Evolution of scalar perturbations near the Cauchy horizon of a charged black hole

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We describe the evolution of a scalar test field on the interior of a Reissner-Nordström black hole. For a wide variety of initial field configurations the energy density in the scalar field is shown to develop singularities in a neighborhood of the geometry's Cauchy horizon, suggesting that for a stellar collapse curvature singularities will develop prior to encountering the Cauchy horizon. The extension to the interior of stationary perturbations due to exterior sources is shown not to disrupt the Cauchy horizon.

I. INTRODUCTION

The Reissner-Nordström geometry[1] is the unique,[2] asymptotically flat, spherically symmetric solution to the Einstein-Maxwell equations that describes the spacetime outside of a spherical star with charge Q and mass M. This geometry may be analytically extended to an electrovacuum solution representing a black hole for $0<|Q|<M$ (Ref. 3). While similar to the Schwarzschild black hole in the exterior region (i.e., outside of the event horizon $r=r_+$), the charged-black-hole interior (i.e., inside of the event horizon) is dramatically different. The Carter-Penrose diagram in Fig. 1 illustrates two distinguishing features: the timelike character of the curvature singularity (cf. the spacelike curvature singularity of a Schwarzschild black hole) and the Cauchy horizon inside of the event horizon at $r=r_-$. (See Hawking and Ellis[3] for the definitions of the global properties and Graves and Brill[4] for details on the analytic structure, coordinate systems, etc., for Reissner-Nordström black holes.)

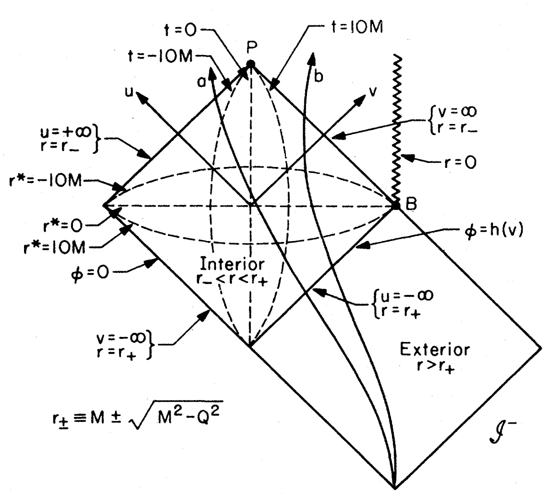

FIG. 1. Part of the conformal Carter-Penrose diagram for a Reissner-Nordström solution with $|Q|<M$. The level surfaces of the coordinates t and r^* are shown by the dashed lines. The coordinates u and v are null coordinates related to t and r^* by $u=-r^*-t$ and $v=-r^*+t$. The Cauchy horizon is the null hypersurface $r=r_-$ with $u=\infty$ and $v=\infty$ on the left and right sides, respectively. The event horizon for the left exterior region is the null hypersurface $r=r_+$ with $u=-\infty$. Paths a and b represent timelike world lines beginning in the exterior, crossing the $u=-\infty$ event horizon, and crossing the $u=\infty$ and $v=\infty$ parts of the Cauchy horizon, respectively.

The Cauchy horizon or (as we will call it) the r_- horizon has the peculiar global property of being the boundary in spacetime where the Einstein-Maxwell equations (or any other physical theory based on partial differential equations) lose their predictive power to describe the future evolution from prior data. However, the r_- horizon also has the property of being a null surface of infinite blue-shift. An observer crossing it will see an arbitrarily large blue-shift of any incoming radiation and the entire history of the exterior region in a finite lapse of her own proper time as she approaches the horizon. These properties suggested to Penrose[5] that the r_- horizon will be unstable, small perturbations will develop into curvature singularities just before it. The future development then stops at a curvature singularity rather than the Cauchy horizon.

The instability of the r_- horizon gives rise to the conjecture that in describing stellar collapse the development of a Cauchy horizon[6-8] is a special feature arising from the assumption of spherical symmetry. Previous studies[9,10] suggest that non-symmetric perturbations from symmetric stellar collapse develop into curvature singularities before the formation of a Cauchy horizon.

Penrose and Simpson[9] have investigated numerically the evolution of a test massless vector field on a charged-black-hole background for a variety of initial field configurations. They found a general divergence of the field energy density as the evolution approached $r=r_-$ and they concluded that this divergence was a generic feature for this background geometry. McNamara[10] has demonstrated the existence of initial data for perturbations by a test scalar field which are bounded by power laws on $\mathscr{I}^-$ and evolve to have unbounded energy densities on $r=r_-$. In this paper we also consider the evolution of test scalar fields, discuss their detailed evolution inside of the black hole, and consider the implications for charged stellar collapse. In a sequel[11] we will discuss the evolution of the coupled electromagnetic and gravitational perturbations on and the final-state problem for the interior black hole left by a charged stellar collapse.

