ scalar field theory, and the operator-product expansion $\phi(\xi)\phi(0)\underset{\xi\to 0}{\simeq}F(\xi^2)N[\phi^2]$, where $N[\phi^2]$ denotes a normal product. For $g<0$, this theory is asymptotically free. We use the asymptotic freedom to calculate explicitly, via the renormalization group, the coefficients $Z(\xi^2)$, $\Delta(\xi^2)$, $\sigma(\xi^2)$, and $F(\xi^2)$. We perform the R transformation $\phi(x)\to\phi(x)+r$ on the finite field equation and obtain the operator part of the change to be proportional to $\lim_{\xi\to 0}Z(\xi^2)F(\xi^2)N[\phi^2]$ which vanishes by our knowledge of the functions $Z(\xi^2)$ and $F(\xi^2)$. We have therefore verified rigorously the partial R invariance of $-|g|\phi^4$ theory. We discuss and solve the technical problem of finding the solution for renormalization-group equations with a matrix $\underline{\gamma}$ function where the lowest-order expansions of the various elements do not begin with the same powers of g.

I. INTRODUCTION

Ultraviolet divergences have plagued quantum field theory ever since its inception. The usual method of removing divergences in perturbation theory by renormalization is perfectly adequate for obtaining unambiguous answers for physical quantities.[1] The esthetic imperfection of this scheme remains, however, and it was felt that quantum field theory should be phrased in terms of only finite objects, without recourse to intermediary divergent quantities.

Such a quest has already been answered for some time now, in the conception of *finite* local field equations written solely in terms of finite field (elementary and composite) operators. The existence of these equations has been rigorously shown to all orders in perturbation theory, and it was conjectured that they exist also for the exact solution of the theory.

These finite field equations involve finite composite operators, which can be written explicitly as the short-distance limits of essentially the product of singular Wilson coefficients and point-separated operator products.[3] Thus for the theory of a scalar field $\phi(x)$ with a $g\phi^4$ interaction, the finite field equation takes the form

$$-(\Box+m^2)\phi(x)=\tfrac{1}{6}g\lim_{\xi\to 0}\left[\frac{1}{E_1(\xi)}:\phi(x+\xi)\phi(x)\phi(x-\xi):-\frac{E_0(\xi)}{E_1(\xi)}\phi(x)-\frac{E_2(\xi)}{E_1(\xi)}(n\cdot\partial_x)^2\phi(x)-\frac{E_3(\xi)}{E_1(\xi)}\Box\phi(x)\right], \quad (1.1)$$

where $n^\mu=\xi^\mu/(\xi^2)^{1/2}$, and E_0, E_1, E_2, E_3 are the singular c-number coefficients. They are fundamental quantities intrinsic to the physical content for the particular theory.

We show in this paper how to calculate these coefficients exactly via the renormalization group,[4] whenever the underlying theory is asymptotically free.[5] Asymptotic freedom gives us information on the exact short-distance behavior for any physical quantity in the theory, and since only the short-distance limit is relevant in the finite field equation, that information suffices.

Our interest in calculating these Wilson coefficients stems initially from the need to investigate symmetries which are not present in the classical (unquantized) field equations, but which arise from infinite renormalization.[6-8] Such symmetries were shown to be instrumental in throwing light on the infrared content of a quantum field theory. Previous investigations relied on an assumed structure in the field equations as follows from an ultraviolet cutoff. With the finite equation in point-separated form with known coefficients, it becomes possible to verify directly the existence of the symmetry in question. The investigation of renormalization-induced symmetries can now be undertaken with considerably enhanced rigor.

The calculation of these coefficients is complicated by the technical problem of operator mixing.[9] We shall solve the mixing problem by a method which is generally applicable. The problem becomes quite horrendous for multi-indicial fields like non-Abelian gauge fields, and in this first paper of a series we solve the problem for $g\phi^4$ theory, for which the coupling constant g is negative.[10]

This theory is asymptotically free,[10] and its presumed nonexistence[11] on energy grounds has been questioned as premature.[12] In any case, we shall use the theory as a warming-up exercise. We tackle the case of non-Abelian gauge field theory in subsequent publications of this series.

It was hoped that the possible existence of asymptotically free purely scalar field theory with negative g would considerably enlarge the class of theories which are candidates for the theory of strong interactions via combination with non-Abelian gauge fields, with the scalar fields acting as Higgs particles. Preliminary investigations indicate[14] that the simpler cases of theories combining non-Abelian gauge fields and scalar fields with a negative coupling constant into one Lagrangian are actually not asymptotically free, even though the two kinds of fields are each asymptotically free by themselves.

