SL(2, C) exterior forms and quantum gravity

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On the basis of a formal generalization of the functional integral to the space of SL(2, C) exterior forms, I obtain gravity as an exact effective theory, after integrating over fundamental spin-1/2 and spin-3/2 fields. The same formalism is suggested for a new frame-independent formulation of quantum gravity.

I. INTRODUCTION

A successful program to quantize gravitation and to obtain finite quantum corrections has not been possible for most conventional systems.[1,2] Recently, great interest has been shown in supergravitational systems,[3,4] many of which have exhibited finite results in few-loop orders. The simplest of these systems consists of a spin-2 gravitational field and a spin-$\frac{3}{2}$ fermion.[5,6] Besides invariance under general coordinate and local Lorentz transformations, the theory admits local supersymmetry.[7] On the other hand, we have noted[8] that this system acquires an elegant and unique expression in Cartan's frame-independent formalism of exterior differential forms.[9] The latter formalism has so far been considered as a mere notational apparatus. However, it is part of our present purpose to give it deeper significance in defining the quantum theory.

We shall consider theories consisting of spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ matter fields and a spin-2 (vierbein) gravitational field. These fields are strikingly the only ones whose general coordinate-invariant and local Lorentz-invariant actions can be formulated in a metric-free and frame-independent language of differential forms.

The quantum theory will be defined by extending the idea of the path integral[10-17] to an integral over the functional space of SL(2, C) exterior forms. Consequently, the Grassmann-type nature[18] of exterior multiplication has the effect of truncating the formal power expansion of the functional determinant and providing an exact result.

As a first application of our formalism we perform, in Sec. II, an integration over the fermionic fields[12-20] and we are led exactly to the usual vierbein action of gravity.[21]

In Sec. III, we discuss the application of our formalism to the construction of the full quantum-gravity theory.[22]

II. GENERALIZED FUNCTIONAL INTEGRAL

In the conventional path-integral approach to quantization,[10-17] one defines $W[J]$, the generating functional of Green's functions, by

$$W[J]=\int [d\phi]\exp\{[S(\phi)+\int d^4x\, J(x)\phi(x)]\}, \qquad (1)$$

where $\phi(x)$ represents all fields of the theory, $S(\phi)$ is the action of these fields, $J(x)$ are external sources, and $[d\phi]$ is a generalized integration measure in the infinite-dimensional functional space. Now if one divides the fields into two classes $\phi(x)=\{A(x), B(x)\}$ and performs integration over one class, say $A(x)$, without including sources for that class, one obtains

$$W[J_B]=\int [dB]\exp\left\{i\left[S_{\mathrm{eff}}(B)+\int d^4x\, J_B(x)B(x)\right]\right\}. \qquad (2)$$

Here $S_{\mathrm{eff}}(B)$ is an effective action expressed solely in terms of $B(x)$. The Green's functions obtained from (2) consist solely of B lines. However, these Green's functions are effectively the same as the corresponding Green's functions obtained from (1) with arbitrary A and B internal lines and only B external lines.

Now we are dealing with a theory described by the following general coordinate-invariant and local Lorentz-invariant action,

$$S=S(e,w)+S(\psi,\chi), \qquad (3)$$

where

$$S(e,w)=\int d^4x -\left(\frac{\tilde{e}}{16\kappa^2}e^{\mu a}e^{\nu b}R_{\mu\nu ab}+\frac{\Lambda}{4}\tilde{e}\right) \qquad (4)$$

is the gravitational field (vierbein) action. Here κ^2 is the gravitational constant, Λ is a cosmological constant, $e_\mu^{\ a}$ is the vierbein field, and

$$\tilde{e}=\det(e_{\mu a}),\quad e^{\mu a}e_{\nu a}=\delta^\mu_\nu,$$
$$R^{ab}_{\mu\nu}=\partial_\mu w^{ab}_\nu - w^{ac}_\mu\, \omega^{b}_{\nu c}-(\mu\leftrightarrow\nu). \qquad (5)$$

w^{ab}_μ is a Lorentz gauge field which is independent of $e_\mu^{\ a}$ but would be expressed in terms of it and the spinor matter fields upon varying (3).

We also have

$$S(\psi,\chi)=\int d^4x(\tilde{e}\bar{\psi}ie^{\mu}\nabla_\mu\psi+\tilde{e}m\bar{\psi}\psi+i\epsilon^{\mu\nu\lambda\rho}\bar{\chi}_\mu e_\nu\gamma_5\nabla_\lambda\chi_\rho+\tilde{e}M\bar{\chi}_\mu ie^{\mu\nu}\chi_\nu)\,, \tag{6}$$

where

$$\nabla_\mu=\partial_\mu-\tfrac{1}{4}iw_\mu^{ab}\sigma_{ab}\,,\qquad e_\mu=e_\mu^a\gamma_a.\quad e^{\mu\nu}=\sigma_{ab}e^{\mu a}e^{\nu b}. \tag{7}$$

Here ψ and χ_μ are spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fields, respectively.

