Kerr-Schild geometry, spinors, and instantons

Hafid Boutaleb Joutei* and Amitabha Chakrabarti
Centre de Physique Théorique de l'Ecole Polytechnique, Plateau de Palaiseau, 91128 Palaiseau Cedex, France
(Received 21 March 1978)

The spin connection is studied for the Kerr-Schild metric. Our choice of tetrad leads to particularly simple results. The associated complex SU(2) gauge fields for many important particular cases, including the axially symmetric stationary Kerr metric, are thus presented in a unified fashion. Static spherically symmetric cases are studied in detail, including a cosmological term. A Lorentz gauge transformation is introduced for this class such that after a further (inverse Finkelstein) coordinate transformation a very simple form is obtained in the diagonal static metric. Using this, we study the passage to the Euclidean section, leading to real (anti) self-dual SU(2) gauge fields for the uncharged case. The role of the cosmological constant concerning the Pontryagin indices is elucidated. Finally a class of solutions of the zero-mass Dirac equation is studied in an appendix. The relation of such solutions possessing stringlike singularities, to similar ones in flat space, in the presence of non-Abelian monopoles and instantons is pointed out.

I. INTRODUCTION

In the following sections we will study the spin connections for metrics belonging to the Kerr-Schild (KS) class. The remarkable properties of the Kerr-Schild metric (see Appendix A) permit a simple unified treatment of important particular cases.

In Sec. II we start by introducing a tetrad and the related representation of the γ matrices in curved space which turn out to be particularly convenient. This leads to a simple form of the spin connection in the KS metric. This can then be used to solve the Dirac equation. In Appendix B we discuss a class of solutions, with stringlike singularities, for the zero-mass case. These solutions were already obtained in Ref. 1. We present a simpler and more general construction and point out that similar solutions also arise in flat space in the presence of monopoles and instantons.

Our main object, however, is the study of the spin connection as an SO(3, 1) gauge field in curved spacetime and its Euclidean SO(4) continuation. Spin connections in general relativity have been discussed extensively by a number of authors.[2] In Sec. II we study associated complex SU(2) gauge fields. Various properties are analyzed including the effect of a cosmological term. A Lorentz gauge transformation is introduced for the static spherically symmetric case. A further coordinate transformation to the diagonal metric ($g_{t0}=0$) gives then directly simple forms for the gauge fields. This is used in Sec. III to study, for static spherical symmetry, the passage to the Euclidean section. Recently Charap and Duff,[3] following an idea of Wilczek,[4] studied the spin connection for the Schwarzschild metric, continued to the Euclidean regime. They used a continuation pointed out by Hawking[5] and obtained SU(2) instantons in curved space with Pontryagin indices ± 1. This becomes possible because the domains of integration of r and (Euclidean) t are defined to be those corresponding to real (continued) Kruskal coordinates. Thus t becomes periodic and $r \geq 2M$, where M is the mass. We will analyze a more general case, including a cosmological term which still leads to (anti) self-dual gauge fields. Topological features will be discussed and the two limits (Schwarzschild and de Sitter) will be compared.

II. SPIN CONNECTION IN KERR-SCHILD GEOMETRY AND ASSOCIATED GAUGE FIELDS

The Dirac equation in curved space is (see Refs. 1, 2, and sources quoted therein)

$$\hat{\gamma}^{\mu}(\partial_{\mu} - B_{\mu})\psi = im\psi , \tag{2.1}$$

where

$$\{\hat{\gamma}_{\mu}, \hat{\gamma}_{\nu}\} = 2g_{\mu\nu} \tag{2.2}$$

and the spin connection B_{μ} satisfies

$$\partial_{\mu}\hat{\gamma}_{\nu} - \Gamma^{\sigma}_{\nu\mu}\hat{\gamma}_{\sigma} - [B_{\mu}, \hat{\gamma}_{\nu}] = 0 . \tag{2.3}$$

(We will not consider more general definitions of B_{μ}.[2]) Given a representation of $\hat{\gamma}_{\mu}$, Eq. (2.3) determines B_{μ} except for a term proportional to the unit matrix $\underline{1}$ which should be a solution of sourceless Maxwell equations. For our purposes we will consider only the traceless part of B_{μ} fixed by (2.3). Moreover, we will restrict ourselves to the Kerr-Schild metric.[6-9] In Appendix A we have summarized many useful results concerning this class of metrics. The key property is, of course, that in

$$g_{\mu\nu} = \eta_{\mu\nu} - 2Ml_{\mu}l_{\nu} \tag{2.4}$$

l is a null vector both with respect to $\eta_{\mu\nu}$ (the Minkowski metric) and $g_{\mu\nu}$, i.e.,

$$l_\mu l_\nu \eta^{\mu\nu} = 0 = l_\mu l_\nu g^{\mu\nu}, \tag{2.4'}$$

where

$$g^{\mu\nu} = \eta^{\mu\nu} + 2Ml^\mu l^\nu . \tag{2.5}$$

The index of the l vector can be raised or lowered by both the η and the g matrix. This permits us to introduce the tetrad

$$L^a_\mu = \eta^a_\mu - Ml_\mu l^a , \tag{2.6}$$

which satisfies

$$L^a_\mu L^b_\nu \eta_{ab} = g_{\mu\nu} \tag{2.7}$$

and

$$L^a_\mu L^b_\nu g^{\mu\nu} = \eta^{ab} . \tag{2.8}$$

This tetrad is particularly suitable for our purpose (though complex null tetrads are often used for Kerr-Newman metrics) and leads to the following canonical realization of $\hat{\gamma}_\mu$.

In terms of the flat-space Dirac matrices γ_μ we define

$$\begin{aligned}\hat{\gamma}_\mu &= L^a_\mu \gamma_a \\ &= \gamma_\mu - M(\gamma_a l^a) l_\mu \equiv \gamma_\mu - M\tau l_\mu .\end{aligned} \tag{2.9}$$

This gives

$$\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2(\eta_{\mu\nu} - 2Ml_\mu l_\nu)$$

and (2.10)

$$\hat{\gamma}^\mu = \gamma^\mu + M\tau l^\mu .$$

In particular, we will choose [with $(+---)$ for $\eta_{\mu\nu}$]

$$\gamma_0 = \begin{pmatrix} & \sigma_0 \\ \sigma_0 & \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} & -\sigma_i \\ \sigma_i & \end{pmatrix},$$

$$\text{and } \gamma_5 = \begin{pmatrix} \sigma_0 & \\ & -\sigma_0 \end{pmatrix}, \tag{2.11}$$

which will lead to a B_μ in block-diagonal form.