Perturbation calculations can only show a solution to be stable. The unbounded growth of a perturbation *suggests* that (through the Einstein equations) a curvature singularity may develop, but higher-order nonlinear terms must be included to provide sufficient conditions for instability. (The exterior of a Reissner-Nordström black hole has been shown to be stable to linear perturbations by Bičák,[12] Sibgatullin and Alekseev,[13] Moncrief,[14] and Zerilli.[15]) As McNamara[10] has done, we will consider the unbounded growth of linear perturbations to be indications of possible instabilities and will use the term "instability" in this sense.

For the electrovacuum solution we will show that in general a massless scalar test field ϕ with arbitrary initial data on $\mathscr{I}^-$ or on the event horizon $r=r_+$ will develop an unbounded energy density as measured by an observer freely falling from rest in the exterior (a "freely falling observer," FFO) in a neighborhood of the r_- horizon. The evolution of the field ϕ proceeds in two steps: First the wave propagates in the exterior from $\mathscr{I}^-$ to the event horizon $r=r_+$. The detailed evolution of ϕ outside the event horizon may be summarized for our purposes by a main wave traveling from $\mathscr{I}^-$ to the horizon along a null ray and a sequence of waves scattered off of the background curvature from the main wave and subsequently scattered waves. The scattered fields superpose to fall off at late times as a pwer law t^{-N}. From the work of Price[16] and Bičák[12] we know that this power-law falloff is a generic feature of wave propagation in the exterior. The field along the event horizon can be characterized by a main wave (which evolved directly along a null ray from data on $\mathscr{I}^-$) and a power-law tail. Second the field on the event horizon evolves through to the interior region $r_-<r<r_+$ to a neighborhood of the Cauchy horizon $r=r_-$. We analyze in detail the evolution of ϕ in the interior region and show how the data on the event horizon evolves to waves running along and across the r_- horizon. The waves running along the r_- horizon are then blue-shifted by an arbitrarily large amount at the r_- horizon.

The formation of power-law tails by waves propagating in the exterior and the blue-shift of the tails on the interior suggest that almost any perturbation in the exterior spacetime will grow into an instability on the r_- horizon. We also find that perturbations that develop inside the black hole (for example a momentary switching on and off of some scalar charge on the surface of a collapsing star after it has crossed the event horizon) evolve subsequently to fields that are regular but with still divergent energy densities at the r_- horizon.

In accord with Israel's theorem,[2] investigations of spherically symmetric collapsing shells of charge[6,8] and dust[7] have Reissner-Nordström geometries as their exterior solutions. Therefore, we may use the previous considerations supplemented with appropriate boundary conditions at the surface of the star to discuss the evolution of a charged collapse. Figure 1 shows two possible world lines for the surfaces of collapsing stars. These lines are to be interpreted as the boundaries of the stars, with the Reissner-Nordström solutions attached smoothly on the right and the stellar interior geometries (not shown in the figure) attached on the left. The development of curvature singularities along the r_- horizon then follows as in the previous case. (We note that McNamara's[10] analysis of instability depended upon the two-sphere P, the singularity in the Killing vector field. The collapse described by the world line to the right in Fig. 1 does not contain P in that spacetime and so his proof is not applicable to that case. Our discussion will show, however, that this section of the r_- horizon in that geometry is also unstable.)

In Sec. II we write the wave equation we will use for the propagation of the field ϕ on the Reissner-Nordström background, define the condition for stability at the r_- horizon, define the appropriate boundary conditions, and set up the characteristic initial-value problem to evolve data in the interior.

In Sec. III we solve for the evolved behavior of the field ϕ at the r_- horizon and present the result of a numerical integration. In Sec. IV we discuss the perturbations from exterior stationary sources. In Sec. V we summarize our results and briefly consider the effects of quantum-mechanical processes on our conclusions.