Now rewrite (4) and (6) as integrals over SL(2, C) exterior differential forms, in the following way:

$$S(e,w)=\int \mathrm{Tr}\gamma_5\left(-\frac{1}{16\kappa^2}e\wedge e\wedge R+\frac{\Lambda}{96}e\wedge e\wedge e\wedge e\right) \tag{8}$$

and

$$S(\psi,\chi)=\int\left(\frac{i}{6}\bar{\psi}\gamma_5 e\wedge e\wedge e\wedge\nabla\psi+i\bar{\chi}\wedge e\gamma_5\wedge\nabla\wedge\chi+\frac{m}{24}\bar{\psi}\gamma_5 e\wedge e\wedge e\wedge e\psi+\frac{M}{2}\bar{\chi}\gamma_5\wedge e\wedge e\wedge\chi\right). \tag{9}$$

In writing (8) and (9), use the following

$$\begin{aligned}&d^4x\epsilon^{\mu\nu\lambda\rho}=dx^\mu\wedge dx^\nu\wedge dx^\lambda\wedge dx^\rho\,,\\ &\chi=\chi_\mu dx^\mu,\quad \nabla=\nabla_\mu dx^\mu\,,\\ &e=\gamma_a e_\mu^a dx^\mu,\quad \omega=\sigma_{ab}\omega_\mu^{ab}dx^\mu\,,\\ &R=\sigma_{ab}R_{\mu\nu}^{ab}dx^\mu\wedge dx^\nu\,.\end{aligned} \tag{10}$$

Thus χ, e, and ω are SL(2, C) one-forms, and ψ is an SL(2, C) spinor zero-form. These forms obey the usual exterior calculus, in particular, the Grassmann-type wedge product ($\wedge$).

Now let us perform, as in (1) and (2), the functional integration over the spinor fields and obtain

$$e^{iS_{\mathrm{eff}}(e,\omega)}=\int[d\psi][\psi\chi]e^{iS(\psi,\chi)}\,, \tag{11}$$

where the meaning of the functional integration is extended to the functional space of the SL(2, C) spinor exterior forms ψ and χ.

In order to integrate over the generalized fermionic variables ψ and χ, note the familiar formula

$$\det A=\int[d\xi d\xi^*]e^{-\int\xi^* A\xi}=e^{\mathrm{Tr}\ln A}\,. \tag{12}$$

In this formula, ξ and ξ^* are Grassmann variables, ξ^* being the Hermitian adjoint of ξ. This formula holds when the space of ξ is a finite-dimensional matrix space. It has been extrapolated in the path-integral approach to the infinite-dimensional functional x space. In the following, we extrapolate this formula further to the functional space of the SL(2, C) exterior forms. In doing so, we must take, corresponding to (11) and (12),

$$e^{iS_{\mathrm{eff}}(e,\omega)}=e^{\mathrm{Tr}(\gamma_5\ln A)}\,. \tag{13}$$

Here A is a kernel belonging to the space of SL(2, C) exterior forms. The logarithm should be interpreted by its formal power series, which in our case consists of few terms. The trace operation Tr denotes integration over the SL(2, C) exterior forms. This means that it involves the usual Grassman-type integral which picks up the terms containing exterior four-forms (in four dimensions) and taking the trace of the Dirac matrices. Since the volume element constructed from an exterior four-form is odd under parity, the Dirac γ_5 matrix should be there, as in (13), in the definition of the trace operation. This is essential in order to ensure that the action integral in the space of SL(2, C) exterior forms is even under parity.

Applying formula (13) to integral (11) and using (9), we obtain

$$S_{\mathrm{eff}}(e,w)=\mathrm{Tr}\gamma_5[\ln(\gamma_5 e\wedge e\wedge e\wedge\nabla+\tfrac{1}{4}m\gamma_5 e\wedge e\wedge e\wedge e)+\ln(e\wedge\gamma_5\nabla+\tfrac{1}{2}M\gamma_5 e\wedge e)]\,. \tag{14}$$

To evaluate the logarithms, note the familiar series

$$\ln x=2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x-1}{x+1}\right)^n,\quad n\text{ is odd.} \tag{15}$$

We need the terms only linear and quadratic in x,

$$\ln x\approx Lx+Jx^2\,, \tag{16}$$

where

$$L=\sum_{n=1}^{\infty}4$$

and

$$J=-4\sum_{n=1}^{\infty}n,$$

denote infinite constants.

Applying the above expansion (16) to Eq. (14), we find, up to perfect differentials,

$$S_{\mathrm{eff}}(e,w)=J\,\mathrm{Tr}\,(\gamma_5 e\wedge e\wedge R)+L\,\mathrm{Tr}\,(e\wedge e\wedge e\wedge e\wedge\gamma_5)\,. \tag{17}$$

This has the same form as the gravitational action (8). Now L and J are infinite constants. They can be absorbed in the redefinitions of the physical gravitational and cosmological constants.