In Ref. 1 one finds a realization which in our notation is

$$\hat{\gamma}_\mu = \gamma_\mu + i\sqrt{2M}\,\gamma_5 l_\mu . \tag{2.12}$$

This looks very simple but leads, as will be seen, to a less simple form for B_μ. Using (2.3), (2.9), and (A5) one can verify that

$$B_\mu = (-\tfrac{1}{2}M)\, i\Sigma^{\alpha\beta}\partial_\alpha(l_\beta l_\mu), \tag{2.13}$$

where

$$\Sigma^{\alpha\beta} = \frac{1}{2i}[\gamma^\alpha, \gamma^\beta] .$$

The corresponding result of (1) involves additional terms. *One should also note both the analogies and differences of (2.13) as compared to a well-known class of solutions for instantons in flat spacetime* [e.g. Eq. (3.52) of Ref. 10 and the corresponding anti-self-dual case]. We will consider only *stationary* cases, where one can define the scalars A and B such that (see Appendix A)

$$\partial_\mu l^\mu = -B$$

and (2.14)

$$l^\mu \partial_\mu l_\nu = -Al_\nu .$$

This leads to, from (2.10) and (2.13),

$$\hat{\gamma}^\mu B_\mu = \tfrac{1}{2}M(A+B)\tau . \tag{2.15}$$

Also, evidently,

$$l^\mu B_\mu = 0 . \tag{2.16}$$

The Dirac equation (2.1) becomes

$$\begin{aligned}(\gamma^\mu\partial_\mu - im)\psi &= -M\tau[l^\mu\partial_\mu - \tfrac{1}{2}(A+B)]\psi \\ &= -M[l^\mu\partial_\mu + \tfrac{1}{2}(A-B)]\tau\psi .\end{aligned} \tag{2.17}$$

This should be compared with Eq. (7.21) of Ref. 1. In Appendix B we make some remarks on a class of solutions of (2.17) for $m=0$. Here we briefly note the following relations. For

$$\hat{\gamma}_\mu = \gamma_\mu + c_1\tau l_\mu + ic_2\gamma_5 l_\mu , \tag{2.18}$$

(2.2) and (2.4) impose

$$(c_2)^2 - 2c_1 = 2M . \tag{2.19}$$

With

$$S = \exp(-\tfrac{1}{2}i\, c_2\gamma_5\tau), \tag{2.20}$$

one obtains for (2.18)

$$S\hat{\gamma}_\mu S^{-1} = \gamma_\mu - M\tau l_\mu , \tag{2.21}$$

with a corresponding transformation

$$B_\mu \to SB_\mu S^{-1} - S\partial_\mu S^{-1} . \tag{2.22}$$

This gives the precise relation between our representation and that of Ref. 1. We have made a systematic study of possible representations of $\hat{\gamma}_\mu$ and their properties. The representation (2.9) seems to be the most convenient one. Let us also note that one can write (2.13) as

$$B_\mu = -\tfrac{1}{4}\, i\Sigma_{ab} B^{ab}_\mu , \tag{2.23}$$

where

$$B^{ab}_\mu = M[\eta^{a\nu}\partial_\nu(l^b l_\mu) - \eta^{b\nu}\partial_\nu(l^a l_\mu)] . \tag{2.24}$$

A formula such as (11) of Ref. 3 holds, of course, also for Lorentz signature and one obtains from it (2.24) on using (2.6). In our approach the formula is already in block-diagonal form; the Dirac equation is also obtained simultaneously.

In this section we continue to study the properties of the SO(3, 1) field tensors corresponding to our B_μ. From (2.13) we obtain

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] = (-\tfrac{1}{2}M)\, i\Sigma^{\alpha\beta}\partial_\alpha[\partial_\mu(l_\beta l_\nu) - \partial_\nu(l_\beta l_\mu)] + \frac{M^2}{4}[\Sigma^{\alpha\beta}, \Sigma^{\rho\sigma}]\partial_\alpha(l_\beta l_\mu)\partial_\rho(l_\sigma l_\nu). \qquad (2.25)$$

The results (2.13) and (2.25) hold for the general KS metric, including even possible time dependence. But the continuation to the Euclidean section[5] to be used does not lead to a real metric even for the stationary Kerr case. For this reason we now restrict ourselves to the *static spherically symmetric metrics* [(A7)–(A14)]. The upper and lower blocks of $\Sigma^{\alpha\beta}$ (corresponding respectively to positive and negative helicities of the zero-mass Dirac spinor) will now be exhibited in terms of the 2×2 matrix

$$\Phi \equiv \frac{\vec{\sigma}\cdot\hat{r}}{2}, \qquad (2.26)$$

where $\hat{r} = \vec{\lambda} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. One obtains [with $(d/dr)l_0^2 \equiv (l_0^2)'$, $\vec{B} \equiv (B_1, B_2, B_3)$]

$$B_0 = \mp M(l_0^2)'\Phi, \qquad (2.27)$$

$$\vec{B} = M\{\mp l_0^2\vec{\nabla}\Phi - l_0^2[\Phi, \vec{\nabla}\Phi] \mp (l_0^2)'\Phi\hat{r}\}. \qquad (2.28)$$

