Comments on the atomic physics of charmonium

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Under very general conditions we show that the masses of ψ', ψ, η_c', and η_c and the partial width for the radiative decay $\psi \to \eta_c \gamma$ cannot all be fitted simultaneously if the η_c' and η_c are assumed to be $^1S_0$ states of charmonium. This is true for a wide class of monotonic confinement potentials (allowed to differ both in strength and radial dependence for singlet and triplet states) and even if all other bound states and radiative transitions are ignored.

I. INTRODUCTION

In this note we wish to concentrate on the atomic physics aspects of the interpretation of properties of the ψ/J-ψ' associated family of particles.[1] We shall be concerned with the spectrum of eigenvalues of the low-lying levels (assumed to be charmonium — bound states of $\bar{c}c$), as well as a particular aspect of the eigenfunctions — the overlap integrals of radial wave functions associated with the calculation of radiative widths for $M1$ transitions. Although the partial widths $\Gamma(\psi \to \bar{e}e)$ and $\Gamma(\psi' \to \bar{e}e)$ are also tests of the atomic physics aspects of charmonium, since these widths are measures of the wave function at the origin of coordinate space, we shall not discuss these quantities, in order to focus our attention on the seeming paradox of the $^1S_0$ states of charmonium, which has been noted by several authors.[2,3,4] Our aim is to sharpen specific aspects of this paradox; we do not resolve it. To accomplish this aim, it is sufficient to emphasize certain qualitative aspects of our predictions, without attempting to make any detailed fits to the data. The loss of such detail enables us to examine general and, to some extent, model-independent, features of the problem.

We will take the interaction to be a static one, and make the conventionally accepted S, L, and J assignments to the spectrum,[1] as indicated in Table I. The question marks in this table indicate the most controversial of the observations and spectral assignments — the objects of our present concern. We use an unconventional spectral labeling: N stands for the number of nodes of the radial wave function, and not for the principal quantum number, which we call n.[5] The spectral table is purposely incomplete. We have excluded levels lying above the $\bar{D}D$ threshold. Such a restriction makes it more likely that, for a qualitative description, we can neglect effects of hadronic decays and of coupling between different levels.

In order to aid the recognition of patterns in the level structure of charmonium we would like to introduce three ratios, $R_{\rm gs}$, $R_{\rm hfs/gs}$, and $R_{\rm fs/hfs}$, where

$$R_{\rm gs} = \frac{0\,^3\bar{P} - 0\,^3S_1}{1\,^3S_1 - 0\,^3S_1}, \tag{1.1}$$

$$R_{\rm hfs/gs} = \frac{N\,^3S_1 - N\,^1S_0}{0\,^3\bar{P} - 0\,^3S_1}, \quad N = 0,\ 1, \tag{1.2}$$

$$R_{\rm fs/hfs} = \frac{0\,^3P_2 - 0\,^3P_1}{N\,^3S_1 - N\,^1S_0}, \tag{1.3}$$

with $0\,^3\bar{P}$ being the center of gravity of the ^3P_J levels,

$$0\,^3\bar{P} = \tfrac{1}{3}(0\,^3P_2 + 0\,^3P_1 + 0\,^3P_0) \tag{1.4}$$

and $N^{(2S+1)}L_J$ in Eqs. (1.1)–(1.3) standing for the corresponding masses. The abbreviations gs, fs, and hfs are for "gross structure," fine structure, and hyperfine structure, respectively. In Table II, the experimental values of the various R_K's are compared to those for positronium and to selected theoretical fits[2,3,6-8] to the charmonium spectrum.

The positronium spectrum has been a guide in the calculation of the charmonium spectrum. Perhaps it has been too restrictive a guide — the very labelings fs and hfs imply that these terms represent higher-order corrections to the gross structure as in the case of positronium. The experimental values of R_K's in Table II indicate otherwise: The fine structure could possibly be a

TABLE I. The spectrum of charmonium.

Particle symbol	M (GeV)	$N^{(2S+1)}L_J^{PC}$
η_c	2.83	$0\,^1S_0^{-+}$?
ψ/J	3.098	$0\,^3S_1^{--}$
χ_0	3.414	$0\,^3P_0^{++}$
χ_1	3.508	$0\,^3P_1^{++}$
χ_2	3.552	$0\,^3P_2^{++}$
η_c'	3.45	$1\,^1S_0^{-+}$?
ψ'	3.684	$1\,^3S_1^{--}$

TABLE II. Comparison of experimental and various theoretical values of the spectral-interval ratios R_K. Unsuccessful fits are in parentheses.