In solving the renormalization-group equations for asymptotically free scalar field theory, we face the problem of evaluating the T-ordered exponential

$$T\exp\left[\int_0^t dt'\underline{\gamma}(\bar{g}(t',g))\right] \tag{1.1}$$

where $\underline{\gamma}$ is a matrix, whose individual elements have expansions in g that start with different powers of g, e.g.,

$$\underline{\gamma}=\begin{bmatrix} Ag & 0 \\ Bg & Cg^2\end{bmatrix}. \tag{1.2}$$

We solve this coupled problem by a method applicable for a $\underline{\gamma}$ of arbitrary complexity. In the present ϕ^4 case, we check the validity of the method by a tedious, direct integration of each element of the T-ordered matrix exponential. This assures us that the problem is correctly solved in the complicated Yang-Mills case, where this direct element-by-element evaluation is not available.

This paper is organized as follows: In Sec. II we give a brief review of the properties of $-|g|\phi^4$ theory where it concerns operator-product expansions, finite local field equations, and the renormalization group. We calculate in Sec. III the Wilson coefficients to lowest order in perturbation theory, and obtain in Sec. IV their exact behavior in the short-distance limit via the renormalization group. In doing so we discuss and solve the technical problem of finding the solution for a renormalization-group equation with a matrix $\underline{\gamma}$ function where the lowest-order expansions of the various elements do not begin with the same powers of g. In Sec. V we demonstrate explicitly the correctness of the solution by direct integration of the matrix renormalization-group equation element by element. In Sec. VI we use the knowledge of the Wilson coefficients to demonstrate that the finite local field equation for $-|g|\phi^4$ is partially invariant under the R transformation $\phi(x)\to\phi(x)+r$. Finally we make some remarks on the possible summability properties of the perturbation series for $g<0$ and their bearing on the existence of $-|g|\phi^4$ as a genuine quantum field theory.

II. ASYMPTOTICALLY FREE SCALAR FIELD THEORY

The asymptotically scalar field theory we consider is specified by the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi_B)^2-\tfrac{1}{2}m_0{}^2\phi_B{}^2-\frac{1}{4!}g_0\phi_B{}^4, \tag{2.1}$$

with the coupling constant $g_0<0$. The theory is asymptotically free, and the β function to lowest order is

$$\beta(g)\equiv\mu\frac{\partial}{\partial\mu}g\bigg|_{\text{fixed } g_0}$$

$$=b_0g^2,\quad b_0=\frac{3}{16\pi^2}. \tag{2.2}$$

The finite field equation takes the form

$$-(\Box+m^2)\phi(x)=\tfrac{1}{6}gN[\phi^3(x)], \tag{2.3}$$

where $N[\phi^3]$ is defined via the short-distance operator-product expansion (: : indicates subtraction of the vacuum expectation value)

$$:\phi(x+\xi)\phi(x)\phi(x-\xi):\ \underset{\xi\to0}{\sim}E_0(\xi)\phi(x)+E_1(\xi)N[\phi^3(x)]+E_2(\xi)(n\cdot\partial_x)^2\phi(x)+E_3(\xi)\Box\phi(x). \tag{2.4}$$

Thus it is crucial to analyze the Wilson expansion of the product of the three operators $\phi\phi\phi$.

We enumerate all the operators that can contribute to the expansion for a given scale dimension. Operators like $\phi\partial_\mu\phi$ which are even in ϕ are excluded by the $\phi\to-\phi$ invariance of the theory. Thus there can only be $N[\phi^3]$ and $\partial_\mu\partial_\nu\phi$. To make a scalar out of the latter we can contract with $g_{\mu\nu}$ or $\xi_\mu\xi_\nu/\xi^2$, and still conserve dimension. So we can use the operators $\{N[\phi^3], (\xi^\mu\xi^\nu/\xi^2)\partial_\mu\partial_\nu\phi, \Box\phi\}$ as a set that is closed under renormalization, and we can define the multiplicative renormalization matirx to be

$$[\phi_B{}^3, (n\cdot\partial)^2\phi_B, \Box\phi_B] = [N[\phi^3], (n\cdot\partial)^2\phi, \Box\phi]\underline{Z}\,, \tag{2.5}$$