The properties of such forms have been discussed in detail elsewhere.[11] One may use those results here, taking care of the changes of convention introduced. Defining $\vec{E} \equiv (F_{01}, F_{02}, F_{03})$ and $\vec{H} \equiv (F_{23}, F_{31}, F_{12})$, we obtain

$$\vec{H} = A_h\vec{\nabla}\Phi + B_h[\Phi, \vec{\nabla}\Phi] + C_h\Phi\hat{r}, \qquad (2.29)$$

$$\vec{E} = A_e\vec{\nabla}\Phi + B_e[\Phi, \vec{\nabla}\Phi] + C_e\Phi\hat{r}, \qquad (2.30)$$

where

$$A_h = (-i)M(1 + Ml_0^2)(l_0^2)', \quad A_e = \pm M(1 - Ml_0^2)(l_0^2)',$$
$$B_h = \mp M^2 l_0^2(l_0^2)', \quad B_e = iM^2 l_0^2(l_0^2)', \qquad (2.31)$$
$$C_h = (-i)2Ml_0^2/r^2, \quad C_e = \pm M(l_0^2)''.$$

Let us now calculate

$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}. \qquad (2.32)$$

Defining

$$\vec{\mathcal{E}} \equiv (F^{01}, F^{02}, F^{03}), \quad \vec{\mathcal{H}} \equiv (F^{23}, F^{31}, F^{12}), \qquad (2.33)$$

one obtains

$$\vec{\mathcal{E}} = \mp i\vec{H} \mp M\left[(l_0^2)'' - \frac{2l_0^2}{r^2}\right]\Phi\hat{r}, \qquad (2.34)$$

$$\vec{\mathcal{H}} = \mp i\vec{E} + iM\left[(l_0^2)'' - \frac{2l_0^2}{r^2}\right]\Phi\hat{r}. \qquad (2.35)$$

The dual of $F_{\mu\nu}$ is

$$*F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \qquad (2.36)$$

where

$$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}\,[\alpha\beta\gamma\delta], \qquad (2.37a)$$

$$\epsilon^{\alpha\beta\gamma\delta} = g^{-1}\epsilon_{\alpha\beta\gamma\delta}, \qquad (2.37b)$$

and $[\alpha\beta\gamma\delta]$ is the usual completely antisymmetric tensor in flat space. In our case

$$g = -1 \text{ and } \epsilon^{\alpha\beta\gamma\delta} = -[\alpha\beta\gamma\delta]. \qquad (2.38)$$

Thus for

$$F_{\mu\nu} = (\vec{E}, \vec{H}), \quad *F^{\mu\nu} = -(\vec{H}, \vec{E}), \qquad (2.39)$$

and hence from (2.34) and (2.35)

$$F^{\mu\nu} = \pm i\, *F^{\mu\nu} + J^{\mu\nu}, \qquad (2.40)$$

where

$$J^{\mu\nu} \equiv M\left[(l_0^2)'' - \frac{2l_0^2}{r}\right](\mp\Phi\hat{r}, i\Phi\hat{r}). \qquad (2.41)$$

Now from (A8) and (A13), we have for

$$N = \left(1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2} - \frac{\Lambda}{3}r^2\right), \qquad (2.42)$$

$$(l_0^2)'' - \frac{2l_0^2}{r^2} = -\frac{2(Q^2 + P^2)}{Mr^4}. \qquad (2.43)$$

Thus for $Q = 0 = P$, i.e., for

$$N = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right), \qquad (2.44)$$

$$F^{\mu\nu} = \pm i\, *F^{\mu\nu}. \qquad (2.45)$$

The factor i arises because we now have SO(3, 1) as the gauge group. After continuation (2.45) will correspond to self-dual and anti-self-dual fields, respectively. Let us note here a particularly simple feature of the KS metric. Covariant derivation gives

$$D_\mu F^{\mu\nu} \equiv \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho}F^{\rho\nu} + \Gamma^\nu_{\mu\rho}F^{\mu\rho} - [B_\mu, F^{\mu\nu}] \qquad (2.46)$$

$$= \partial_\mu F^{\mu\nu} - [B_\mu, F^{\mu\nu}]. \qquad (2.47)$$

Using evident symmetry properties (valid for all torsion-free metrics) and (A6) holding for the KS metric the Christoffel symbols disappear altogether. The curvature, however, enters through the definition (2.32). Flat-space solutions cannot be carried over.

In any case one has the Bianchi identity

$$D_\mu *F^{\mu\nu} = 0. \qquad (2.48)$$

Thus from (2.40),

$$D_\mu F^{\mu\nu} = D_\mu J^{\mu\nu} \equiv j^\nu, \qquad (2.49)$$

where

$$j^\nu \equiv \partial_\mu J^{\mu\nu} - [B_\mu, J^{\mu\nu}]\,. \tag{2.50}$$

Thus for $Q=P=0$ we have (2.44), (2.45), and

$$D_\mu F^{\mu\nu} = 0\,. \tag{2.51}$$

For the charged case there is a "source term" j^ν given by (2.50) with

$$J^{\mu\nu} = -\frac{2(Q^2+P^2)}{r^4}\,(\mp\Phi\hat{r}, i\Phi\hat{r})\,. \tag{2.52}$$

From (2.29)–(2.35) one obtains, for both the upper and lower signs,

$$-\tfrac{1}{8}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{M^2}{r^2}\left\{[(l_0^2)']^2 + l_0^2(l_0^2)'' + \frac{1}{4}\left[(l_0^2)'' - \frac{2l_0^2}{r^2}\right]^2\right\} \tag{2.53}$$

and

$$-\tfrac{1}{8}\,\mathrm{Tr}(F_{\mu\nu}\,{}^*F^{\mu\nu}) = \mp i\,\frac{M^2}{r^2}\{[(l_0^2)']^2 + l_0^2(l_0^2)''\}\,. \tag{2.54}$$

There are simple relations between the invariants such as $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, $\mathrm{Tr}(F_{\mu\nu}\hat{\gamma}^\mu\hat{\gamma}^\nu)$, and those constructed from the Riemann tensor (see Loos, Ref. 2).

