Path integrals for a particle in curved space

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We consider a particle obeying the Schrödinger equation in a general curved n-dimensional space, with arbitrary linear coupling to the scalar curvature of the space. We give the Feynman path-integral expressions for the probability amplitude, $\langle x,s|x',0\rangle$, for the particle to go from x' to x in time s. This generalizes results of DeWitt, Cheng, and Hartle and Hawking. We show in particular, that there is a one-parameter family of covariant representations of the path integral corresponding to a given amplitude. These representations are different in that the covariant expressions for the incremental amplitudes, $\langle x_{l+1}, s_l+\epsilon|x_l,s_l\rangle$, appearing in the definition of the path integral, differ even to first order in ϵ (after dropping common factors). Finally, using the proper-time representation, we give the corresponding generally covariant expressions for the propagator of a scalar field with arbitrary linear coupling to the scalar curvature of the spacetime.

I. INTRODUCTION

The dynamical system under consideration obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial s}\langle x,s|\psi\rangle=\left[-\frac{\hbar^2}{2\mu}g^{\alpha\beta}(x)\nabla_\alpha\nabla_\beta+\frac{\hbar^2}{2\mu}\xi R\right]\langle x,s|\psi\rangle\,, \tag{1}$$

where $g_{\alpha\beta}(x)$ is the metric of the n-dimensional space, R is the scalar curvature, and ∇_α is the covariant derivative formed from that metric, with the wave function $\langle x,s|\psi\rangle$ regarded as a scalar at x (summation over α and β from 1 to n is understood). Here ξ is an arbitrary dimensionless coupling constant. This equation can be viewed as the Schrödinger equation of a particle of mass μ moving in an n-dimensional curved space (or more generally of a dynamical system having an n-dimensional configuration space, with the metric determined by the expression for the kinetic energy). Equation (1) also appears in connection with the proper-time representation of the propagator of a scalar field $\varphi(x)$ satisfying the field equation (in units with $\hbar=c=1$ and $\mu=\frac{1}{2}$)

$$(-\nabla^\alpha\nabla_\alpha+\xi R+m^2)\varphi(x)=0\,, \tag{2}$$

as will be discussed in the final section.

We show that the probability amplitude $\langle x,s|x',0\rangle$ satisfying Eq. (1) and the boundary condition

$$\lim_{s\to 0}\langle x,s|x',0\rangle=[g(x)]^{-1/2}\delta(x-x') \tag{3}$$

can be written in the path-integral form

$$\langle x,s|x',0\rangle=\int d[x(s')][\Delta^p]\exp\left(\frac{i}{\hbar}\int_0^s ds'\left\{\tfrac{1}{2}\mu g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}-\frac{\hbar^2}{2\mu}[\xi+\tfrac{1}{3}(p-1)]R(x)\right\}\right), \tag{4}$$

where p is a dimensionless parameter which can be chosen at will. The meaning of the notation $[\Delta^p]$ and the measure of the path integration will be defined in detail in Sec. II.

The case $p=0$, $\xi=\frac{1}{3}$, in which Eq. (4) reduces to

$$\langle x,s|x',0\rangle=\int d[x(s')]\exp\left(\frac{i}{\hbar}\int_0^s ds'\,\tfrac{1}{2}\mu g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}\right), \tag{5}$$

has been given before by DeWitt,[1] who was the first to notice that a scalar curvature term appears in Eq. (1), and by Cheng.[2] They used units with $\mu=1$. The same case, in units with $\mu=\frac{1}{2}$, was also discussed by Hartle and Hawking,[3] who made use of Riemann normal coordinates, a technique which we will use in Sec. II. The case $\xi=\frac{1}{6}$ was also considered in Ref. 1. The case with ξ general was briefly considered in Ref. 3, but the expression suggested there does not yeild a result for $\langle x,s|x',0\rangle$ with the correct transformation properties.