with $n^\mu = \xi^\mu/(\xi^2)^{1/2}$, and where the subscript B denotes bare quantities. We define the $\underline{\gamma}$ function for the $\phi\phi\phi$ expansion in the usual manner:

$$-\gamma_{ij} = \sum_k \mu\frac{\partial}{\partial\mu}\underline{Z}_{ik}\bigg|_{g_0\ \text{fixed}} \underline{\tilde{Z}}_{kj}\,, \tag{2.6}$$

where

$$\underline{\tilde{Z}} = Z_3{}^{-3/2}\underline{Z}\,. \tag{2.7}$$

$\underline{\tilde{Z}}$ is the quantity which satisfies the renormalization-group equation.[9] As it turns out, Z_3 is finite in the $-|g|\phi^4$ theory, and so the distinction is minor.

Equivalently (2.5) can be written as the Wilson expansion Eq. (2.4), so that the behaviors of the E's as $\xi\to 0$ are the same as those of $\underline{\tilde{Z}} = \underline{Z}Z_3{}^{-3/2}$ (see Ref. 9) as $\Lambda\to\infty$:

$$\begin{aligned} E_1 &\sim \tilde{Z}_{11}\,,\\ E_2 &\sim \tilde{Z}_{21}\,,\\ E_3 &\sim Z_{31}\,, \end{aligned} \tag{2.8}$$

and so we shall use them interchangeably.

For obvious reasons we refer to the $(n\cdot\partial)^2\phi$ term in the expansion as the *direction-dependent* contribution. These direction-dependent terms were missed in the formal field equations written in terms of renormalized fields and renormalization constants Z_1 and Z_3. The inclusion of these terms in our renormalization-group calculations therefore puts our resulting finite field equation on a footing more rigorous than heretofore.

The renormalization matrix $\underline{\tilde{Z}}$ satisfies the renormalization-group equation

$$\sum_j\left\{\delta_{ij}\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right] + \gamma_{ij}(g)\right\}\underline{\tilde{Z}}_{jk}\left(\frac{\Lambda}{\mu}\right) = 0\,, \tag{2.9}$$

which is the same equation satisfied by the singular functions $F_j(\xi,\mu)$:

$$\sum_j\left\{\delta_{ij}\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right] + \gamma_{ij}(g)\right\}E_j(\xi,\mu) = 0\,. \tag{2.10}$$

In the following, we shall always identify the $\xi\to 0$ and $\Lambda\to\infty$ limits.

The $\phi(\xi)\phi(0)$ operator product has a similar expansion,

$$\phi(\xi)\phi(0) \underset{\xi\to 0}{\sim} F(\xi)N[\phi^2]\,, \tag{2.11}$$

whose structure is very simple since $N[\phi^2]$ is the only operator that can contribute.

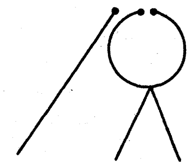

FIG. 1. Lowest-order diagrammatic contribution to the singular function $E_1(\xi)$.

III. PERTURBATIVE CALCULATION OF THE SINGULAR COEFFICIENTS

The lowest-order calculation of the Wilson coefficients in the $\phi\phi\phi$ operator-product expansion follows standard procedure.[9] The lowest-order contribution to E_1 is given by the Feynman diagram of Fig. 1 (plus permutations), and we have

$$\begin{aligned} E_1^{(1)}(\xi) &= \frac{3g}{32\pi^2}\ln\xi^2\\ &\sim -\frac{3g}{16\pi^2}\ln\frac{\Lambda}{\mu}\,, \end{aligned} \tag{3.1}$$

where the superscript indicates the order in g, or equivalently, by (2.8), that to first order,

$$\tilde{Z}_{11} = 1 - \frac{3g}{16\pi^2}\ln\frac{\Lambda}{\mu}\,, \tag{3.2}$$

since there is no first-order contribution to Z_3. Z_{11} is to this order identical to Z_1^{-1}, the inverse of the coupling-constant renormalization. This means that the formal field equation with explicit renormalization constants, where one writes the interaction terms as

$$N[\phi^3] \sim Z_1 g\phi^3\,, \tag{3.3}$$

retains its validity to this order.

The contribution to $E_2(\xi)$ and $E_3(\xi)$ is given by the matrix element of $\phi\phi\phi$ between the vacuum $|0\rangle$ and the one-ϕ-state $|p\rangle$, as illustrated in Fig. 2. It is convenient to hold the limiting separation ξ finite and use it as cutoff parameter. The relevant contribution is obtained by differentiating the diagram of Fig. 2 twice with respect to p, and then

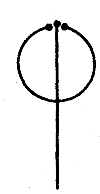

FIG. 2. Lowest-order diagrammatic contribution to the singular functions $E_2(\xi)$ and $E_3(\xi)$.