K	N	R_K				
		Exp	Positronium	Refs. 2, 6, 7	Ref. 3	Ref. 8
gs		0.67	$(1-0.2\alpha^2)$	0.61	0.63	0.61
hfs/gs	0	0.68	$(3.1\alpha^2)$	(0.20)	(0.16)	0.84
	1	0.60	$(0.4\alpha^2)$	•••	•••	0.70
fs/hfs	0	0.16	(0.02)	(1.2)	(0.5)	(0.42)
	1	0.19	(0.17)	•••	•••	(0.50)

higher-order correction to the gross structure, but gross structure and hyperfine structure are of the same order. Incidentally, the prediction of the experimental value of R_{gs} constitutes a successful use of a confinement potential. Such a potential is at least sufficient to predict the roughly equal spacing of the ground S state, the lowest-lying P state, and the first radial excitation of the S state.[6,9]

A popular approach in predicting the charmonium spectrum has been to assume a particular covariant structure to the confinement potential (four-scalar, four-vector[2,3,6,7,10] with a possible additional anomalous gluon magnetic-moment correction,[8] in various combinations). The radial dependence is taken as linear,[2,3,6-8] logarithmic,[11] or quadratic[2,12] (simple harmonic oscillator) in r. The effect of single-vector-gluon exchange is often also included in the potential. The system is assumed to be nonrelativistic, and spin-orbit, spin-spin, and tensor forces are calculated to $O((v/c)^2)$. Good fits to the spectrum of ψ, ψ', and the various ^3P states, χ_0, χ_1, and χ_2, as well as $|\psi(0)|^2$ are frequently obtained. There are various degrees of success in fitting η_c and η_c' masses, but estimates of the $M1$ transition $\psi \to \gamma\eta_c$, using wave functions corresponding to the ψ and η_c eigenvalue fit, are too large.[4]

An alternative approach, focusing on the rough equality of the gross and hyperfine structure, is to assume that the problem is a highly relativistic one. Preliminary attempts[13] in this direction have not been successful. The hyperfine- and fine-structure results are closely related, as in the case of $(v/c)^2$ expansions: A large hfs interval results in unacceptably large fs intervals since the ratio $R_{\mathrm{fs/hfs}}$ is too large. In addition, a relativistic system requires small masses for the charmed quark, leading to low lepton pair branching ratios for ψ' and ψ decay.

Our point in the current note will be to limit ourselves to the S states (ψ', ψ, η_c', η_c) alone. We will assume the existence of a confining potential, different in strength and even in radial dependence for the triplet and singlet states. Even for such arbitrary parametrization, we find, subject to certain qualifications stated below, that it is impossible both to fit the spectrum of the four low-lying S states and to obtain wave-function overlaps which predict a small enough transition rate for the $M1$ transition.

II. STUDY OF LOW-LYING S STATES

We shall assume that the (ψ, ψ') and (η_c, η_c') represent the S-state solutions to the radial Schrödinger equations,[14]

$$\frac{d^2}{dr^2}u_N^{(S)}(r) + [k_N^{(S)}(r)]^2 u_N^{(S)}(r) = 0\,, \tag{2.1}$$

where

$$[k_N^{(S)}(r)]^2 = 2[E_N^{(S)} - V^{(S)}(r)]\,. \tag{2.2}$$

In the above $S=1$ refers to the ψ (triplet) case and $S=0$ to the η (singlet) case. Further, we have

$$u_N^{(S)}(0) = 0\,. \tag{2.3}$$

For our model of confinement we take

$$V^{(S)}(r) = \frac{1}{2R_{\gamma_S}^{\,2}}\left(\frac{r}{R_{\gamma_S}}\right)^{\gamma_S} + c_S\,, \tag{2.4}$$

with

$$\gamma_S \geq 0\,. \tag{2.5}$$

The parameters γ_S, R_{γ_S}, c_S give different potentials for the different spin states and represent a rather general class of monotonic confining potentials. For a suitable choice of c_S and R_{γ_S} the logarithmic potential[15] is given in the limit $\gamma_S \to 0$. The results for the square well can be obtained in the limit $\gamma_S \to \infty$.