II. PROOF THAT SCHRÖDINGER EQUATION IS SATISFIED

The path integral in Eq. (4) can be defined[4] by breaking the time interval from 0 to s into $N+1$ equal increments of length ϵ, and writing

$$\langle x, s|x', 0\rangle = \lim_{N\to\infty} \left(\frac{\mu}{2\pi i\epsilon}\right)^{n(N+1)/2} \int \prod_{j=1}^{N} \{d^n x_j [g(x_j)]^{1/2}\} \times \exp\left\{\sum_{l=0}^{N} \left[i\int_{l\epsilon}^{(l+1)\epsilon} \left(\tfrac{1}{2}\mu g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'} - \frac{\lambda}{2\mu}R\right)ds' + p\ln\Delta(x_{l+1}, x_l)\right]\right\}, \tag{6}$$

where $d^n x_j = dx_j^1 dx_j^2 \cdots dx_j^n$, $g = \det(g_{\alpha\beta})$, $s = (N+1)\epsilon$, $x_0 = x'$, $x_{N+1} = x$, and

$$\lambda = \xi + \tfrac{1}{3}(p-1). \tag{7}$$

The integral from $s' = l\epsilon$ to $s' = (l+1)\epsilon$ is along the (shortest) geodesic path $x(s')$, from x_l to x_{l+1}. We are using units with $\hbar = c = 1$, and taking the spatial metric signature as $(+,+,\ldots)$. If the signature of $g_{\alpha\beta}$ were $(-,+,+,\ldots)$, then a further factor of $(1/i)^{N+1}$ would be present in Eq. (6), and $(g)^{1/2}$ would be replaced by $(-g)^{1/2}$. The quantity $\Delta(x_{l+1}, x_l)$ is defined as

$$\Delta(x_{l+1}, x_l) \equiv [g(x_{l+1})]^{-1/2} \det\left[-\frac{\partial^2\sigma(x_{l+1}, x_l)}{\partial x_{l+1}\partial x_l}\right][g(x_l)]^{-1/2}, \tag{8}$$

where

$$\sigma(x_{l+1}, x_l) = \tfrac{1}{2}\int_{l\epsilon}^{(l+1)\epsilon} ds' \left[g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}\right]^{1/2} \tag{9}$$

is $\frac{1}{2}$ of the proper arc length along the geodesic from x_l to x_{l+1}. The symbol $[\Delta^p]$ in Eq. (4) indicates that each term of the form $\exp[i\int_{l\epsilon}^{(l+1)\epsilon} ds'(\;)]$ in Eq. (6) is multiplied by a factor of $[\Delta(x_{l+1}, x_l)]^p$. The quantity $\Delta(x_{l+1}, x_l)$ transforms as a scalar at x_{l+1} and at x_l. In flat space $\Delta(x_{l+1}, x_l) = 1$ and $R = 0$, so that Eq. (6) reduces to the correct flat-space limit. In the classical limit, $\hbar \to 0$ [see Eq. (4) for the placement of $\hbar$], the path integral is clearly dominated by the geodesic from x' to x for which the classical action $\int_0^s \frac{1}{2}\mu g_{\alpha\beta}(dx^\alpha/ds')(dx^\beta/ds')ds'$ is stationary.

From Eq. (6) it follows that

$$\langle x, s+\epsilon|x', 0\rangle = \left(\frac{\mu}{2\pi i\epsilon}\right)^{n/2} \int d^n x_{N+1}[g(x_{N+1})]^{1/2}[\Delta(x, x_{N+1})]^p \times \exp\left[i\int_{(N+1)\epsilon}^{(N+2)\epsilon} ds' \left(\tfrac{1}{2}\mu g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'} - \frac{\lambda}{2\mu}R\right)\right]\langle x_{N+1}, s|x', 0\rangle, \tag{10}$$

where now $x_{N+2} = x$. To show that Eq. (1) is satisfied, introduce Riemann normal coordinates[5] with origin at x. [We assume that these coordinates are good out to x_{N+1}. As the main contribution to the integral in Eq. (10) comes from x_{N+1} close to x as $\epsilon \to 0$, this appears to be justified if the geodesics near x are continuous and do not intersect.] Let y^μ be the Riemann normal coordinate of the point x_{N+1}. The geodesic joining x to x_{N+1} is linear in these coordinates, and it can be shown that

$$\int_{(N+1)\epsilon}^{(N+2)\epsilon} ds' g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'} = \epsilon^{-1}\delta_{\alpha\beta}y^\alpha y^\beta, \tag{11}$$