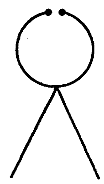

FIG. 3. Lowest-order diagrammatic contribution to the singular function $F(\xi)$.

setting $p=0$. Thus we obtain

$$E_2^{(1)} n^\alpha n^\beta + 2E_3^{(1)} g^{\alpha\beta} = -g\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\left[\frac{8(p+q)^\alpha (p+q)^\beta}{(p+q)^6 p^2 q^2} - \frac{2g^{\alpha\beta}}{(p+q)^4 p^2 q^2}\right] e^{i\xi\cdot(q+2p)}. \quad (3.4)$$

Segregating the terms proportional to $g^{\alpha\beta}$ and to $n^\alpha n^\beta$, we get

$$E_2(\xi) = -\frac{g}{48\pi^4}\ln\xi^2 \sim +\frac{g}{24\pi^4}\ln\frac{\Lambda}{\mu}, \quad (3.5)$$

$$E_3(\xi) = +\frac{g}{192\pi^4}\ln\xi^2 \sim -\frac{g}{96\pi^4}\ln\frac{\Lambda}{\mu}. \quad (3.6)$$

The Wilson coefficient for the $\phi\phi$ operator-product expansion is simply given by Fig. 3, and we have

$$F(\xi) = 1 + \frac{g}{16\pi^2}\ln\xi^2 \sim 1 - \frac{g}{8\pi^2}\ln\frac{\Lambda}{\mu}. \quad (3.7)$$

IV. RENORMALIZATION-GROUP CALCULATIONS

The renormalization matrix $\underline{\tilde{Z}}$ for $\phi\phi\phi$ thus satisfies the renormalization-group equation

$$\sum_j \left[\delta_{ij}\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right) + \gamma_{ij}\right]\underline{\tilde{Z}}_{jk}\left(\frac{\Lambda}{\mu}\right) = 0, \quad (4.1)$$

with

$$-\gamma = \begin{bmatrix} Ag & 0 & 0 \\ Bg & Cg^2 & 0 \\ Dg & 0 & Cg^2 \end{bmatrix}, \quad (4.2)$$

where

$$A = \frac{3}{16\pi^2}, \quad B = -\frac{1}{24\pi^4}, \quad C = \frac{1}{2^{11}3\pi^4}, \quad D = \frac{1}{96\pi^4}. \quad (4.3)$$

We see immediately that this matrix equation has the characteristic that the lowest-order contributions in $\underline{\gamma}$ are of *different* powers in g. This feature is not of critical importance in the present theory, where $\bar{g}(t) \sim_{t\to\infty} -(b_0 t)^{-1}$, with $t = \ln(\Lambda/\mu)$, and

$$\bar{g}^2(t) \underset{t\to\infty}{\sim} \frac{1}{b_0{}^2 t^2}. \quad (4.4)$$

With this rapid falloff in t in the effective coupling constant $\bar{g}$, it turns out to be justifiable to neglect the $O(\bar{g}^2)$ contribution. We have, however, developed a general method for solving this mixing problem, which, when applied in this case, gives extra information. This general method will be seen to be indispensible in the case of non-Abelian gauge theories, when

$$\bar{g}_{NA} \sim \frac{1}{(2b_{NA}t)^{1/2}} \quad (4.5)$$

and

$$\bar{g}_{NA}{}^2 \sim \frac{1}{2b_{NA}t}, \quad (4.6)$$

and where the $O(\bar{g}^2)$ terms cannot be dropped.[13]

Formally the solution to the renormalization-group equation (4.1) is as usual:

$$\underline{\tilde{Z}}_{ik}(t,g) = \sum_j \left\{T\exp\left[\int_0^t dt'\underline{\gamma}(\bar{g}(t',g))\right]\right\}_{ij} \times \underline{\tilde{Z}}_{jk}(1,\bar{g}(t,g)). \quad (4.7)$$

The difficulty with the mixing problem lies in the evaluation of the T-ordered exponential of the $\underline{\gamma}$ matrix needed for the solution. We show below the more generally applicable method; in Sec. V we shall present the alternative way of direct evaluation of the matrix exponential.