II. THE WAVE EQUATION

The Reissner-Nordström black-hole interior for $r_- < r < r_+$ is described by the metric

$$ds^2 = \frac{r^2}{(r_+ - r)(r - r_-)}\,dr^2 - \frac{(r_+ - r)(r - r_-)}{r^2}\,dt^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \tag{2.1}$$

where $r_\pm \equiv M \pm (M^2 - Q^2)^{1/2}$, r is a temporal coordinate, and t is a spatial coordinate (see Fig. 1). It is convenient to define a "tortoise" coordinate r^* by the equation

$$r^* = -r - \frac{1}{\kappa_+}\ln(r_+ - r) + \frac{1}{\kappa_-}\ln(r - r_-)\,, \tag{2.2}$$

where $\kappa_\pm \equiv (r_+ - r_-)/r_\pm^2$ are the surface gravities at the two null surfaces. We will regard r as an implicit function of r^* with the asymptotic limits given by $r \sim r_\pm$ as $r^* \to \pm\infty$, and define the null coordinates u and v by the equations $u = -r^* - t$ and $v = -r^* + t$. These definitions for u and v are the natural extensions to the interior of Price's[16] exterior null coordinates. The event horizon is the null hypersurface $u = -\infty$ and the left and right r_- horizons are the null hypersurfaces $u = \infty$ and $v = \infty$, respectively.

The propagation of the scalar test field on this background is taken to be governed by the scalar wave equation

$$\phi_{;\alpha;\beta}\, g^{\alpha\beta} = 0\,. \tag{2.3}$$

To exploit the symmetries of the background we expand ϕ in spherical harmonics and Fourier transform in t to obtain

$$\phi(r^*, t, \theta, \varphi) = \sum_{l,m} \int_{-\infty}^{\infty} dk\, e^{-ikt}\, Y_{lm}(\theta, \varphi)\, \frac{1}{r}\, \psi_{lmk}(r^*)\,. \tag{2.4}$$

Substituting this expression for ϕ, the wave equation is reduced to an ordinary differential equation in r^* for the modes ψ_{lmk} and given by

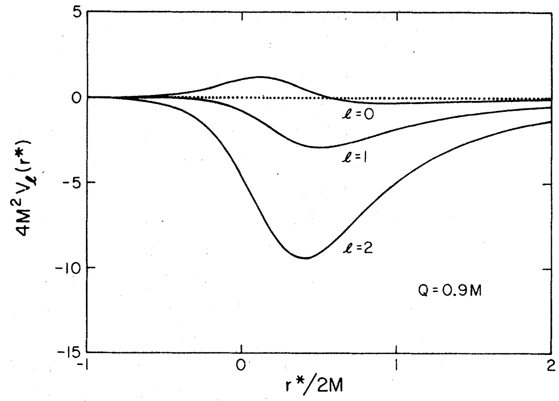

FIG. 2. The potential of the separated wave equation.

$$\frac{d^2\psi_{lmk}}{dr^{*2}} + [k^2 - V_l(r^*)]\psi_{lmk}(r^*) = 0\,, \tag{2.5}$$

where the scattering "potential" $V_l(r^*)$ is given by the equation

$$V_l(r^*) = \frac{-(r_+ - r)(r - r_-)}{r^2} \times \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4}\right]. \tag{2.6}$$

The potential $V_l(r^*)$ (shown in Fig. 2) is sharply localized in r^* and falls to zero exponentially with the asymptotic forms given by

$$V_l(r^*) \sim \exp(\mp\kappa_\pm r^*) \quad \text{as} \quad r^* \to \pm\infty\,. \tag{2.7}$$

The solutions to Eq. (2.5) as $r^* \to -\infty$ have the asymptotic forms given by

$$e^{-ikt}\psi_{lmk}(r^*) \sim \begin{bmatrix} e^{-ikv} \\ e^{iku} \end{bmatrix} [1 + O(e^{-\kappa_- r^*})]\,. \tag{2.8}$$

Near the r_- horizon these solutions may be described as left-going (e^{-ikv}) and right-going (e^{iku}) waves with exponentially vanishing corrections in r^*. Similar expressions hold for ψ near the r_+ horizon.

The energy density in the ϕ field (we are not considering the conformally invariant scalar field so the $\frac{1}{6}R$ term is absent) as measured by our FFO is a quadratic function of the quantity $\phi_{,\alpha}U^\alpha$ where U^α is the FFO's four-velocity. A FFO falling from rest in the exterior towards the left $u = \infty$ horizon has a four-velocity given by

$$\vec{U}_{(L)} = \left\{\frac{r^2}{-(r_+ - r)(r - r_-)}\left[1 - \left(1 + \frac{(r_+ - r)(r - r_-)}{r^2}\right)^{1/2}\right]\right\}\left(\frac{\partial}{\partial v}\right) - \left\{\frac{r^2}{-(r_+ - r)(r - r_-)}\left[1 + \left(1 + \frac{(r_+ - r)(r - r_-)}{r^2}\right)^{1/2}\right]\right\}\left(\frac{\partial}{\partial u}\right). \tag{2.9}$$