where we have, without loss of generality, taken the metric tensor at $y=0$ to be $\delta_{\alpha\beta}$ in these coordinates. One also has

$$\sigma(x, x_{N+1}) = \tfrac{1}{2}\delta_{\alpha\beta}y^\alpha y^\beta, \tag{12}$$

so that Eq. (8) yields

$$\Delta(x, x_{N+1}) = [g(y)]^{-1/2}. \tag{13}$$

From Eq. (11) it is clear that the contributions to the integral in Eq. (10) come mainly from the region in which $\delta_{\alpha\beta}y^\alpha y^\beta \lesssim \mu\epsilon$ Thus, we expand the quantities appearing in Eq. (10) about $y=0$ (only the terms up to order y^2 will contribute to order ϵ) and extend the range of integration from $-\infty$ to ∞ (if it is not already over that range). One has[5]

$$[g(y)]^{(1-p)/2} = 1 - \tfrac{1}{6}(1-p)R_{\mu\lambda}y^\mu y^\lambda + O(y^3), \tag{14}$$

where $R_{\mu\lambda}$ is evaluated at $y=0$. Also

$$\langle x_{N+1}, s|x', 0\rangle = \langle x, s|x', 0\rangle + y^\lambda\partial_\lambda\langle x, s|x', 0\rangle + \tfrac{1}{2}y^\lambda y^\mu\partial_\lambda\partial_\mu\langle x, s|x', 0\rangle + \cdots, \tag{15}$$

where

$$\partial_\lambda\langle x, s|x', 0\rangle \equiv [\partial\langle x_{N+1}, s|x', 0\rangle/\partial y^\lambda]_{y=0}.$$

Furthermore, expanding $R(y)$ about $y=0$ and integrating along the geodesic gives

$$\int_{(N+1)\epsilon}^{(N+2)\epsilon} ds' R(y) = \epsilon[R + \tfrac{1}{2}y^\mu R_{;\mu} + O(y^2)], \tag{16}$$

where R and $R_{;\mu}$ are evaluated at $y=0$. Therefore

$$\exp\left[-\frac{i\lambda}{2\mu}\int_{(N+1)\epsilon}^{(N+2)\epsilon} ds' R(y)\right] = 1 - \frac{i\lambda}{2\mu}\epsilon[R + \tfrac{1}{2}y^\mu R_{;\mu} + O(y^2)]. \tag{17}$$

Substituting Eqs. (14), (15), and (17) into Eq. (10), using y as the variable of integration, and retaining only terms which will give contributions up to order ϵ, one finds

$$\langle x, s+\epsilon | x', 0\rangle = \left(\frac{\mu}{2\pi i\epsilon}\right)^{n/2} \int_{-\infty}^{\infty} d^4y \exp[i\tfrac{1}{2}\mu\epsilon^{-1}\delta_{\alpha\beta}y^\alpha y^\beta]$$
$$\times\{\langle x, s|x', 0\rangle - [\tfrac{1}{6}(1-p)R_{\mu\lambda}y^\mu y^\lambda + i\epsilon(\lambda/2\mu)R]\langle x, s|x', 0\rangle$$
$$+ \tfrac{1}{2}y^\mu y^\lambda \partial_\mu\partial_\lambda\langle x, s|x', 0\rangle + \cdots\}. \tag{18}$$

Terms odd in y have been dropped because their integrals vanish. The remaining Gaussian integrals are

$$\int_{-\infty}^{\infty} d^n y \exp(i\tfrac{1}{2}\mu\epsilon^{-1}\delta_{\alpha\beta}y^\alpha y^\beta) = \left(\frac{2\pi i\epsilon}{\mu}\right)^{n/2} \tag{19}$$

and

$$\int_{-\infty}^{\infty} d^n y y^\mu y^\lambda \exp(i\tfrac{1}{2}\mu\epsilon^{-1}\delta_{\alpha\beta}y^\alpha y^\beta) = \left(\frac{2\pi i\epsilon}{\mu}\right)^{n/2}\delta^{\mu\lambda}\left(\frac{i\epsilon}{\mu}\right). \tag{20}$$

