Kinks and the Heisenberg uncertainty principle

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The Lieb form of the Heisenberg relations is applied to the kinks of a nonlinear Dirac field. It is found that these relations are violated for large enough values of the self-coupling constant. A close relation between this violation and the norm of the kinks is established.

I. INTRODUCTION

The important development of nonlinear classical theories of extended particles poses a very interesting problem: their compatibility with the Heisenberg uncertainty principle. In fact the velocity and the position of the center of a kink, soliton, or solitary wave can be known with arbitrary precision. Even if the width of the kink is taken as the value of the uncertainty of the position, the Heisenberg relations appear violated. In this paper we will show that, in two simple models, such a violation does occur when the self-coupling constant corresponds to a small value of the norm of the kink. From this we will conclude that the violation takes place when the norm of the kink is smaller than a quantity of the order of $\hbar$, a situation which deserves the qualification of "ultra-quantum."

II. THE LIEB FORM OF THE HEISENBERG RELATIONS

In order to understand better the problem let us stress that the Heisenberg relations can be considered from two different points of view:

(a) as a mathematical statement which expresses a property of the Fourier transformation,

(b) as a physical statement which prevents the simultaneous determination of the momentum and the position of a particle.

While the first statement is always correct, provided that the Fourier transformation can be defined, the physical one is true only if the use of the operator $(\hbar/i)\vec{\nabla}$ as a representation of the momentum is adequate. As we will see, this is no longer the case when the norm of the kink is small enough.

In order to study the Heisenberg relations in a nonlinear classical theory it is very convenient to use the form proposed by Lieb,[1]

$$\langle x^2 \rangle_\psi T_\psi \geq \frac{9}{4} \frac{\hbar^2}{2m} , \qquad (1)$$

where T_ψ is the mean value of $\vec{p}^2/2m$, the non-relativistic kinetic energy, and $\langle x^2 \rangle_\psi$ is the mean

square radius. In a classical field theory the left-hand side can be calculated from the energy-momentum tensor, independently of $\hbar$. Equation (1) thus gives a condition on the parameters of the model in order that the Heisenberg relations be verified.

III. NONLINEAR DIRAC FIELDS

We will consider two cases:

(a) The classical nonlinear Dirac field described by the Lagrangian[2,3,4] density

$$\mathcal{L} = \frac{i}{2}[\bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi] - \mu\bar{\psi}\psi + \lambda(\bar{\psi}\psi)^2 , \qquad (2)$$

with $\lambda > 0$. Using a system in which $c = 1$, the dimensions are the following:

$$[\psi] = M^{-1/2}L^{-1}, \quad [\mu] = L^{-1},$$

$$[\lambda] = M^{-1}L, \quad [\hbar] = ML .$$

The field equation is

$$i\gamma^\mu \partial_\mu \psi - \mu\psi + 2\lambda(\bar{\psi}\psi)\psi = 0 . \qquad (3)$$

This equation admits stable, localized solutions of the form

$$\psi = e^{-i\mu\Omega t} \left(\frac{\mu}{2\lambda}\right)^{1/2} \begin{bmatrix} G(\rho)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iF(\rho)\begin{pmatrix} \cos\theta \\ \sin\theta\, e^{-i\varphi} \end{pmatrix} \end{bmatrix}, \quad \rho = \mu r. \qquad (4)$$

Soler[3] used Eq. (3) to construct a model of an elementary fermion and calculated numerically the functions F, G which depend on Ω but not on λ. He found a family of solutions which depend continuously of the frequency Ω. It turns out that the energy has a minimum for $\Omega = 0.936$, which is taken as the frequency of the ground state. The energy and norm of the kink are

$$E = \frac{2\pi}{\lambda\mu}\left[\Omega \int_0^\infty (F^2 + G^2)\rho^2 d\rho \right.$$

$$\left. + \frac{1}{2}\int_0^\infty (F^2 - G^2)^2\rho^2 d\rho\right], \tag{5}$$

$$N = \frac{2\pi}{\lambda\mu^2}\int_0^\infty (F^2 + G^2)\rho^2 d\rho . \tag{6}$$

As we see the coupling constant λ is a scale parameter, inversely proportional to the norm of the solution, which can be interpreted either as a wave function or as a classical field. With the second interpretation the left-hand side of (1) can be calculated as

$$T_\psi = \frac{1}{2}\int \frac{\vec{p}^2}{\sigma}d^3 r , \tag{7}$$

where

$$\sigma = T^{00}, \quad \vec{p} = (T^{01}, T^{02}, T^{03}) ,$$

and $T^{\mu\nu}$ is the symmetric energy-momentum tensor

$$T^{\mu\nu} = \frac{i}{4}\left[\bar{\psi}\gamma^\mu \partial^\nu \psi - (\partial^\nu \bar{\psi})\gamma^\mu \psi\right.$$

$$\left. - (\partial^\mu \bar{\psi})\gamma^\nu \psi + \bar{\psi}\gamma^\nu \partial^\mu \psi\right] - g^{\mu\nu}\mathcal{L} . \tag{8}$$