At $u=\infty$ the FFO will measure a $\phi_{,\alpha}U^\alpha$ given by

$$\phi_{,\alpha}U^\alpha_{(L)} \sim \frac{\partial\phi}{\partial u} e^{\kappa_-(v+u)/2} + (\text{const})\frac{\partial\phi}{\partial v} \quad \text{as} \quad u\to\infty. \tag{2.10}$$

Similarly a FFO falling to the right $v=\infty$ horizon will measure a $\phi_{,\alpha}U^\alpha$ given by

$$\phi_{,\alpha}U^\alpha_{(R)} \sim \frac{\partial\phi}{\partial v} e^{\kappa_-(v+u)/2} + (\text{const})\frac{\partial\phi}{\partial u} \quad \text{as} \quad v\to\infty. \tag{2.11}$$

Hence, for the FFO's to measure physically nonsingular fields near the r_- horizon, the appropriate derivatives of the field times the exponential blue-shift factor must be bound. [The condition Eq. (2.11) is identical with McNamara's[10] condition derived by an exponential boost to a nonsingular coordinate system at $v=\infty$.]

The higher-order terms in the solutions in Eq. (2.8) all fall off as fast or faster than $e^{\kappa_- r^*}$ as $r^*\to-\infty$ and the physical features described by these solutions are dominated by the leading terms $\psi\sim e^{-ikv}$ and $\psi\sim e^{iku}$. From Eqs. (2.11) and (2.10) we see that the e^{-ikv} waves are singular along the right ($v=\infty$) r_- horizon and the e^{iku} waves are singular along the left ($u=\infty$) r_- horizon.

The development of ϕ in the interior region from data on the r_+ horizon is most naturally stated as a characteristic initial-value problem. Since we are basically concerned with the history of a stellar collapse we take ϕ to be 0 on the left ($v=-\infty$) r_+ horizon and take ϕ to be some initial-value function $h(v)$ on the right ($u=-\infty$) r_+ horizon—the event horizon for the collapsing star. In the interior ($r_-<r<r_+$) r^* is a timelike coordinate (as r^* goes from ∞ to $-\infty$ time increases in a positive sense) and Eq. (2.5) describes the temporal evolution between the horizons. In this respect the calculation is more like a cosmological problem than a scattering problem (cf. scattering in the exterior). The final evolution of ϕ is given by its values on the $u=\infty$ and $v=\infty$ r_- horizons.

III. EVOLUTION OF THE SCALAR FIELD

To impose the initial conditions on $r^*=+\infty$ it is convenient to write for a particular l,m-spherical harmonic mode the expression

$$\phi_{lm}(r^*,t)=\int_{-\infty}^{\infty} dk\, e^{-ikt}H_{lm}(k)\frac{1}{r}\,\psi^{(-)}_{lmk}(r^*)\,, \tag{3.1}$$

where $\psi^{(-)}_{lmk}(r^*)$ is the solution to Eq. (2.5) with the asymptotic form at the r_+ horizons given by

$$e^{-ikt}\psi^{(-)}_{lmk}(r^*)\sim e^{-ikv} \quad \text{as} \quad r^*\to\infty. \tag{3.2}$$

[The absence of the conjugate function $\psi^{(+)}_{lmk}(r^*)$, which has the asymptotic behavior e^{iku} as $r^*\to\infty$, in Eq. (3.1) is due to the initial condition $\phi=0$ on the null surface $v=-\infty$.] At the r_- horizon $\psi^{(-)}_{lmk}(r^*)$ has the asymptotic form given by

$$e^{-ikt}\psi^{(-)}_{lmk}(r^*)\sim A_{lm}(k)e^{-ikv}+B_{lm}(k)e^{iku} \quad \text{as} \quad r^*\to-\infty\,, \tag{3.3}$$

where

$$|A_{lm}(k)|^2-|B_{lm}(k)|^2=1\,, \tag{3.4}$$

which follows from the Wronskian condition. In the sequel[11] we will show that $A_{lm}(k)$ and $B_{lm}(k)$ are analytic in k in a neighborhood of $k=0$, take the values at $k=0$ given by

$$A_{lm}(k=0)=\frac{(-1)^l}{2}\left(\frac{r_+}{r_-}+\frac{r_-}{r_+}\right), \qquad B_{lm}(k=0)=-\frac{(-1)^l}{2}\left(\frac{r_+}{r_-}-\frac{r_-}{r_+}\right), \tag{3.5}$$

and as $k\to\infty$, $[A_{lm}(k)-1]$ and $B_{lm}(k)$ decay exponentially. [A strict proof of the regularity of $A_{lm}(k)$ and $B_{lm}(k)$ at all intermediate points for real k has been given by McNamara.[17]]