The potentials (2.4) can be written as

$$V(r) = \frac{1}{4}\left\{\frac{1}{2R_{\gamma_0}^{\,2}}\left(\frac{r}{R_{\gamma_0}}\right)^{\gamma_0} + \frac{3}{2R_{\gamma_1}^{\,2}}\left(\frac{r}{R_{\gamma_1}}\right)^{\gamma_1} + c_0 + 3c_1 + \vec{\sigma}_1\cdot\vec{\sigma}_2\left[\frac{1}{2R_{\gamma_1}^{\,2}}\left(\frac{r}{R_{\gamma_1}}\right)^{\gamma_1} - \frac{1}{2R_{\gamma_0}^{\,2}}\left(\frac{r}{R_{\gamma_0}}\right)^{\gamma_0} + c_1 - c_0\right]\right\}, \tag{2.6}$$

where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli matrices for the quark and the antiquark, respectively.

For any given choice of γ_0 and γ_1, we can determine the R_{γ_S} from the experimental result that the triplet and singlet low-lying states are approximately equally spaced; i.e., we require

$$\Delta E^{(0)} = \Delta E^{(1)}, \tag{2.7}$$

where

$$\Delta E^{(S)} = E_1^{(S)} - E_0^{(S)}. \tag{2.8}$$

Having found the parameters R_{γ_S}, we can then determine the ground-state and first-excited-state wave functions $u_N^{(S)}(r)$, and compute the overlap integrals

$$\mathcal{O}_{00} \equiv \left(\int_0^\infty u_0^{(1)*} u_0^{(0)} dr \right)^2 \tag{2.9}$$

and

$$\mathcal{O}_{10} \equiv \left(\int_0^\infty u_1^{(1)*} u_0^{(0)} dr \right)^2. \tag{2.10}$$

If there were no spin dependence in (2.6) we would have

$$\mathcal{O}_{00} = 1, \quad \mathcal{O}_{10} = 0. \tag{2.11}$$

However, the experimental result for the $M1$ transition of ψ to η_c gives an upper limit to $\mathcal{O}_{00}$, indicating it is very small,[16]

$$\mathcal{O}_{00} < \tfrac{1}{15}. \tag{2.12}$$

The solutions to Eq. (2.1) are known exactly for $\gamma_S = 1, 2, \infty$. We can find very good approximations to the solutions for these values of γ_S and for all others by making use of the WKB approximation. The WKB quantization condition[17] for these potentials is[18]

$$\int_0^{L_N^{(S)}} k_N^{(S)}(r) dr = (N + \tfrac{3}{4})\pi, \quad N = 0, 1, 2, \ldots, \tag{2.13}$$

where $L_N^{(S)}$ is the position of the turning point

$$E_N^{(S)} = V^{(S)}(L_N^{(S)}) \tag{2.14}$$

and N is the number of nodes of the wave function. The integral in (2.13) can be evaluated (dropping the subscripts and superscripts) to give

$$E_N = \frac{1}{2R_\gamma^2} [b(\gamma)(N + \tfrac{3}{4})]^{(2\gamma/\gamma+2)} + c, \tag{2.15}$$

where

$$b(\gamma) = \pi \Gamma(1/\gamma + \tfrac{3}{2}) / \Gamma(1/\gamma + 1)\Gamma(\tfrac{3}{2}), \tag{2.16}$$

and $\Gamma(z)$ is the Γ function.

This approximation is, of course, excellent for large quantum numbers N. In addition, Eq. (2.15) agrees with the exact eigenvalues for $\gamma = 2$ [and the modified[18] Eq. (2.15) with the exact solution for the square well]. For $\gamma = 1$, the low-lying $N = 0$ (1) values of Eq. (2.15) are correct to 0.7 (0.1) percent.[19]

TABLE III. The turning points L_0, L_1 for the ground state and first excited state for different potentials, all having the same energy gap.

γ	L_0/R_2	L_1/R_2
0	1.22	2.86
$\frac{1}{10}$	1.26	2.83
$\frac{1}{4}$	1.31	2.79
$\frac{1}{3}$	1.35	2.79
$\frac{1}{2}$	1.40	2.76
1	1.53	2.71
2	1.73	2.65
∞	2.72	2.72

For each choice of γ we use (2.15) to calculate ΔE. Assuming ΔE is the same for any γ, we can find R_γ in terms of some standard R_γ (say R_2). Using (2.14) we determine L_N for each choice of γ. The results are presented in Table III.