Hence

$$\langle x, s+\epsilon | x', 0\rangle = \langle x, s|x', 0\rangle - \frac{i\epsilon\lambda R}{2\mu}\langle x, s|x', 0\rangle + \frac{i\epsilon}{\mu}\delta^{\mu\lambda}[\tfrac{1}{2}\partial_\mu\partial_\lambda\langle x, s|x', 0\rangle - \tfrac{1}{6}(1-p)R_{\mu\lambda}\langle x, s|x', 0\rangle] + O(\epsilon^2), \tag{21}$$

and in the limit $\epsilon\to 0$,

$$i\frac{\partial}{\partial s}\langle x, s|x', 0\rangle = \left\{-\frac{1}{2\mu}\delta^{\alpha\beta}\partial_\alpha\partial_\beta + \left[\frac{1}{6\mu}(1-p) + \frac{\lambda}{2\mu}\right]R\right\}\langle x, s|x', 0\rangle. \tag{22}$$

Using Eq. (7) for λ, and returning to general coordinates yields

$$i\frac{\partial}{\partial s}\langle x, s|x', 0\rangle = \left\{-\frac{1}{2\mu}\nabla^\alpha\nabla_\alpha + \frac{1}{2\mu}\xi R\right\}\langle x, s|x', 0\rangle, \tag{23}$$

which shows that the path-integral expression for $\langle x, s|x', 0\rangle$ satisfies the Schrödinger equation [Eq. (1)].

To show that the boundary condition of Eq. (3) is satisfied, write Eq. (6) in the form

$$\langle x, s|x', 0\rangle = \lim_{N\to\infty}\int\prod_{j=1}^{N}\{d^n x_j[g(x_j)]^{1/2}\}\langle x, s|x_N, N\epsilon\rangle\langle x_N, N\epsilon|x_{N-1}, (N-1)\epsilon\rangle\cdots\langle x_1, \epsilon|x', 0\rangle, \tag{24}$$

with

$$\langle x_{l+1}, (l+1)\epsilon|x_l, l\epsilon\rangle = \left(\frac{\mu}{2\pi i\epsilon}\right)^{n/2}[\Delta(x_{l+1}, x_l)]^p \exp\left[i\int_{l\epsilon}^{(l+1)\epsilon} ds'\left(\tfrac{1}{2}\mu g_{\alpha\beta}\frac{dx_\alpha}{ds'}\frac{dx_\beta}{ds'} - \frac{\lambda}{2\mu}R\right)\right], \tag{25}$$

where the integration is along the geodesic path $x(s')$ from x_l to x_{l+1}. Working in Riemann normal coordinates at x_{l+1} and using the previously given expansions about $y=0$, one finds for an arbitrary smooth function $f(x)$ that

$$\int d^n x_l[g(x_l)]^{1/2}\langle x_{l+1}, (l+1)\epsilon|x_l, l\epsilon\rangle f(x_l) = f(x_{l+1}) + O(\epsilon). \tag{26}$$

It then follows from Eq. (24) by repeated use of Eq. (26) that

$$\int d^n x'[g(x')]^{1/2}\langle x, s|x', 0\rangle f(x') = \lim_{N\to\infty}[f(x) + O(N\epsilon)] = f(x) + O(s),$$

since $s=(N+1)\epsilon$ is held fixed as $N\to\infty$. Taking the limit $s\to 0$ now yields Eq. (3).

Thus, the path integral of Eq. (4), with p having any given value, satisfies Eqs. (1) and (3). Therefore that path integral, which gives $\langle x,s|x',0\rangle$, is independent of the value of p. By choosing $p=0$ one can eliminate the $[\Delta^p]$ term in Eq. (4), or by choosing $p=1-3\xi$ one can eliminate the scalar curvature term in Eq. (4). When the path integral is written as in Eq. (24) each factor $\langle x_{l+1},(l+1)\epsilon|x_l,l\epsilon\rangle$ does depend on p. Each factor has the form of Eq. (25), and is thus a scalar at x_{l+1} and at x_l for all values of p. It is interesting that for different values of p, the corresponding expressions for