We write the component equations of (4.1), in an obvious notation, as

$$D_\mu\tilde{Z}_{11} = Ag\tilde{Z}_{11}, \quad (4.8a)$$

$$D_\mu\tilde{Z}_{21} = Bg\tilde{Z}_{11} + Cg^2\tilde{Z}_{21}, \quad (4.8b)$$

$$D_\mu\tilde{Z}_{31} = Dg\tilde{Z}_{11} + Cg^2\tilde{Z}_{31}, \quad (4.8c)$$

$$D_\mu\tilde{Z}_{22} = Cg^2\tilde{Z}_{22}, \quad (4.8d)$$

$$D_\mu\tilde{Z}_{33} = Cg^2\tilde{Z}_{33}. \quad (4.8e)$$

Equations (4.8a), (4.8d), and (4.8e) can be solved independently, giving the results

$$\tilde{Z}_{22}, \tilde{Z}_{33} = \text{finite} \tag{4.9}$$

and

$$\tilde{Z}_{11} \sim \exp\left(\int^{\bar{g}} dg' - \frac{A}{b_0}\right) = (b_0 t)^{A/b_0} \underset{t\to\infty}{\longrightarrow} \infty . \tag{4.10}$$

Thus the renormalization-group equation for $\tilde{Z}_{11}$ can be solved by itself, and there is no effect from operator mixing. Since $\tilde{Z}_{11} = Z_1^{-1}$ to lowest nontrivial order, this means that

$$\tilde{Z}_{11} = Z_1^{-1} \tag{4.11}$$

to all orders according to the renormalization-group calculations. The only effect of the introduction of operator mixing is to add extra additive direction-dependent terms to the finite field equation. The coefficient multiplying the $\phi\phi\phi$ singularity remains the same as the formal field equation in terms of Z_1 and Z_3.

For $\tilde{Z}_{21}$ we consider the quantity

$$\begin{aligned} D_\mu(gZ_{21}) &= (b_0 g^2 + Cg^3)\tilde{Z}_{21} + Bg^2\tilde{Z}_{11} \\ &\cong b_0 g(g\tilde{Z}_{21}) + Bg(g\tilde{Z}_{11}) , \end{aligned} \tag{4.12}$$

keeping only lowest-order terms. We have used the definition $D_\mu g = \beta(g)$. We see here that the Cg^3 term is ignorable for the renormalization-group calculation.

We write an *arbitrary* decomposition

$$B = B_1 + B_2 \tag{4.13}$$

and get, using (4.8a),

$$\begin{aligned} D_\mu(g\tilde{Z}_{21}) &= b_0 g(g\tilde{Z}_{21}) + \frac{B_1}{A} gD_\mu\tilde{Z}_{11} \\ &\quad + B_2 g(g\tilde{Z}_{11}) . \end{aligned} \tag{4.14}$$

Similarly,

$$D_\mu(g\tilde{Z}_{11}) = b_0 g(g\tilde{Z}_{11}) + g(D_\mu\tilde{Z}_{11}) , \tag{4.15}$$

and so

$$\begin{aligned} D_\mu&\left(g\tilde{Z}_{21} - \frac{B_1}{A} g\tilde{Z}_{11}\right) \\ &= b_0 g\left[g\tilde{Z}_{21} + \frac{1}{b_0}\left(B_2 - \frac{B_1 b_0}{A}\right) g\tilde{Z}_{11}\right] . \end{aligned} \tag{4.16}$$

This equation can now be diagonalized by requiring that

$$-\frac{B_1}{A} = \frac{1}{b_0}\left(B_2 - \frac{B_1 b_0}{A}\right) , \tag{4.17}$$

which eliminates the arbitrariness in (4.13) to give

$$\begin{aligned} B_1 &= B , \\ B_2 &= 0 . \end{aligned} \tag{4.18}$$

Thus we can solve the diagonalized equation

$$D_\mu\left(g\tilde{Z}_{21} - \frac{B}{A} g\tilde{Z}_{11}\right) = b_0 g\left(g\tilde{Z}_{21} - \frac{B}{A} g\tilde{Z}_{11}\right) \tag{4.19}$$

by

$$\begin{aligned} g\tilde{Z}_{21} - \frac{B}{A} g\tilde{Z}_{11} &\sim \left[\bar{g}\tilde{Z}_{21}(1,\bar{g}) - \frac{B}{A}\bar{g}\tilde{Z}_{11}(1,\bar{g})\right]\bar{g}^{-1} \\ &= \text{finite} , \end{aligned} \tag{4.20}$$

and so we have, using (4.10),

$$\tilde{Z}_{21} \underset{t\to\infty}{\longrightarrow} \infty . \tag{4.21}$$

Note that we have also obtained the *ratio*

$$\frac{\tilde{Z}_{21}}{\tilde{Z}_{11}} = \frac{B}{A} = -\frac{1}{3\pi^2} . \tag{4.22}$$

