Dual transformation in non-Abelian gauge theories

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The dual transformation is applied to non-Abelian gauge theories in four dimensions. It is shown that after the dual transformation the SU(2) Higgs-Kibble models give the same partition functions as the relativistic hydrodynamics of Freedman coupled to Higgs scalars and that these two theories have various dual relations. The hydrodynamics thus obtained is naturally defined as an expansion in the inverse of the original coupling constant. There exists a simple classical solution representing a circulation flow of fluid (vorticity) in the hydrodynamics with two triplets of Higgs scalars. This vorticity solution satisfies a flux quantization rule and corresponds to the vortex solution of Nielsen and Olesen in the non-Abelian Higgs-Kibble model.

I. INTRODUCTION

The purpose of the present paper is to study a dual transformation in non-Abelian gauge theories, especially in the Higgs-Kibble models, with the help of the technique developed in a previous paper by one of us on the Abelian gauge theories.[1] The dual transformation originally discussed in the study of critical phenomena in solid-state physics[2,3] connects two different models with different coupling constants (or temperatures in solid-state physics) related inversely to each other. Therefore, the transformation is very useful to determine a critical coupling constant, or to investigate a model in the strong-coupling region by using the result of the dually transformed model in the weak-coupling region. There are several works in which this dual transformation is discussed in the lattice version of the U(1)-invariant[4,5] or $Z(N)$-invariant[6,7] gauge theories. These works intend to embody the duality of Mandelstam and 't Hooft[8] in the lattice gauge theories.

In the previous study of conventional field theory[1] it has been noted that the dual transformation is nothing but a *Fourier transformation* performed in the integrand of a partition function defined by the path-integral method. It has also been shown that the dually transformed Abelian Higgs model is identical to the relativistic hydrodynamics of Kalb and Ramond[9] and of Nambu,[10] coupled to a Higgs scalar. The correspondence between the classical solutions and that between the Green's functions in the two models (Higgs vs Kalb, Ramond, and Nambu) have been clarified there.

In this paper, we shall apply the technique developed in the previous paper[1] to *non-Abelian gauge theories*, especially to the Higgs-Kibble models with an internal-symmetry group of SU(2). We shall discuss the models with one doublet or two triplets of Higgs scalars. It will be shown that the dually transformed non-Abelian Higgs-Kibble models are identical to the non-Abelian hydrodynamical models of Freedman[11] coupled complicatedly to Higgs scalars. Freedman has found his model in the course of generalizing the model of Kalb and Ramond[9] and of Nambu[10] to a non-Abelian case. It will be noted that these models represent relativistic hydrodynamics with (Freedman) or without (Kalb-Ramond-Nambu) internal symmetry. It will also be noted that the gauge principle of Kalb and Ramond (of Freedman), written in terms of an antisymmetric tensor field $W^a_{\mu\nu}$ (velocity potential), has to do with the dually transformed Abelian (non-Abelian) Higgs-Kibble models and that this gauge principle looks very different from the original gauge principle of Yang and Mills,[12] written in terms of a vector potential A^a_μ. The Lagrangian of our hydrodynamics, obtained from the original non-Abelain Higgs-Kibble model with the help of the dual transformation, will be defined as an expansion in the inverse of the original guage coupling constant, $1/e$, so that this hydrodynamical model may play an important role in the future study of the non-Abelian gauge theories in the strong-coupling region. We shall also derive various dual relations between the Green's functions of our hydrodynamical models and those of the original non-Abelian Higgs-Kibble models. In order to derive these we shall introduce an external tensor source and perform the dual transformation in the presence of this source. The hydrodynamical model with two Higgs scalars in the triplet representation has a classical soltuion representing a circulation flow around the x^3 axis possessing the third isospin index, whose rapid stream is localized around the x^3 axis. This solution corresponds to the vortex solution of Nielsen and Olesen[13] in the non-Abelian

Higgs-Kibble models. It is, therefore, natural to have a flux quantization rule also in our hydrodynamics. The quantized flux in our hydrodynamics is "electric" [the (0, 3) component of the velocity potential] since the electric components of the velocity potential correspond to the magnetic components of the Higgs-Kibble models.

In the next section, we shall show that the dually transformed non-Abelian Higgs-Kibble models are identical to the relativistic hydrodynamics of Freedman, a non-Abelian version of the theory of Kalb and Ramond and of Nambu coupled complicatedly to the Higgs scalars. In Sec. III, we shall study several dual relations between the Green's functions of the non-Abelian Higgs-Kibble models and those of the hydrodynamics. A classical solution, representing a circulation flow around the x^3 axis and corresponding to the vortex solution of Nielsen and Olesen, will be obtained and discussed there.