So far we have obtained the results by systematically exploiting the beautiful properties of the KS metric (A1). This permitted us to treat the spin connections in a unified way. But, as already mentioned, passage to the Euclidean section will be considered for static spherical symmetry only. Moreover, we will use for that purpose the metric corresponding to (A7). Though values of the invariants [such as (2.53)] will not be affected, it is interesting to try to obtain a particularly simple form for B_μ after a coordinate transformation inverse to (A10). For this purpose we first introduce a *spin gauge transformation*. We again separate the two cases in (2.27) and (2.28) from the start and define

$$S = \exp(\mp\beta(r)\Phi)\,, \tag{2.55}$$

where

$$\tanh\beta(r) = -\frac{Ml_0^2}{1-Ml_0^2} = \frac{N-1}{N+1} \tag{2.56}$$

from (A13). Now $B_\mu \to SB_\mu S^{-1} - S\partial_\mu S^{-1}$ gives for the transformed spin connection

$$B_0 = \pm\tfrac{1}{2}N'\Phi\,, \tag{2.57}$$

$$\vec{B} = (N^{1/2}-1)[\Phi, \vec{\nabla}\Phi] \mp \frac{N'}{2}\left(\frac{1-N}{N}\right)\Phi\hat{r}\,. \tag{2.58}$$

This result follows from a direct application of Eqs. (1.22) of Ref. 11 (taking care to supply factors i due to altered conventions).

Next we perform a coordinate transformation [inverse to (A10), acting on the index μ of B_μ], namely

$$t' = t + r - \int \frac{dr}{N}\,, \tag{2.59}$$

and obtain finally

$$B_0 = \pm\frac{N'}{2}\,\Phi\,, \tag{2.60}$$

$$\vec{B} = (N^{1/2}-1)[\Phi, \vec{\nabla}\Phi]\,. \tag{2.61}$$

After continuation this will provide the simple generalization of the Schwarzschild case discussed by Charap and Duff.[3] The results (2.45) for the uncharged case are, of course, conserved. These results may also be easily obtained directly for the metrics in question.

Before continuation, however, we have complex SU(2) gauge fields. In particular, taking note of our convention, we see that B_0^a is pure imaginary (for real N) and $\vec{B}^a$ are purely real (for $N \geq 0$). This should be compared with the example in Ref. 12. Complex monopoles have also been studied by Manton.[13] On the other hand, the 't Hooft-Polyakov type of monopole in curved space has been studied by several authors.[14] In curved space possible singularities at $r\to 0$ or $r\to\infty$ may possibly be avoided by restricting the domain appropriately. This will be made more explicit in the next section, where, moreover (as a consequence of the continuation procedure) the time integration will be over a finite period.

III. PASSAGE TO THE EUCLIDEAN SECTION

To construct SO(4) gauge fields [leading to $SU(2)_\pm$ ones] one has to complexify the spacetime and go to the Euclidean section in a suitable fashion.[5] The pure Schwarzschild case has been discussed in Ref. 3. Let us consider the (uncharged) case with a nonzero cosmological constant, namely (A7) with

$$N = 1 - \frac{2M}{r} - \frac{\Lambda}{3}\,r^2\,. \tag{3.1}$$

The maximal analytic extension has been discussed recently.[15] We will utilize those results. We will start with the interesting case[15]

$$0 < 9M^2\Lambda < 1\,. \tag{3.2}$$

(Certain limiting features will be treated eventually.) For (3.2),

$$N = \left(-\frac{\Lambda}{3r}\right)(r-r_-)(r-r_1)(r-r_2)\,, \tag{3.3}$$

where, defining $\cos 3\alpha = (9M^2\Lambda)^{1/2}$, $(0<\alpha<\pi/6)$,

$$r_- = -2\Lambda^{-1/2}\cos\alpha < 0\,, \tag{3.4}$$

$$r_1 = 2\Lambda^{-1/2}\cos\left(\alpha - \frac{\pi}{3}\right) > 0\,,$$
$$r_2 = 2\Lambda^{-1/2}\cos\left(\alpha + \frac{\pi}{3}\right) > 0\,, \tag{3.5}$$

and

$$r_1 > r_2\,. \tag{3.6}$$

Transforming to Kruskal-type coordinates gives in (A37), for the region $r_2 < r < r_1$,

$$\begin{aligned}(\eta^2 - \xi^2) &= 4\exp(2cr^*)\\ &= 4(r_1 - r)^{-6cA_1/\Lambda}(r-r_2)^{-6cA_2/\Lambda}\\ &\quad\times (r-r_-)^{-6cA_-/\Lambda}\,,\end{aligned} \tag{3.7}$$

where

$$A_1 = \frac{r_1}{(r_1-r_2)(r_1-r_-)}\,,\quad A_2 = \frac{r_2}{(r_2-r_1)(r_2-r_-)}\,,$$
$$A_- = \frac{r_-}{(r_- - r_1)(r_- - r_2)}\quad (A_1>0\,,\ A_2<0\,,\ A_-<0)\,. \tag{3.8}$$

Choosing

$$c = -\Lambda/6A_1\,, \tag{3.9}$$

the metric corresponding to (A36) becomes regular at r_1 but remains singular at r_2. Similarly, choosing

$$c = -\Lambda/6A_2\,, \tag{3.10}$$

the metric becomes regular at r_2 but remains singular at r_1 (the root r_- being negative does not lie in the region of interest). For

$$M\Lambda^{1/2} \equiv \epsilon \ll 1\,, \tag{3.11}$$

$$\begin{aligned} r_- &= \Lambda^{-1/2}[-\sqrt{3} - \epsilon + O(\epsilon^2)]\,,\\ r_1 &= \Lambda^{-1/2}[+\sqrt{3} - \epsilon + O(\epsilon^2)]\,,\\ r_2 &= 2M + O(\epsilon^2)\,.\end{aligned} \tag{3.12}$$

Thus it is seen that as $\epsilon \to 0$ (for $\Lambda \to 0$) it is the choice (3.10) that leads to the Schwarzschild limit regularized at $r = 2M$.