Similarly, we can solve the other pair (4.8a) and (4.8c) to get

$$\tilde{Z}_{31} \underset{t\to\infty}{\longrightarrow} \infty . \tag{4.23}$$

and obtain the ratio

$$\frac{\tilde{Z}_{31}}{\tilde{Z}_{11}} = \frac{D}{A} = \frac{1}{12\pi^2} . \tag{4.24}$$

In the language of Wilson coefficients, the above results are stated as

$$E_i(\xi) \underset{\xi\to 0}{\sim} \epsilon_i[b_0 \ln(\xi^2)^{1/2}]^{A/b_0} , \quad i = 1, 2, 3 , \tag{4.25}$$

where the constants ϵ_1, ϵ_2, and ϵ_3 obey

$$\frac{\epsilon_2}{\epsilon_1} = -\frac{1}{3\pi^2} , \tag{4.26}$$

$$\frac{\epsilon_3}{\epsilon_1} = +\frac{1}{12\pi^2} . \tag{4.27}$$

The Eqs. (4.25)–(4.27) form the main result of this paper.

The mass-shift term E_0 in the $\phi\phi\phi$ expansion was evaluated in Ref. 12, and we have

$$E_0(\xi) \underset{\xi\to 0}{\sim} \frac{\epsilon_0}{\xi^2}(\ln\xi^2)^{-1} + \text{smaller terms} . \tag{4.28}$$

The results of our calculation, Eqs. (4.25)–(4.28), thus give explicitly all the c-number coefficients in our finite field equation (2.3) and (2.4):

$$\begin{aligned} -(\Box + m^2)\phi(x) = \tfrac{1}{6} g \lim_{\xi\to 0}\Big\{&\epsilon_1^{-1}[b_0\ln(\xi^2)^{1/2}]^{-A/b_0} :\phi(x-\xi)\phi(x)\phi(x+\xi): \\ &- \frac{1}{\xi^2}\frac{\epsilon_0}{\epsilon_1}\frac{\ln\xi^2}{b_0[\ln(\xi^2)^{1/2}]^{A/b_0}}\phi(x) - \frac{1}{3\pi^2}(n\cdot\partial)^2\phi(x) - \frac{1}{12\pi^2}\Box\phi(x)\Big\} . \end{aligned} \tag{4.29}$$

V. INTEGRATION OF THE T-ORDERED EXPONENTIAL

For our $-|g|\phi^4$ theory, the mixing problem is of manageable proportions, as we saw in Sec. IV. Indeed, for the simple 2×2 $\underline{\gamma}$ matrix, the renormalization-group equation can be directly integrated by evaluating the T-ordered exponential of $\underline{\gamma}$'s. This we shall do in the following; we doubt very much that this direct evaluation, complicated as it is in the 2×2 case, can be generalized for use in the mixing problem of Yang-Mills theories involving $\underline{\gamma}$ matrices of very large dimension.

We shall solve the equation pair (4.8a) and (4.8b) by direct integration. The solution is given by the T-ordered exponential

$$\tilde{Z}_{i1}=\sum_{j=1}^{2}\left\{T\exp\left[\int_0^t dt'\gamma(\bar{g}(t',g))\right]\right\}_{ij}\times\tilde{Z}_{j1}(1,\bar{g}(t,g)),\qquad(5.1)$$

with

$$-\underline{\gamma}=\begin{bmatrix}Ag & 0\\ Bg & Cg^2\end{bmatrix}.\qquad(5.2)$$

Let $\underline{Q}$ be the T product in Eq. (5.1). It is easy to see that the Q_{11} and Q_{21} elements are elementary to evaluate:

$$Q_{11}\underset{t\to\infty}{\sim}\sum_{n=0}^{\infty}\left(\frac{A}{b_0}\right)^n\int^t dt_1\cdots\int^{t_{n-1}}dt_n t_1^{-1}\cdots t_n^{-1}=T\exp\left(\frac{A}{b_0}\int^t\frac{dt'}{t'}\right)=t^{A/b_0},\qquad(5.3)$$

$$Q_{22}=\text{finite},\qquad(5.4)$$

both being for $t\to\infty$.