Note that in integrating over the spinor fields, the spin-$\frac{3}{2}$ field kinetic term has mainly contributed to the first term of (17) while the mass terms of both spinor fields have contributed to the cosmological terms. The kinetic terms of the spin-$\frac{1}{2}$ field did not make any contribution.

III. DISCUSSION

What we have done, in the previous section, is to integrate over the spinor fields and obtain an exact effective theory described by the usual vierbein action of gravity. This means that all quantum corrections, which consist of Feynman diagrams with only gravitational-field external lines and arbitrary spinor-field internal lines, are effectively given by the corresponding diagrams in pure gravity. However, this conclusion has nothing to do with the diagrams containing external spinor lines. These diagrams and the Green's functions of the full quantum theory, in general, can be studied through the generating functional (1).

Hence we introduce external sources

$$\bar{\eta}^{\alpha}=(\bar{\eta}_{\mu\nu\lambda}\gamma_5)^{\alpha}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda}\,,$$
$$\bar{\lambda}^{\alpha}=(\bar{\lambda}_{\mu\nu\lambda\rho}\gamma_5)^{\alpha}dx^{\mu}\wedge dx^{\mu}\wedge dx^{\lambda}\wedge dx^{\rho}\,, \qquad (18)$$

corresponding to χ^{α} and ψ^{α}, respectively. Actually there are sources corresponding to the spinor fields as well as their Hermitian adjoints.

Define the generating functional

$$W(\bar{\eta},\bar{\lambda})=\int[d\psi d\chi]\,\exp\left[i\,S(\psi,\chi)+i\int(\bar{\eta}\wedge\psi+\bar{\lambda}\chi)\right]. \qquad (19)$$

It is interesting to note that, again by the power of the formalism, the above integration can be formally done, by the method of completing the square and shifting the integration variables. We obtain

$$W[\bar{\eta},\bar{\lambda}]=\exp\left[iS_{\text{eff}}(e,w)+\int\bar{\eta}\wedge(\gamma_5e\wedge\nabla+\tfrac{1}{2}M\gamma_5e\wedge e)^{-1}\wedge\eta+\int\bar{\lambda}\wedge(\gamma_5e\wedge e\wedge e\wedge\nabla+\tfrac{1}{2}m\gamma_5e\wedge e\wedge e\wedge e)^{-1}\wedge\lambda\right]. \qquad (20)$$

In order to give meaning to the inverses of the expressions occurring in (20) involving the exterior forms, we shall introduce the natural notion of division of forms. This is comprised of operations like

$$[dx^{\nu}]^{-1}dx^{\mu}=\delta^{\mu}_{\nu}\,,$$
$$[dx^{\lambda}]^{-1}(dx^{\mu}\wedge dx^{\nu})=\delta^{\mu}_{\lambda}dx^{\nu}-\delta^{\nu}_{\lambda}dx^{\mu}\,, \qquad (21)$$

etc.

Thus from (20) one can define propagators for the forms χ and ψ. These represent the complete propagators of the interacting theory.

It is also possible to apply the above ideas to the theory including the gravitational field. We shall consider here the pure gravitational action with external sources.

We define the generating functional

$$W[N,K]=\int[de][dw]\,\exp[iS(e,w)+i\,\mathrm{Tr}(N\wedge e+K\wedge w)]\,, \qquad (22)$$

where the functional integral is taken over the indicated SL(2, C) exterior forms and $S(e,w)$ is given by (8). We have introduced the sources

$$N=\gamma_aN^{a}_{\mu\nu\lambda}\gamma_5dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda}\,,$$
$$K=\sigma_{ab}\gamma_5K^{ab}_{\mu\nu\lambda}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda}\,, \qquad (23)$$

corresponding to e and w, respectively.

The integration over e and w does not seem to be formally straightforward in general. However, we note the special case when the cosmological constant is zero. In that case, integration over e is Gaussian and can be done easily to give

$$W[N,K]=\int dw\,\exp\{\mathrm{Tr}[N\wedge(\gamma_5R)^{-1}\wedge N+K\wedge w]\}\,, \qquad (24)$$

where we have neglected the familiar surface term

$$\mathrm{Tr}\gamma_5\ln R=J\,\mathrm{Tr}(R\wedge R\gamma_5)$$
$$=J\int d^4x\,e^{\mu\nu\lambda\rho}R^{ab}_{\mu\nu}R^{cd}_{\lambda\rho}\epsilon_{abcd}\,. \qquad (25)$$

The next integration in (24) over w is not straightforward. It could be dealt with by developing a perturbative scheme. However, we would like to make the suggestion that w may be left as a classical background, corresponding to nontrivial solutions of Einstein's equations. The integration in (24) becomes a summation over classical w's. In that case, one has propagators for the vierbein field and other spinor fields as implied by Eq. (20) and (24).

Needless to say that the above formal developments have to be made more precise and more useful for practical quantum calculations. At this stage we feel that the impressive frame-independent formalism of differential forms has an important role to play in formulating an unorthodox approach to quantum gravity.

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