Let us now introduce the continuation[3,5]

$$\xi \to -i\xi\,,\quad t \to -it \tag{3.13}$$

when [*for (3.10)*] the metric remains real for

$$r_2 \le r < r_1 \tag{3.14}$$

and t is now an angular variable with a period

$$T = \frac{12\pi}{\Lambda}(-A_2) = \frac{12\pi}{\Lambda}\,\frac{r_2}{(r_1-r_2)(r_2-r_-)}\,. \tag{3.15}$$

(As compared to Refs. 3 and 5 our line element has an opposite overall sign. This may be reabsorbed. The difference is trivial anyhow.) The corresponding continued fields are now [replacing B_0 by iB_0 in (2.60) and (2.61)]

$$\begin{aligned} iB_0 &= \pm\tfrac{1}{2}N'\Phi\,,\\ i\vec{B} &= i(N^{1/2}-1)[\Phi\,,\vec{\nabla}\Phi]\,.\end{aligned} \tag{3.16}$$

Our convention now corresponds to

$$B_\mu = B_\mu^a\,\frac{\sigma_a}{2i}\quad (a = 1,2,3)\,,$$

and the components B_μ^a are now real for $N>0$. Instead of (2.45) we now have

$$F^{\mu\nu} = \pm{}^*F^{\mu\nu} \tag{3.17}$$

[since $N'' = (2/r^2)(N-1)$], i.e., self-dual and anti-self-dual solutions, respectively. In (2.53) and (2.54) we now obtain

$$\frac{M^2}{r^2}\{[(l_0^2)']^2 + l_0^2(l_0^2)''\} = \frac{3M^2}{r^6} + \frac{\Lambda^2}{6}\,. \tag{3.18}$$

We have for the Pontryagin index (after continuation and simplifications)

$$\begin{aligned} P &= \frac{4\pi}{32\pi^2}\int_0^T dt\int_{r_2}^{r_1} r^2 dr(\mathrm{Tr}F_{\mu\nu}{}^*F^{\mu\nu})\\ &= \pm\frac{1}{2M}\,\frac{(6M-r_-)}{(r_2-r_-)}\,r_2\,.\end{aligned} \tag{3.19}$$

For a coupling constant e (instead of 1) the action is[3] $(8\pi^2/e^2)|P|$.

[Though the metric is not regular at $r=r_1$, the integrand is so and one may integrate up to $(r_1-\delta)$ and take the limit $\delta\to 0$.] When (3.11) holds we have from (3.19)

$$P = \pm 1 + O(\epsilon^2)\,. \tag{3.20}$$

Using (3.9) one obtains similarly

$$P = \pm\frac{1}{2M}\,\frac{(6M-r_-)}{(r_1-r_-)}\,r_1\,. \tag{3.21}$$

This diverges as ϵ^{-1} as $\epsilon\to 0$.

Let us now go back to (3.10) and (3.20) and compare two different limiting possibilities. We have the result that $\epsilon \to 0$ both for

$$\Lambda \to 0$$

and for (3.22)

$$M \to 0$$

and for both cases $P \to \pm 1$. Though P has the same limiting behavior one should study the extreme cases separately by starting respectively with

$$N = \left(1 - \frac{2M}{r}\right)\ \text{(Schwarzschild)} \tag{3.23}$$

and

$$N=\left(1-\frac{\Lambda r^2}{3}\right) \text{ (de Sitter limit of the static case)}. \quad (3.24)$$

The first case is the one studied in Ref. 3. Here we consider the second case where the situation is fundamentally different. The corresponding KS metric is now (substituting $Ml_0^2=\Lambda r^2/6$),

$$g_{\mu\nu}=\eta_{\mu\nu}-\tfrac{1}{3}\Lambda r^2\lambda_\mu\lambda_\nu . \quad (3.25)$$

There is no essential singularity at $r=0$.

We will choose as the region of interest

$$r<(3/\Lambda)^{1/2} .$$

One obtains (after continuation)

$$\tfrac{1}{4}(\eta^2+\xi^2)=\exp(2cr^*)$$
$$=\left[\left(\frac{3}{\Lambda}\right)^{1/2}+r\right]^{c(3/\Lambda)^{1/2}}\left[\left(\frac{3}{\Lambda}\right)^{1/2}-r\right]^{-c(3/\Lambda)^{1/2}} \quad (3.26)$$

and the line element (A36) involves

$$N\exp(-2cr^*)=\frac{\Lambda}{3}\left[\left(\frac{3}{\Lambda}\right)^{1/2}-r\right]^{1-c(3/\Lambda)^{1/2}}\times\left[\left(\frac{3}{\Lambda}\right)^{1/2}+r\right]^{1+c(3/\Lambda)^{1/2}} . \quad (3.27)$$

Choosing

$$c=(\Lambda/3)^{1/2} , \quad (3.28)$$

the metric is regular at $r=(3/\Lambda)^{1/2}$ and the (continued) time has a period

$$T=2\pi(3/\Lambda)^{1/2} . \quad (3.29)$$

Also for both SU(2),

$$\tfrac{1}{8}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})=\frac{\Lambda^2}{6} . \quad (3.30)$$

Now we notice that if we integrate as before from r_2 to r_1, we get

$$P=\pm\frac{T}{\pi}\frac{\Lambda^2}{6}\int_{r_2=0}^{r_1=(3/\Lambda)^{1/2}} r^2dr=\pm 1 . \quad (3.31)$$

This is the limit one gets from (3.20). Direct substitution in (3.15) and (3.19) shows that $T\to 0$ and the radial integral $\to\infty$ leading to (3.20). This is why we have treated this case separately. Moreover, there is now no essential singularity at $r=0$ even before continuation and η, ξ are real in the Euclidean section for

$$r_1=\left(\frac{3}{\Lambda}\right)^{1/2}>r>-\left(\frac{3}{\Lambda}\right)^{1/2}=r_- . \quad (3.32)$$

If the domain of integration is defined to be this whole region, one obtains

$$P=\pm\frac{T}{\pi}\frac{\Lambda^2}{6}\lim_{\delta\to 0}\int_{r_- -\delta}^{r_1} r^2dr=\pm 2 . \quad (3.33)$$

IV. REMARKS

The possible origin and consequences of a cosmological term have recently been studied by several authors in different contexts and from different points of view. One aspect is the study of quantum vacuum fluctuations (related to the Casimir effect) as the source of a nonzero cosmological term. (See, for example, Refs. 16 and 17 which contain other references.)