The function $H_{lm}(k)$ is determined from the initial data $h_{lm}(v)$ on the right ($u=-\infty$) r_+ horizon [here $h_{lm}(v)$ are the multipole moments of the initial value data $h(v)$, i.e., $h(v)=\sum_{l,m}h_{lm}(v)Y_{lm}(\theta,\varphi)$]. Evaluating Eq. (3.1) as $u\to-\infty$, with the help of Eq. (3.2) and using the Fourier inversion theorem we may write

$$H_{lm}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}h_{lm}(v)e^{ikv}dv\,. \tag{3.6}$$

The general structure of the solution in a neighborhood of the r_- horizon has the form

$$\phi_{lm}(r^*,t)=\frac{1}{r_-}\left[h_{lm}(v)+\phi_{1lm}(v)+\phi_{2lm}(u)+O(r-r_-)\right]\,, \tag{3.7}$$

where

$$\phi_{1lm}(v)=\int_{-\infty}^{\infty}dk\, e^{-ikv}H_{lm}(k)[A_{lm}(k)-1]\,, \tag{3.8}$$

$$\phi_{2lm}(u)=\int_{-\infty}^{\infty}dk\, e^{iku}H_{lm}(k)B_{lm}(k)\,. \tag{3.9}$$

[Note that the integrals in Eqs. (3.8) and (3.9) are convergent for a wide class of data $H_{lm}(k)$ due to the exponential fall off of the $A_{lm}(k)-1$ and $B_{lm}(k)$ coefficients.] [A formal mathematical proof that the fourth term in Eq. (3.7) is $O(r-r_-)$ has also been given by McNamara.[17]]

If $H_{lm}(k)$ is regular on the real k axis, then since $A_{lm}(k)$ and $B_{lm}(k)$ have no irregular points in a neighborhood of the real k axis the contour of integration in Eq. (3.8) may be deformed into the lower half k plane in the case $v\to\infty$, and in Eq. (3.9) the contour may be deformed into the upper half plane in the case $u\to\infty$. Hence we obtain that $\phi_{1lm}(v)$ and $\phi_{2lm}(u)$ decay exponentially as v and $u\to\infty$, respectively.

We now specialize to an $h_{lm}(v)$ as a sum of a δ-function burst and a power-law tail after the burst, viz.,

$$h_{lm}(v)=\lambda\delta(v-v_0)+\mu\theta(v-v_0)v^{-\alpha},\quad \alpha>1 \quad (3.10)$$

where λ, μ, α, and v_0 are all constants characterizing the initial data. Substituting into Eq. (3.6) we find $H_{lm}(k)$ to be given by

$$H_{lm}(k)=\frac{1}{2\pi}e^{ikv_0}\left[\lambda+\mu\int_0^\infty \frac{e^{ikz}}{(z+v_0)^\alpha}dz\right]. \quad (3.11)$$

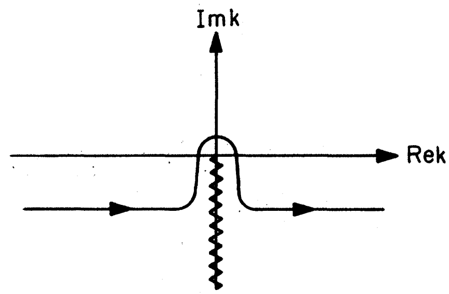

FIG. 3. Complex k-plane contour for integrating Eq. (3.8) for the contribution from the power-law tail of the initial-value data. There is a branch cut from $k=0$ to $k=-i\infty$ in the lower half plane along the imaginary k axis.