A bit of algebra leads to

$$T_\psi = \frac{2\pi}{3}\frac{1}{\lambda\mu}\int_0^\infty \frac{(\Omega G + F/2\rho)^2 F^2}{\Omega(G^2 + F^2) + \frac{1}{2}(G^2 - F^2)^2}\rho^2 d\rho ,$$

$$\langle x^2\rangle_\psi = \frac{1}{\mu^2}\frac{\int_0^\infty (F^2 + G^2)\rho^4 d\rho}{\int_0^\infty (F^2 + G^2)\rho^2 d\rho} . \tag{9}$$

In the ground state ($\Omega = 0.936$) one has

$$T_{\psi_0} = \frac{0.537}{\lambda\mu}, \quad \langle x^2\rangle_{\psi_0} = \frac{10.62}{\mu^2}, \quad N = \frac{22.98}{\lambda\mu^2} .$$

From (1) we get

$$\frac{\hbar^2}{m} \leqslant \frac{5.07}{\lambda\mu^3} . \tag{10}$$

In order to compare the classical and quantum theories we express μ in units of mass by the change $\mu = m/\hbar$ which simplifies the relation (10) without affecting the results of this work. Thus (1) can be written in two forms,

$$\lambda m^2 \leqslant 5.07\hbar , \tag{11a}$$

$$N \geqslant 4.53\hbar . \tag{11b}$$

These inequalities give the condition for the association $\vec{p} \cong (\hbar/i)\vec{\nabla}$, which gives physical meaning to the Heisenberg relations, to be adequate. As we see it, this happens if the theory is near the linear limit (small λ) or, equivalently, if N is big with respect to $\hbar$. However, if we use this model to represent an elementary particle, as was done

in Ref. 3, $N = \hbar$ and the Heisenberg relations no longer apply. In other words, a particle described by the Soler kink with $N = \hbar$ does not have quantum behavior.

(b) Nonlinear model of the hydrogen atom[5]: If the nonlinear Dirac field is submitted to a Coulomb potential we have a model of the hydrogen atom. It was solved numerically with the result that every level is represented by a solitary wave or kink. The energy difference with the usual quantum mechanics model is not observable if $\lambda m_e^2 < 10^{-5}\hbar$, where m_e is the electron mass. A family of solutions depending on the frequency is found. There is no minimum of the energy, but the solution tends to zero when $\Omega \to \Omega_e$, where Ω_e is an eigenvalue of the linear problem. The reader is referred to Ref. 5 for more details.

Proceeding as in the Soler model we find

$$\lambda m^2 \leqslant 1.87\times 10^{-5}\frac{\Omega - \Omega_e}{\alpha^6}\hbar , \tag{12a}$$

$$N \geqslant 0.033\hbar . \tag{12b}$$

As we see (12b) is a much weaker condition than (11b), corresponding to the fact that as the electron is very extended in the atom the nonlinearities become less important. The frequency must adjust itself in such a way that $N = \hbar$, from which we conclude that the Heisenberg relations are verified. In other words, the classical model of the hydrogen atom has a quantum behavior.

IV. CONCLUSIONS

We have seen that the physical statement implied in the Heisenberg relations applies to the kinks of some nonlinear classical equations if the self-coupling constant is small enough or if the norm is large as compared with $\hbar$.

The first condition is easily interpreted: if λ is small we are near the weak-field limit, where the Heisenberg relations apply.

The second condition suggests an attractive possibility. As is well known, classical mechanics is the limit of quantum mechanics when $\hbar \to 0$, which means when $\hbar$ is small with respect to the actions or norms of the problem. If the conditions (10b) or (11b) do not hold, $\hbar$ is larger than a quantity of the order of the norm of the kink. It can be even much larger, a situation which we could term as "ultraquantum" and which is characterized by the violation of the Heisenberg relations. In this context there are three kinds of physical systems:

(a) The classical systems in which the Heisenberg relations are verified in a trivial way because the bounds that they impose are smaller than the experimental errors.

(b) The quantum systems in which the Heisenberg relations hold in a nontrivial way.

(c) The "ultraquantum" systems in which the Heisenberg relations are violated in the above explained sense.

When the "ultraquantum" theory admits the linear approximation we obtain the quantum one, which in some cases can be approximated by the classical theory. It is paradoxical that in the study of the "ultraquantum" world the classical consideration of the problem seems to be an essential step.

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