The only difficulty is the Q_{21} element:

$$Q_{21}\underset{t\to\infty}{\sim}\sum_{n=0}^{\infty}(-1)^n\int^t dt_1\cdots\int^{t_{n-1}}dt_n B\sum_{l=1}^{n-1}A^{n-l}C^{l-n}b_0^{-2(l-1)}(-b_0)^{-(n-l+1)}(t_1^{-2}\cdots t_{l-1}^{-2})(t_l^{-1}\cdots t_n^{-1}).\qquad(5.5)$$

We first evaluate the $n-l+1$ integrations involving $t_l,\ldots,t_n$:

$$Q_{21}=\sum_{n=0}^{\infty}(-1)^n\int^t dt_1\cdots\int^{t_{l-1}}dt_{l-1}B\sum_{l=1}^{n-1}A^{n-l}C^{l-1}b_0^{-2(l-1)}(-b_0)^{-(n-l+1)}(t_1^{-2}\cdots t_{l-1}^{-2})\frac{1}{(n-l+1)!}(\ln t_{l-1})^{n-l+1}.\qquad(5.6)$$

To perform the remaining integrations, we use repeatedly the formula

$$\int dx\,x^n(\ln x)^m=\frac{x^{n+1}}{n+1}\sum_{k=0}^{m}\left(\frac{-1}{n+1}\right)^{m-k}\frac{m!}{k!}(\ln x)^k,\qquad(5.7)$$

and we get

$$Q_{21}=B\sum_{n=0}^{\infty}(-1)^n\sum_{l=1}^{n-1}A^{n-l}C^{l-1}b_0^{-2(l-1)}(-b_0)^{-(n-l+1)}\frac{(-1)^{l-1}}{(l-1)!}t^{-(l-1)}$$
$$\times\sum_{k_1=0}^{n-l+1}\cdots\sum_{k_{l-2}=0}^{k_{l-3}}\sum_{k_{l-1}=0}^{k_{l-2}}\left(\frac{1}{l-1}\right)^{k_{l-2}-k_{l-1}}\left(\frac{1}{l-2}\right)^{k_{l-3}-k_{l-2}}\cdots\left(\frac{1}{2}\right)^{k_1-k_2}\frac{1}{k_{l-1}!}(\ln t)^{k_{l-1}}.\qquad(5.8)$$

It is clear now that the largest contribution comes from the coefficient of $l=1$; all other contributions are smaller by integral powers of t as $t\to\infty$. Thus we have

$$Q_{21}\underset{t\to\infty}{\sim}B\sum_{n=0}^{\infty}(-1)^nA^{n-1}(-b_0)^{-n}\frac{1}{n!}(\ln t)^n=Bt^{A/b_0}.\qquad(5.9)$$

In terms of E_1 and E_2, the solution of Eq. (5.1) is then

$$\begin{bmatrix}E_1\\ E_2\end{bmatrix}\sim\begin{bmatrix}t^{A/b_0} & 0\\ Bt^{A/b_0} & (\text{finite})\end{bmatrix}\begin{bmatrix}1\\ \bar{g}\end{bmatrix}=\begin{bmatrix}\text{const}\times t^{A/b_0}\\ \text{const}\times t^{A/b_0}\end{bmatrix},\qquad(5.10)$$

in complete agreement with our previous results (4.24) and (4.25).

VI. R INVARIANCE

By an R transformation[6] on a field theory, we mean a constant shift in a particular (not necessarily all) field

$$R:\ \Phi_i(x) \rightarrow \Phi_i(x) + r_i\,, \tag{6.1}$$

with i enumerating all the quantum numbers of the field. The theory is said to be R invariant if the (renormalized) finite field equation is invariant under the R transformation.

Classically, the invariance holds in massless derivative-coupled theories. Many more theories become R invariant, however, when renormalization is taken into account to all orders. The (renormalized) finite field equation is R invariant even though the classical field equation is not.