$$(2\pi i\epsilon/\mu)^{n/2}\exp\left(-i\int_0^\epsilon ds'\tfrac{1}{2}\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right)\langle x,\epsilon|x',0\rangle$$

differ even to first order in ϵ. For example, one has from Eq. (24), using Riemann normal coordinates at x, that

$$\left(\frac{2\pi i\epsilon}{\mu}\right)^{n/2}\exp(-i\tfrac{1}{2}\mu\epsilon^{-1}\delta_{\alpha\beta}y^\alpha y^\beta)\langle x,\epsilon|x',0\rangle$$
$$=1+\tfrac{1}{6}pR_{\mu\nu}y^\mu y^\nu-i\epsilon\frac{\lambda}{2\mu}R+\cdots, \quad (27)$$

with λ given by Eq. (7). It is only after integration over $d^ny[g(y)]^{1/2}$ in Eq. (10) that p drops out of $\langle x,s+\epsilon|x',0\rangle$ to first order in ϵ. We know of no other system in which expressions for $\langle x_{l+1},(l+1)\epsilon|x_l,l\epsilon\rangle$ in a path integral have been shown to exist which differ (after common factors are dropped) to order ϵ, but nevertheless yield the same result for $\langle x,s|x',0\rangle$. That the additional requirement of covariance under general transformations of the coordinates x_l can also be met by the $\langle x_{l+1},(l+1)\epsilon|x_l,\epsilon\rangle$ for all values of p is rather surprising.

III. PROPAGATOR FOR SCALAR FIELD

Consider the following curved spacetime generalization of the Minkowski space scalar field equation:

$$(-g^{\alpha\beta}\nabla_\alpha\nabla_\beta+\xi R+m^2)\phi(x)=0, \quad (28)$$

where m is the mass of the particles associated with the field, and the metric now has the signature $(-,+,+,\dots)$ in this n-dimensional spacetime. For $\xi=0$ one has minimal coupling, and for $\xi=\frac{1}{6}$ one has conformal coupling. The Green's function satisfying

$$(-\nabla^\alpha\nabla_\alpha+\xi R+m^2)G(x,x')=[-g(x)]^{-1/2}\delta(x-x') \quad (29)$$

can be written in the proper-time representation[6,7] as

$$G(x,x')=\int_0^\infty ids\, e^{-im^2s}\langle x,s|x',0\rangle, \quad (30)$$

where $\langle x,s|x',0\rangle$ satisfies Eq. (23) with $\mu=\frac{1}{2}$, and the boundary condition of Eq. (3) with $g^{-1/2}$ replaced by $(-g)^{-1/2}$. The parameter s is known as the "proper time" because it does not change under transformation of the spacetime coordinates x^μ. With the metric signature $(-,+,+,\dots)$, one can show that Eqs. (4) and (6) remain valid, except that $(g)^{\pm 1/2}$ is replaced by $(-g)^{\pm 1/2}$ in Eq. (6) and in the definition of Δ in Eq. (8), and a further normalization factor $(1/i)^{N+1}$ appears in Eq. (6), as well as a further factor of (-1) on the right-hand side of Eq. (8). Thus, one has the generally covariant expression

$$G(x,x')=\int_0^\infty ids\, e^{-im^2s}\int d[x(s')][\Delta^p]\exp\left[i\int_0^s ds'\left(\tfrac{1}{4}g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}-[\xi+\tfrac{1}{3}(p-1)]R(x)\right)\right]. \quad (31)$$

If we replace $\exp(-im^2s)$ by $\exp(-im^2s-s^{-1}\delta)$ where δ is a small positive quantity to be taken to zero at the end of the calculation, then Eq. (31) can be shown to yield the Feynman propagator in the flat-spacetime limit. One is always free to eliminate the scalar curvature term in Eq. (31) by choosing $p=1-3\xi$, or to eliminate the Δ term by choosing $p=0$. The propagator $G(x,x')$ is independent of the value of p. It is interesting that if one uses the simplest path-integral expression, corresponding to Eq. (5), for $\langle x,s|x',0\rangle$ in Eq. (30), then one obtains the propagator for Eq. (28) with $\xi=\frac{1}{3}$, which is neither the minimal nor the conformal coupling.[8]

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