The remarkable feature of Table III is the fact that, **once the energy** gap in the singlet and triplet systems is made equal, the positions of the turning points L_0 and L_1 are very insensitive to the power γ in the potential. Since the turning point gives the range for the classical path, we can expect the overlap integral ($\mathcal{O}_{00}$) between the two ground states $u_0^{(1)}$ and $u_0^{(0)}$ to be almost 1. To see this in more detail we evaluate the wave functions using the WKB approximation. They are

$$u_N(r) = \begin{cases} \dfrac{c}{(k_N(r))^{1/2}} \sin \displaystyle\int_0^r k_N(r) dr, & r < L_N \\ \dfrac{c}{(|k_N(r)|)^{1/2}} \exp\left(- \displaystyle\int_{L_N}^r |k_N(r)| dr \right), & r > L_N. \end{cases} \tag{2.17}$$

The WKB approximation is good for those values of r for which

$$\left| \frac{d|k_N|^{-1}}{dr} \right| \ll 1. \tag{2.18}$$

There is a small region of r about the turning point L_N for which (2.18) does not hold. (The point at $r = 0$ does not give rise to serious problems.) For these values of r near L_N we use a linear extrapolation between the two functions in (2.17). The integrals in (2.17) can be evaluated in closed form if $1/\gamma$ is an integer. For those

cases ($\gamma=1$, 2, ∞) for which there exist exact solutions, the WKB approximations for the wave functions never differ from the exact wave functions by more than 10% for any value of r. We have evaluated the overlap integrals associated with different potentials in the singlet and triplet systems. The values of γ we have used are 0, $\frac{1}{3}$, $\frac{1}{2}$, 1, 2, and ∞. For all combinations (excluding the square well, $\gamma=\infty$) the overlap function $\mathcal{O}_{00}$ is never smaller than 0.85. Even for the extreme case of the square well ($\gamma=\infty$) and a logarithmic potential ($\gamma=0$) for the singlet and triplet potentials, the overlap, $\mathcal{O}_{00}$ is 0.76. For any of a large number of cases we have studied it is more difficult to get a believable value for the overlap between a first excited state and the ground state, $\mathcal{O}_{10}$, since here the overlap should be near zero and we are very sensitive to any approximation.

III. CONCLUSION

We have seen that, for a whole range of confining potentials, the overlap between the two ground states ψ and η_c is very close to 1, if we impose the experimental result of approximately equal spacing between (ψ, ψ') and (η_c, η_c'). This is in apparent contradiction to the experimental results for the $M1$ transition $\psi \to \eta_c + \gamma$.

We have made no attempt to use the partial widths of ψ, ψ' into e^+e^- pairs to determine the quantities $u_N^{(1)}(0)$. The wave function at the origin will be strongly dependent upon a possible $1/r$ potential. Such an additional potential will have small influence on the determination of energy gaps and overlaps of wave functions.

As was pointed out in Sec. II, the ground states for $S=0$ and 1 have almost perfect overlap because their classically allowed regions $0 \leq r \leq L_0^{(S)}$ have almost perfect overlap. It is easy to see how one may alter the potentials in such a way that the energy gap remains the same, while the overlap between ground states becomes vanishingly small. One need only have a potential[20] (for the singlet case, say) for which the classically allowed region is $x_0 \leq r \leq x_0 + L_0^{(0)}$ with $x_0 \geq L_0^{(1)}$. This implies a singlet potential with a large barrier at small distances. Although it is difficult to see how quantum chromodynamics could predict such spin-dependent interactions, nevertheless, it may be worth pursuing the effects of such potentials. This will be the subject of a future publication.[23]

If we accept the above results concerning the impossibility of obtaining a small overlap between the ψ and η_c states, we must abandon the singlet S-state assignments of one or both of η_c, η_c'. If η_c' is not the first excited singlet state, we may ask, "what is it?"[21,22] and then, "where is the true η_c'?" If the particle η_c is not really as reported, we may take η_c' as the singlet ground state. The first excited state could then be anywhere and we can make no predictions based on any overlaps or energy differences.