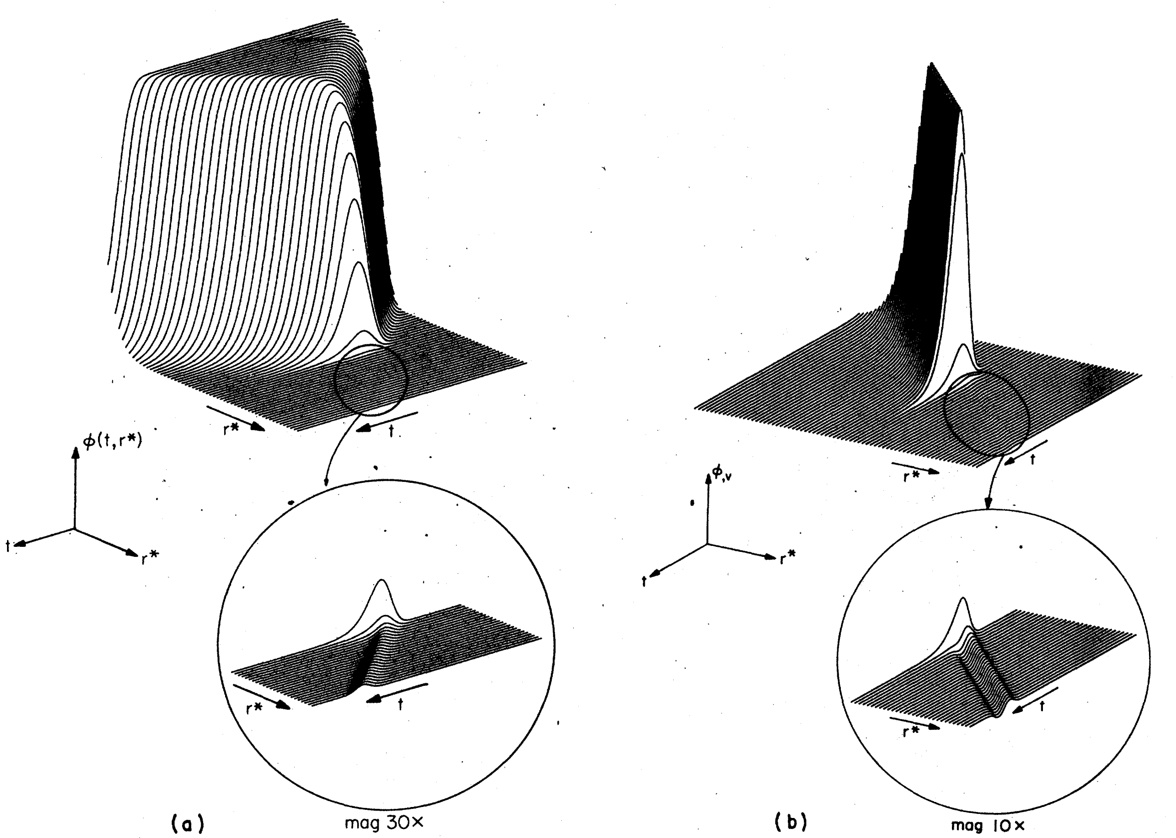

FIG. 4. These figures summarize the results of a numerical integration of the equation $\phi_{l,r^*r^*}-\phi_{l,tt}=V_l(r^*)\phi_l$. The results presented are for the particular case $Q=0.9M$ and $l=2$; however, additional integrations with alternative parameters support the qualitative features illustrated here. Figures 4(a) and 4(b) show ϕ_l and $\phi_{l,v}$ in relief as functions of r^* and t. The r^* coordinate ranges from $50M$ (near the event horizon) to $-60M$ (near the Cauchy horizon) and t ranges from $-50M$ to $50M$. The separation between lines of constant r^* is $2M$ for Fig. 4(a) and $1.6M$ for Fig. 4(b). The initial-wave form is taken to be a Gaussian of unit amplitude. These are shown in the inserts magnified 30 × for ϕ and 10× for $\phi_{,v}$. The asymptotic exponential falloff of the derivatives of ϕ near the Cauchy horizon are shown in Figs. 4(c) and 4(d). Figure 4(c) shows the exponential behavior of $\phi_{,u}$ at constant v in the neighborhood of the right-hand side of the r_- horizon. Figure 4(d) shows the analogous behavior of $\phi_{,v}$ at constant u near the left-hand side of the r_- horizon.

In this case $H_{lm}(k)$ is not analytic at the point $k=0$. The nonanalytic part of $H_{lm}(k)$ is proportional to $k^{\alpha-1}$ if α is nonintegral and $k^{\alpha-1}\ln k$ if α is an integer. In either case $H_{lm}(k)$ has a cut in the complex k plane and we shall place this cut along the imaginary k axis in the lower half plane in the case of Eq. (3.6) and in the upper half plane in the case of Eq. (3.9).

We now consider the contribution to $\phi_{1lm}(v)$ and $\phi_{2lm}(u)$ from the power-law tail [second term in Eq. (3.11)]. Due to the cut, the main contribution to $\phi_{1lm}(v)$ and $\phi_{2lm}(u)$ for large values of their arguments comes from the integration in the vicinity of the origin. The general form of the contour for Eq. (3.8) is shown in Fig. 3. Owing to the analyticity of $A_{lm}(k)$ and $B_{lm}(k)$ at $k=0$ we can substitute for them their values at $k=0$ [Eqs. (3.5)] in Eqs. (3.8) and (3.9) and we obtain

$$\tilde{\phi}_{1lm}(v)=\mu v^{-\alpha}[A_{lm}(0)-1] \quad \text{as} \quad v\to\infty\,, \tag{3.12}$$