Speculations have been made concerning micro-de Sitter universes as possible models for particles.[18] It is to be noted that zero-point fluctuations are also evoked in the MIT bag model[19] in order to furnish a supplementary parameter. The formulation of supersymmetry in de Sitter space also has interesting properties (see Ref. 20 and references quoted therein).

In this paper we point out certain topological features one obtains (in a macrouniverse or a microuniverse) in the presence of a cosmological term and in particular for the de Sitter limit of the static case.

We have analyzed in Appendix B a class of singular solutions of the Dirac equation, pointing out how similar solutions arise also (in flat space) in the presence of monopoles and instantons. We hope to present elsewhere a more complete study of different types of solutions for spinors in the Kerr-Schild metric.

Since the first version of this paper was written several important references have been brought to our attention.

Charap and Duff[21] and Duff and Madore[22] study certain classes of solutions directly in the Euclidean section with appropriate changes in the signs of certain parameters.

Duff and Madore[22] analyze the singularity structure of their types of solutions and obtain quantization conditions necessary to remove the string singularities (electric and magnetic). These conditions are found to be just those which lead to integral values of the Pontryagin integral.

This aspect may be compared with one arising in our discussion of the Schwarzschild-de Sitter case. For this case we cannot desingularize simultaneously both the horizons (r_1 and r_2 in Sec. III). The corresponding Pontryagin integral (3.19) is not an integer. Only the limiting cases

(i) $9M^2\Lambda=0$ ($\Lambda=0$ or $M=0$) and

(ii) $9M^2\Lambda=1$ give integral values.

We have discussed case (i) in detail, and it may be verified that as $9M^2\Lambda\to 1$, $r_1\to r_2$ and $T\to\infty$ in such a way that

$$P \to \pm 2 .$$

These limits are thus seen to correspond to a quantization condition.

Spin structures in curved spaces and self-dual solutions to Euclidean gravity have been studied in several recent papers.[23,21,25]

ACKNOWLEDGMENTS

We thank Professor M. J. Duff for helpful remarks concerning the relation between singularities and nonintegral values of the Pontryagin integral.

APPENDIX A

Here we collect together some useful results concerning the Kerr-Schild metric. The essential references are quoted in Sec. II. The Kerr-Schild metric is given by

$$g_{\mu\nu}=\eta_{\mu\nu}-2Ml_\mu l_\nu , \tag{A1}$$

where $\eta_{\mu\nu}$ is the Lorentz metric $(+1,-1,-1,-1)$ and l_μ is a null four-vector with respect to $\eta_{\mu\nu}$ (and hence also with respect to $g_{\mu\nu}$), i.e.,

$$l_\mu l_\nu \eta^{\mu\nu}=l_\mu l_\nu g^{\mu\nu}=0 . \tag{A2}$$

M is a constant parameter. One obtains

$$g^{\mu\nu}=\eta^{\mu\nu}+2Ml^\mu l^\nu \tag{A3}$$

and

$$\sqrt{-g}=+1 . \tag{A4}$$

The connection coefficients are

$$\Gamma^\mu_{\alpha\beta}=(-M)[\partial_\alpha(l^\mu l_\beta)+\partial_\beta(l^\mu l_\alpha)-\partial^\mu(l_\alpha l_\beta)] \tag{A5}$$

and in particular

$$\Gamma^\mu_{\mu\alpha}=0 . \tag{A6}$$

Let us first note the relation of (A1) to some well-known static spherically symmetric metrics. Let

$$ds^2=Ndt^2-N^{-1}dr^2-r^2d\Omega , \tag{A7}$$

where $d\Omega=(d\theta^2+\sin^2\theta d\varphi^2)$ and

$$N=1-\frac{2M}{r}+\frac{Q^2+P^2}{r^2}-\frac{\Lambda r^2}{3} . \tag{A8}$$

(M is the mass of the central body with electric charge Q and magnetic monopole charge P. Λ is the cosmological constant.)

We now introduce Eddington-Finkelstein type of coordinates as follows. Let

$$r^*=\int\frac{dr}{N} \tag{A9}$$

and

$$\tilde{t}=t+r^*-r . \tag{A10}$$

Then

$$ds^2=(d\tilde{t}^2-r^2d\Omega-dr^2)+(N-1)(d\tilde{t}+dr)^2 \tag{A11}$$

and hence, in these coordinates,

$$g_{\mu\nu}=\eta_{\mu\nu}-2Ml_\mu l_\nu , \tag{A12}$$

where

$$l_0^2=\frac{1}{2M}(1-N) \tag{A13}$$

and

$$l_\mu=l_0\lambda_\mu\equiv l_0(1,\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta) . \tag{A14}$$

Let us next consider the stationary axially symmetric Kerr metric with angular momentum. We will consider mainly the uncharged case without cosmological constant, though solutions exist including both.[15,26] Now we do not assume spherical symmetry but starting with (A1) we solve the equation

$$R_{\mu\nu}=0 \tag{A15}$$

by expanding $R_{\mu\nu}$ in powers of M. One obtains

$$l^\mu\partial_\mu l_\nu=-Al_\nu \tag{A16}$$

and

$$\partial_j\lambda_i=\alpha(\delta_{ij}-\lambda_i\lambda_j)+\beta\epsilon_{ijk}\lambda_k \quad (i,j,k=1,2,3) . \tag{A17}$$

We also define the scalar

$$\partial_\mu l^\mu\equiv -B . \tag{A18}$$

The scalars A, B, α, β are determined as follows. The Einstein equations are satisfied if

$$\vec{\nabla}^2(\alpha+i\beta)\equiv\vec{\nabla}^2\gamma=0 \tag{A19}$$

and

$$(\vec{\nabla}\gamma^{-1})^2\equiv(\vec{\nabla}\omega)^2=1 \tag{A20}$$

and further

$$l_0^2=\alpha , \tag{A21}$$

$$(A+B)l_0=\alpha^2+\beta^2 , \tag{A22}$$

and

$$(A-B)=-2l_0\alpha . \tag{A23}$$

For the static case

$$A=\partial_r l_0 , \quad B=A+2l_0/r \tag{A23$'$}$$

where l_0 is given by (A13).