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[1]M. Chanowitz and F. J. Gilman, Phys. Lett. 63B, 178 (1976); Particle Data Group, Phys. Lett. 68B, 1 (1977); G. J. Feldman, in *Proceedings of the 1976 Summer Institute on Particle Physics*, edited by Martha C. Zipf (SLAC, Stanford, California, *1976*); B. H. Wiik, in *Proceedings of the XVIII International Conference on High Energy Physics, Tbilisi, 1976*, edited by N. N. Bogolubov *et al.* (JINR, Dubna, U.S.S.R., 1977); B. H. Wiik and G. Wolf, DESY Report No. DESY 77/01, 1977 (unpublished); W. Braunschweig *et al.* Phys. Lett. 67B, 243 (1977); J. S. Whitaker *et al.* Phys. Rev. Lett. 37, 1596 (1976); W. D. Apel *et al.* Phys. Lett. 72B, 500 (1978).

[2]Howard J. Schnitzer, Phys. Rev. D 13, 74 (1975).

[3]A. B. Henriques, B. H. Kellett, and R. G. Moorhouse, Phys. Lett. 64B, 85 (1976).

[4]H. Fritzsch and J. D. Jackson, Phys. Lett. 66B, 365 (1977); K. Gottfried, in *Proceedings of the 1977 International Symposium on Lepton and Photon Interactions at High Energies, Hamburg*, edited by F. Gutbrod (DESY, Hamburg, 1978).

[5]For the Coulomb potential, $n=N+L+1$, and for the isotropic simple harmonic oscillator in three dimensions, $n=2N+L$.

[6]E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K. D. Lane, and T.-M. Yan, Phys. Rev. Lett. 34, 369 (1975).

[7]Howard J. Schnitzer, Phys. Rev. Lett. 35, 1540 (1975).

[8]Howard J. Schnitzer, Phys. Lett. 65B, 239 (1976).

[9]A. Martin, Phys. Lett. 67B, 330 (1977).

[10]A. Billoire and A. Morel, CEN-Saclay, Report No. DPh-T/77/98, 1977 (unpublished).

[11]C. Quigg and J. L. Rosner, Phys. Lett. 71B, 153 (1977).

[12]J. D. Jackson, in *Proceedings of the 1976 Summer Institute on Particle Physics* (Ref. 1).

[13]T. Fulton, report, 1977 (unpublished).

[14]We have set $\hbar=m=1$, where m is the effective mass of the quark.

[15]For $\gamma_S \to 0$, the potential is slightly modified to

$$V^{(S)}(r)=\frac{1}{2\gamma_S R_S^{\gamma_S}}\left[\left(\frac{r}{R_{\gamma_S}}\right)^{\gamma_S}-1\right]+c_S \xrightarrow[\gamma_S\to 0]{\lim} \frac{1}{2R_S^2}\ln\frac{r}{R_{\gamma_S}}+c_S.$$

[16]It is difficult to make a prediction for the branching ratio of the forbidden $M1$ transition $\psi' \to \eta_c\gamma$, since contributions from pair terms, which are strongly charmonium-potential-dependent, must also be taken into account. See G. Feinberg and J. Sucher, Phys. Rev.

Lett. 35, 1740 (1975). In any case, we must expect the $\mathcal{O}_{10}$ to be very small, smaller than $\mathcal{O}_{00}$.

[17]A recent detailed discussion of the WKB approximation is given in A. B. Migdal and V. P. Krainov, *Approximation Methods of Quantum Mechanics* (Neo Press, Ann Arbor, 1968), Chap. III.

[18]For the square-well potential ($\gamma_s \to \infty$), the quantization rule is altered: $\frac{3}{4} \to 1$ on the right-hand side of Eq. (2.13). This is so because the potential can no longer be considered slowly varying at the turning point.

[19]$E_{\text{exact}} > E_{\text{WKB}}$.

[20]We thank L. Madansky for making this point.

[21]H. Harari, Phys. Lett. 64B, 469 (1976).

[22]W. Celmaster, Harvard Univ. report, 1976 (unpublished).

[23]G. Feldman and T. Fulton, Phys. Lett. 77B, 89 (1978).

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Erratum

Erratum: Flavor and baryon quantum numbers and their nondiffractive renormalizations of the Pomeron [Phys. Rev. D 18, 303 (1978)]

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