$$\tilde{\phi}_{2lm}(u)=\mu u^{-\alpha}B_{lm}(0) \quad \text{as} \quad u\to\infty\,, \tag{3.13}$$

where the tilde means these are the contributions from the second term in Eq. (3.11). Therefore, in the case of a power-law tail ϕ is bounded at the r_- horizon and vanishes at the point P $(u\to\infty, v\to\infty)$, but the invariants in Eqs. (2.10) and (2.11) diverge according to the equation

$$\begin{bmatrix} v^{-(\alpha+1)} \\ u^{-(\alpha+1)} \end{bmatrix} \exp\left\{\frac{\kappa_-}{2}\begin{bmatrix} v \\ u \end{bmatrix}\right\} \tag{3.14}$$

on the respective horizons. Hence, the r_- horizon is unstable against scalar perturbations with power-law initial data at all points along the horizon except the point B $(u=-\infty, v=\infty)$. (The properties and behavior of perturbations at the point B will be discussed in the subsequent paper.[11])

It is interesting to note that the δ-function part of $h_{lm}(v)$ [i.e., the first term in Eq. (3.11)] also gives rise to singularities along the r_- horizon. In this case $H_{lm}(v)=(1/2\pi)e^{ikv_0}$ and hence it is analytic in the whole complex plane. Therefore the contour in Eqs. (3.8) and (3.9) can be deformed into the lower and upper half planes, respectively, until the contour intersects the nearest nonanalytic point of $A_{lm}(k)$ or $B_{lm}(k)$. Then we obtain the result that $\tilde{\tilde{\phi}}_{1lm}(v)$ and $\tilde{\tilde{\phi}}_{2lm}(u)$ [the corresponding contribution to $\phi_{1lm}(v)$ and $\phi_{2lm}(u)$ from the first term in Eq. (3.11)] decay exponentially for large values of their arguments. An estimate of the index γ for the exponential decay depends upon the

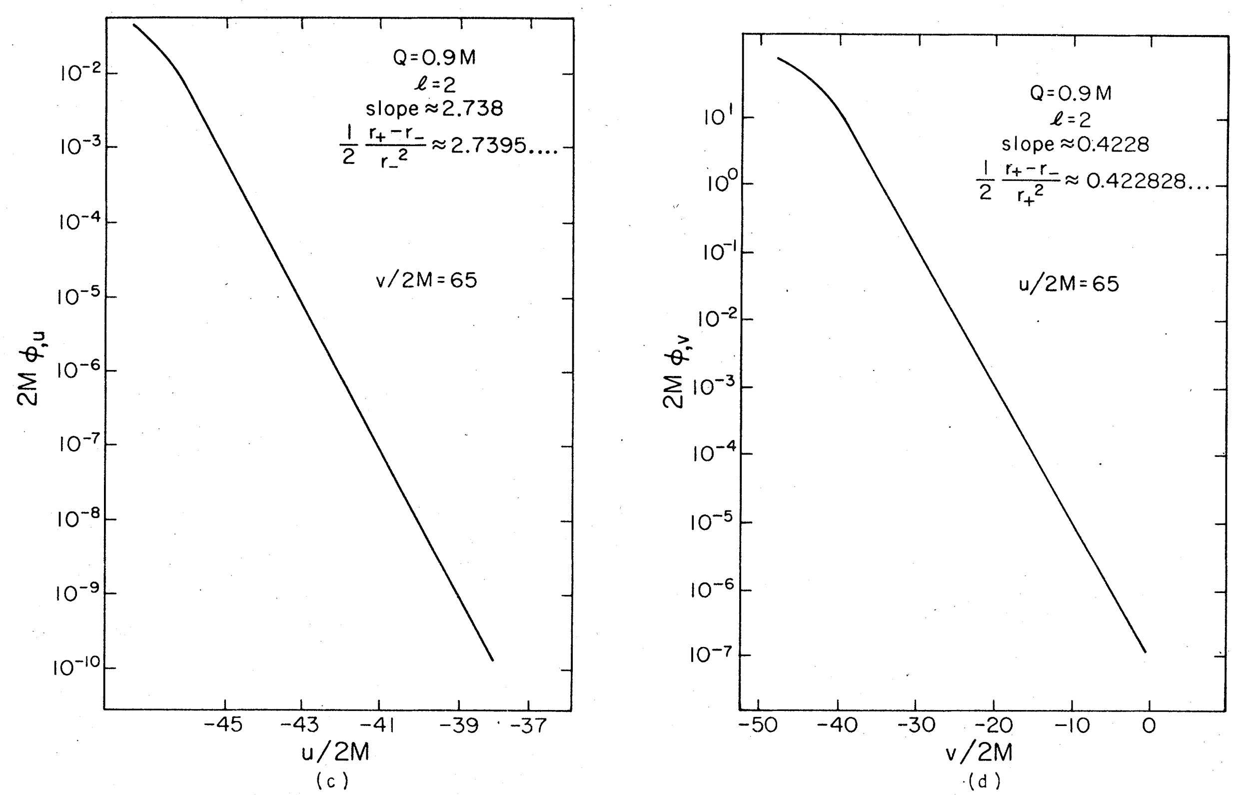

FIG. 4. (*Continued*)