This renormalization-induced aspect of R invariance is thus particularly useful in illuminating problems which cause confusion either classically or order by order in perturbation theory. A rigorous consequence[15] of R invariance is the validity of low-energy theorems. Typically, the theorem states that the n-point one-particle irreducible vertices of the field Φ_i vanish in the limit when any m ($\leq n$) of the particle momenta approach zero. For non-Abelian gauge theories where a conserved color current exists, this has been shown to lead to an infinite effective coupling in the infrared limit,[8] which furnishes a plausible mechanism for color confinement. For our $-|g|\phi^4$ theory, the low-energy information has been used to cast doubt[12] on the statement, valid on a semiclassical level, that $-|g|\phi^4$ does not have a well-defined quantized theory, as the energy of the system is not bounded from below.[11]

All of these assertions connected with R invariance presuppose the existence of finite local field equations for the relevant theory. Now that we have the finite field equation for $-|g|\phi^4$ with known Wilson coefficients, we can verify directly its transformation properties under R.

Thus we make the R transformation

$$R:\ \phi(x) \rightarrow \phi(x) + r \tag{6.2}$$

in our field equation (4.28), and the only terms that do not obviously vanish are the following: (1) the (infinite) mass-shift term; this does not change[12] the low-energy theorems which follow from R invariance because it is a c number and can therefore be neglected[12]; when R invariance is violated by a mass term as in here, we refer to the existence of a *partial* R invariance; (2) the terms proportional to

$$\lim_{\xi\rightarrow 0} E_1(\xi)^{-1}\phi(x+\xi)\phi(x) = \lim_{\xi\rightarrow 0}\frac{F(\xi)}{E_1(\xi)}N[\phi^2]\,. \tag{6.3}$$

We therefore require the knowledge of the singularities in the $\phi\phi$ operator-product expansion in order to decide on the magnitude of (6.3). The lowest-order calculation, (3.7), yields, via the renormalization group, the result

$$F(\xi) \underset{\xi\rightarrow 0}{\sim} [(\xi^2)^{1/2}]^{-1/8\pi^2 b_0} \sim \xi^{1/8\pi^2 b_0}\,. \tag{6.4}$$

So we finally see that

$$\frac{F(\xi)}{E_1(\xi)}N[\phi^2] \underset{\xi\rightarrow 0}{\sim} [(\xi^2)^{1/2}]^{1/16\pi^2 b_0} N[\phi^2] \underset{\xi\rightarrow 0}{\rightarrow} 0\,, \tag{6.5}$$

showing that the $-|g|\phi^4$ theory is indeed partially R invariant.

VII. REMARKS

Asymptotic freedom has enabled us to obtain a finite field equation in point-separated form for $-|g|\phi^4$ theory. The real question is whether the quantum theory exists. We have already mentioned the difficulty with the ground state. A further difficulty arises in another quarter, from the estimation of the large-order behavior of its perturbation series. By exploiting the existence of an instanton solution for $g<0$, the usual $+g\phi^4$ theory was shown to have a Borel-summable perturbation series, whereas the series for $-|g|\phi^4$, having terms all of the same sign, seems to be not even Borel summable.[16] If the perturbation series were not even summable, it would throw grave doubts upon the consequences of its asymptotic freedom, which is after all predicted on the series being at least an asymptotic expansion for small g of the true solution. The same danger is therefore present also for asymptotically free non-Abelian gauge theories.

However, there may be a glimmer of hope: instead of considering the usual Borel transform $B_1(\beta)$,

$$B_1(\beta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\left(\frac{1}{g}\right) e^{\beta/g} W(g)\,, \tag{7.1}$$

for the generating functional $W(g)$ say, one might consider the *second* Borel transform[7]

$$B_2(\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz\, e^{z\nu} W(g = z^{-2})\,. \tag{7.2}$$

The difficulty with the first Borel transform lies in the fact that the instanton yields a singularity of the form[18]

$$B_1(\beta) \sim (\beta - \tfrac{1}{3})^{-3/2} \tag{7.3}$$

on the integration path of the Laplace integral

$$W(g) = \int_0^\infty d\beta\, e^{-\beta/g} B_1(\beta)\,, \tag{7.4}$$

inverting $B_1(\beta)$ to give $W(g)$. In the second Borel transform, the instanton gives an exponential damping

$$B_2(\nu) \underset{\nu \to \infty}{\sim} e^{-\alpha\nu^2} \qquad (7.5)$$

as $\nu \to \infty$, which does not prevent the existence of the integral even beyond the cut at $|\arg g| = 180°$. Apparently there is an extra imaginary part of the form

$$W(g) \sim ie^{-|g|^{-1}}, \qquad (7.6)$$

which does not imperil the predictions of asymptotic freedom. Nothing really goes wrong on the ray $|\arg g| = 180°$, which merely corresponds to a Stokes's ray[19] of the asymptotic expansion.

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