Introducing the (Boyer-Lindquist) coordinates through

$$x+iy=(\rho+ia)\sin\theta e^{i\varphi} , \tag{A24}$$

$$z=\rho\cos\theta\,, \tag{A25}$$

one obtains for the Kerr metric

$$\omega=\gamma^{-1}=\rho-ia\cos\theta\equiv\rho+i\sigma\,, \tag{A26}$$

$$l_0^2=\alpha=\frac{\rho}{\rho^2+\sigma^2}\,,\quad \beta=-\frac{\sigma}{\rho^2+\sigma^2}\,, \tag{A27}$$

and once again

$$\lambda_\mu=(1,\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)\,. \tag{A28}$$

The parameter a is interpreted through the relation that the angular momentum

$$L=Ma\,. \tag{A29}$$

Let us just briefly mention that for the charged case

$$l_0^2=\left(1-\frac{Q^2+P^2}{2M\rho}\right)\alpha\,. \tag{A30}$$

Let us now go back to the case of spherical symmetry and indicate briefly the passage to Kruskal-type coordinates.[21] Let

$$U=t-r^* \text{ and } V=t+r^*\,, \tag{A31}$$

where r^* is defined by (A9). Then (A7) becomes

$$ds^2=NdUdV-r^2d\Omega\,. \tag{A32}$$

Now defining

$$u=-e^{-cU}\,,\quad v=e^{cV}\,, \tag{A33}$$

one obtains

$$ds^2=c^{-2}N\exp(-2cr^*)dudv-r^2d\Omega\,. \tag{A34}$$

Again with the definitions

$$u=\tfrac{1}{2}(\xi-\eta)\,,\quad v=\tfrac{1}{2}(\xi+\eta)\,, \tag{A35}$$

we get

$$ds^2=\tfrac{1}{4}c^{-2}N\exp(-2cr^*)(d\xi^2-d\eta^2)-r^2d\Omega\,, \tag{A36}$$

where

$$\exp(2cr^*)=\tfrac{1}{4}(\eta^2-\xi^2) \tag{A37}$$

and

$$\exp(ct)=\left(\frac{\eta+\xi}{\eta-\xi}\right)^{1/2}. \tag{A38}$$

The roots of $N=0$ determine the spacetime regions. A proper choice of c may be used to remove a chosen apparent singularity. Examples are given in Sec. III.

APPENDIX B

In our representation the Dirac equation in the KS metric is [see the steps leading to (2.17)]

$$(\gamma^\mu\partial_\mu-im)\psi=-M[l^\mu\partial_\mu+\tfrac{1}{2}(A-B)]\tau\psi\,. \tag{B1}$$

For $m=0$ let us consider the class of solutions satisfying

$$\tau\psi=0\,, \tag{B2}$$

$$\gamma^\mu\partial_\mu\psi=0\,. \tag{B3}$$

This is the class studied in Ref. 1, though our representation simplifies the situation from the very beginning.

One of us has studied, elsewhere, a class of singular solutions for spinors coupled to non-Abelian magnetic monopoles[27] and instantons (unpublished). These solutions possess *stringlike singularities*. The same type of solutions reappear in the context of Kerr-Newman space.[1]

For comparison, let us briefly indicate the structure of the solutions. We use the following notations and definitions. We start with the case of *spherical symmetry*.

Let

$$\zeta=\tan\frac{\theta}{2}\,e^{i\varphi}\,,\quad \bar\zeta=\tan\frac{\theta}{2}\,e^{-i\varphi}\,, \tag{B4}$$

where (r,θ,φ) are the spherical coordinates. Let

$$\Phi_+=\begin{pmatrix}1\\ \zeta\end{pmatrix},\quad \Phi_-=\begin{pmatrix}-\bar\zeta\\ 1\end{pmatrix} \tag{B5}$$

be two 2-components spinors, singular on the axis

$$\theta=\pi\,.$$

(For compactness sometimes we use the notations $\zeta_+\equiv\zeta$, $\zeta_-\equiv\bar\zeta$.) Let

$$\sigma_r\equiv\vec\sigma\cdot\vec\nabla r\,,\quad \sigma_\theta\equiv\vec\sigma\cdot(r\vec\nabla\theta)\,,\quad\text{and } \sigma_\varphi\equiv r\sin\theta\vec\sigma\cdot\vec\nabla\varphi\,. \tag{B6}$$

Then (with $\epsilon=\pm$)

$$\sigma_r\Phi_\epsilon=\epsilon\Phi_\epsilon\,,\quad \sigma_\theta\Phi_\epsilon=e^{i\epsilon\varphi}\Phi_{-\epsilon}\,,\quad\text{and } \sigma_\varphi\Phi_\epsilon=i\epsilon e^{i\epsilon\varphi}\Phi_{-\epsilon}\,. \tag{B7}$$

Moreover, for *arbitrary* functions $f(\zeta)$ and $g(r,t)$

$$(\vec\sigma\cdot\vec\nabla)\left[(\sin\theta)^\lambda f(\zeta_\epsilon)\frac{g(r,t)}{r}\,\Phi_\epsilon\right]=(\sin\theta)^\lambda f(\zeta_\epsilon)\frac{1}{r}\left[\epsilon g'(r,t)\Phi_\epsilon+\lambda g(r,t)\left(\frac{\cot\theta}{r}\right)e^{i\epsilon\varphi}\Phi_{-\epsilon}\right]\quad (g'\equiv\partial_r g)\,. \tag{B8}$$

The final singularity structure, of course, depends on the choice of f and g.

Using such properties [with $\lambda=0$ in (B8) to start with] one can easily verify that one has (away from possible singularities) solutions for (B2) and (B3) of the form

$$\Psi=\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}, \tag{B9}$$

where

$$\chi_+=\frac{g_+(r+t)}{r}f_+(\zeta)\Phi_+\,,\qquad \chi_-=\frac{g_-(r+t)}{r}f_-(\bar\zeta)\Phi_-\,; \tag{B10}$$

the functions g_ϵ, f_ϵ are *arbitrary*.