imaginary value of the point where $A_{lm}(k)$ or $B_{lm}(k)$ becomes nonanalytic. Numerical computations of the evolution of a Gaussian wave packet are shown and described in Fig. 4. These computations indicate numerically that

$$\begin{aligned} \bar{\bar{\phi}}_{1lm}(v) &\sim e^{-\kappa_+ v/2} \quad \text{as} \quad v \to \infty , \\ \bar{\bar{\phi}}_{2lm}(u) &\sim e^{-\kappa_- u/2} \quad \text{as} \quad u \to \infty . \end{aligned} \tag{3.15}$$

We have not yet obtained an analytical proof of this result. Since $\kappa_+ < \kappa_-$ the invariants in Eqs. (2.11) and (2.10) diverge at the $v = \infty$ horizon.

IV. STATIONARY EXTERNAL SOURCES

In this section we consider the nonradiative fields ϕ which are connected with external sources outside of the r_+ horizon. We assume that these sources are at rest in the exterior with respect to the charged black hole. This means that in the exterior part of the black-hole geometry ($r > r_+$) the field ϕ is independent of the exterior time and is well behaved at the r_+ horizon (its behavior at $r = \infty$ is unimportant for this discussion). To extend this field inside ($r \leq r_+$) we must solve Eq. (2.3) for the case that ϕ is independent of t on $r = r_+$. This means finding the solutions to Eq. (2.5) for $k = 0$.

The $k = 0$ solutions to Eq. (2.5) can be written in closed form as

$$\phi_{1l}(r) = P_l\left(\frac{2r - r_+ - r_-}{r_+ - r_-}\right) , \tag{4.1}$$

$$\phi_{2l}(r) = Q_l\left(\frac{2r - r_+ - r_-}{r_+ - r_-}\right) , \tag{4.2}$$

where P_l is the Legendre polynomial and Q_l is the Legendre function of the second kind.[18] Of these two independent solutions only P_l is regular at r_+ and hence describes the extension inside the hole of fields due to external sources. With this solution we see that near the r_- horizon the ϕ field behaves as $P_l(-1)$ and the energy density as measured by one of our FFO is proportional to the expression

$$\phi_{,\alpha} U^\alpha \sim \frac{\partial P_l}{\partial r^*} e^{\kappa_- r^*} + (\text{const}) \sim \frac{\partial P_l}{\partial r} + (\text{const}) ,$$

which is *finite*.

V. CONCLUSION

This work has described the dynamical development of a test scalar field with the aim of demonstrating that for a wide class of physically reasonable initial conditions (really, for all the conditions we considered) the energy density in the field grows singular along the Reissner-Nordström geometry's Cauchy horizon. Such behavior suggests that for the real collapse of a charged star curvature singularities will develop in the interior of the forming black hole before the Cauchy horizon and timelike singularity of the Reissner-Nordström black hole are encountered by the developing spacetime geometry of the collapse. These conclusions for the development of perturbations that began in the exterior are in agreement with the results of Penrose and Simpson[9] and McNamara.[10] An interesting feature of the present calculation is that even a δ-function initial distribution on the outer horizon leads to the unbounded growth in the energy density of the scalar field at the inner horizon.

In this paper we have restricted our attention to classical fields. To examine the physical evolution of a more realistic stellar collapse we must take into consideration quantum mechanical effects: viz., pair creation by the gravitational field (all particles are created, including massless particles) and the creation of charged particle pairs by the electromagnetic field. The former process takes place for all values of M and Q, the latter process is possible if $|eQ| > 4GmM$. If $|eQ| \gg G^2 mM/\hbar$ then the creation of charged pairs by the electromagnetic field is much more rapid than the creation of pairs by the gravitational field. We expect these and associated processes to drive perturbations that also disrupt the Cauchy horizon, but their contribution to the total energy-momentum tensor will be proportional to Planck's constant and in general small as compared to the classical perturbations. Work on these problems is currently underway.

ACKNOWLEDGMENTS

When completing the work described herein, the authors received a report of a similar calculation by McNamara.[17] We have included his paper as a reference and tried to note where his work has overlapped with ours in this paper.

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