For the axially symmetric Kerr-Newman case we use the Boyer-Lindquist coordinates and the notations of Appendix A [(A24)–(A30)]. The following results are useful (with $\vec\nabla$ still denoting derivatives with respect to x, y, z, while ρ, θ, φ are the BL coordinates):

$$(\vec\sigma\cdot\vec\nabla\rho)=\sigma_\rho+a\sin\theta(\alpha\sigma_\varphi-\beta\sigma_\theta)\,, \tag{B11}$$

$$(\vec\sigma\cdot\vec\nabla\theta)=\frac{1}{\rho}\,\sigma_\theta-\frac{\sigma}{\rho}\,(\alpha\sigma_\varphi-\beta\sigma_\theta)=\beta\sigma_\varphi+\alpha\sigma_\theta\,, \tag{B12}$$

$$(\vec\sigma\cdot\vec\nabla\varphi)=\frac{1}{\rho\sin\theta}\,\sigma_\varphi+\frac{\sigma}{\rho\sin\theta}\,(\beta\sigma_\varphi+\alpha\sigma_\theta)=\frac{1}{\sin\theta}\,(\alpha\sigma_\varphi-\beta\sigma_\theta)\,, \tag{B13}$$

$$(\vec\sigma\cdot\vec\nabla\zeta)=\frac{1}{\rho\sin\theta}\,[\sigma_\theta+i\omega(\alpha\sigma_\varphi-\beta\sigma_\theta)]\,, \tag{B14}$$

and hence

$$(\vec\sigma\cdot\vec\nabla\zeta)\Phi_+=0\,. \tag{B15}$$

Similarly

$$(\vec\sigma\cdot\vec\nabla\bar\zeta)\Phi_-=0\,. \tag{B16}$$

Using such results one may again verify that

$$\chi_+=\frac{g_+(\bar\omega+t)}{\bar\omega}f_+(\zeta)\Phi_+$$

and

$$\chi_-=\frac{g_-(\omega+t)}{\omega}f_-(\bar\zeta)\Phi_- \tag{B17}$$

are the necessary generalizations of (B10). Thus, for this class of solutions, the technique of a simple complex translation[7,1] works also for the spinors, namely

$$r\to\bar\omega=\rho+ia\cos\theta$$

and

$$r\to\omega=\rho-ia\cos\theta \tag{B18}$$

for χ_+ and χ_-, respectively.

Now our

$$\zeta=\left(\frac{1+\sigma/a}{1-\sigma/a}\right)^{1/2}e^{i\varphi}\,. \tag{B19}$$

Thus we see that taking, as particular cases,

$$g_+(\bar\omega+t)=\exp[ik_+(t+\bar\omega)]\,,\qquad g_-(\omega+t)=\exp[ik_-(t+\omega)]\,, \tag{B20}$$

$$f_+(\zeta)=(\zeta)^{m_+}\,,\qquad f_-(\bar\zeta)=-i(\bar\zeta)^{m_-}\,, \tag{B21}$$

we get respectively the solutions (9.12+) and (9.12–) of Ref. 1.

Let us now indicate very briefly this type of solutions for isodoublet massless spinors coupled to instantons in *flat Euclidean* spacetime (for the restricted case of spherical symmetry in the three-space). The Dirac equation is now

$$\gamma\cdot(i\partial-W)\psi=0\,, \tag{B22}$$

where the *Hermitian* W_μ are [compare with (2.27) and (2.28)]

$$W_0=d\Phi\,,\qquad \vec W=a\vec\nabla\Phi+(b-1)i[\Phi,\vec\nabla\Phi]+c\Phi\hat r\,, \tag{B23}$$

where

$$\Phi\equiv\tfrac12\sigma_r\,,$$

and with $f(r,t)=\ln\rho(r,t)$

$$a=rc=r\dot f,\quad (b-1)=-rd=-f'\quad(\dot f\equiv\partial_t f,\quad f'\equiv\partial_r f)\,. \tag{B24}$$

For

$$\rho=(\lambda^2+x^2)^{-1}=(\lambda^2+r^2+t^2)^{-1} \tag{B25}$$

one gets the Belavin-Polyakov-Schwartz-Tyupkin (BPST)[10] regular solution. The isodoublet solution can now be written as

$$\psi=\begin{pmatrix}e^{-i\varphi/2}\left(\cos\frac{\theta}{2}\,\psi_1-\sin\frac{\theta}{2}\,\psi_2\right)\\ e^{i\varphi/2}\left(\sin\frac{\theta}{2}\,\psi_1+\cos\frac{\theta}{2}\,\psi_2\right)\end{pmatrix}, \tag{B26}$$

where the Dirac spinors $\Psi_{1,2}$ are

$$\Psi_1=r^{-1}(\sin\theta)^{-1/2}f_1(\zeta)\times\begin{pmatrix}\rho^{1/2}g_1(r-it)\Phi_+\\ \rho^{-1/2}h(r+it)\Phi_+\end{pmatrix} \tag{B27}$$

and

$$\Psi_2 = r^{-1}(\sin\theta)^{-1/2} f_2(\bar{\zeta}) \begin{pmatrix} \rho^{1/2} g_2(r+it)\Phi_- \\ \rho^{-1/2} h_2(r-it)\Phi_- \end{pmatrix}; \tag{B28}$$

the functions $f_{1,2}, g_{1,2}, h_{1,2}$ are arbitrary. These functions may be chosen so as to make the energy density finite and integrable at the cost of introducing a branch cut so that the density is single valued only in a cut plane. These solutions can, moreover, be extended to Witten type configurations[28] and also to more general representations of SU(2) ($j=\frac{1}{2}, 1, \frac{3}{2}, \ldots$). We will not enter into these details here.[29]

The lesson of solution (B17) is that in a very particular case a passage to *axial* symmetry is possible through a complex translation.

For the well-known n-instanton solutions in the singular gauge,[10] already referred to after (2.13), the solution of a set of nonlinear equations is again reduced to one of a linear equation, namely

$$\Box\rho = 0 . \tag{B29}$$

Here again one can obtain new types of solutions through a complex translation. But the action is then found to be divergent.

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*Détaché du Laboratoire de Physique Théorique de la Faculté des Sciences de Rabat (Maroc).

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