II. DUAL TRANSFORMATION IN NON-ABELIAN GAUGE THEORIES

The Lagrangian densities for the vector fields A_μ^a and the scalars ϕ^a (and ψ^a) discussed in this paper are the following:

$$\mathcal{L}_{\rm I} = -\tfrac{1}{4}(F_{\mu\nu}^a)^2 + |(\partial_\mu - ie\tfrac{1}{2}\tau^a A_\mu^a)\phi|^2 - V_{\rm I}(\phi^\dagger\phi) \tag{2.1}$$

and

$$\mathcal{L}_{\rm II} = -\tfrac{1}{4}(F_{\mu\nu}^a)^2 + \tfrac{1}{2}(\partial_\mu\phi^a + e\epsilon^{abc}A_\mu^b\phi^c)^2 + \tfrac{1}{2}(\partial_\mu\psi^a + e\epsilon^{abc}A_\mu^b\psi^c)^2 - V_{\rm II}(\phi,\psi), \tag{2.2}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\epsilon^{abc}A_\mu^b A_\nu^c, \tag{2.3}$$

$$V_{\rm I}(\phi^\dagger\phi) = -c_2\phi^\dagger\phi + c_4(\phi^\dagger\phi)^2, \tag{2.4}$$

and

$$V_{\rm II}(\phi,\psi) = -c_2(\phi\cdot\phi) + c_4(\phi\cdot\phi)^2 - d_2(\psi\cdot\psi) + d_4(\psi\cdot\psi)^2 - e_2(\phi\cdot\psi) + e_4(\phi\cdot\psi)^2. \tag{2.5}$$

In the above equations $(\phi\circ\psi) = \sum_{a=1}^3 \phi^a\psi^a$, c_2, c_4, d_2, d_4, e_2, and e_4 are all positive numbers, and τ^a $(a = 1-3)$ and ϵ^{abc} denote the Pauli matrices and the structure constants of the SU(2) group, respectively.

A. Model I

First, we shall study model I described by the Lagrangian density (2.1) since it is simpler than model II described by (2.2). The partition function of model I is written as

$$Z_{\rm I} \propto \int \mathfrak{D}A_\mu^a(x)\int \prod_{i=1}^4 \mathfrak{D}\phi_i(x)\prod_x \times[\Delta_{\rm FP}^{\rm I}(x)\delta(\phi_2(x))\,\delta(\phi_3(x))\,\delta(\phi_4(x))] \times \exp\left[i\int d^4x\,\mathcal{L}_{\rm I}(x)\right], \tag{2.6}$$

where the components of the Higgs scalar $\phi(x)$ are defined by

$$\phi(x) = \begin{pmatrix} \phi_1(x) + i\phi_2(x) \\ \phi_3(x) + i\phi_4(x) \end{pmatrix}, \tag{2.7}$$

and $\Delta_{\rm FP}(x)$ denotes the Faddeev-Popov determinant corresponding to the gauge-fixing conditions

$$\phi_2(x) = \phi_3(x) = \phi_4(x) = 0. \tag{2.8}$$

It is more convenient to set the gauge-fixing conditions on the Higgs scalars as above rather than on the gauge fields A_μ^a since we need to integrate over $A_\mu^a(x)$ completely after performing the dual transformation. Let us define the dual transformation by

$$\exp\left\{i\int d^4x[-\tfrac{1}{4}(F_{\mu\nu}^a)^2]\right\} \propto \int \mathfrak{D}W_{\mu\nu}^a(x)\exp\left(i\int d^4x\{-\tfrac{1}{4}[m^2(W_{\mu\nu}^a)^2 - 2m\tilde{W}_{\mu\nu}^a F^{a,\mu\nu}]\}\right), \tag{2.9}$$

where $W_{\mu\nu}^a$ are antisymmetric tensor fields. This is a non-Abelian version of the dual transformation (2.5) in Ref. 1. As was stressed in Ref. 1, this transformation (2.9) is a kind of Fourier transformation and is analogous to the transformation in quantum mechanics from the coordinate representation to the momentum representation. In Eq. (2.9), the parameter m is arbitrary and nonvanishing, and

$$\tilde{W}_{\mu\nu}^a \equiv \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}W^{a,\lambda\rho}, \tag{2.10}$$

where $\epsilon_{\mu\nu\lambda\rho}$ equals $+1$ or -1 according to whether $(\mu\nu\lambda\rho)$ is an even or odd permutation of (1234). The Faddeev-Popov determinant[14] $\Delta_{\rm FP}^{\rm I}(x)$, for ϕ subject to the conditions (2.8), is given by

$$\Delta_{\rm FP}^{I}(x) = 1\Big/\int\prod_{a=1}^3 d\omega^a(x)\prod_{k=2}^4\delta(\phi_k(x)^\omega) \propto [\phi_1(x)]^3, \tag{2.11}$$

where $\phi_k(x)^\omega$ is the kth component of $\phi(x)^\omega$ obtained from $\phi(x)$ by the infinitesimal gauge transformation, with parameters $\omega^a(x)$, namely,

$$\phi(x)^\omega \approx \phi(x) + i\tfrac{1}{2}\tau^a\omega^a(x)\phi(x)\,. \tag{2.12}$$

After performing the dual transformation (2.9) in the integrand of the partition function (2.6), we carry out the following integration over A^a_μ:

$$\int \mathfrak{D}A^a_\mu(x)\exp\Big\{i\int d^4x\Big[\tfrac{1}{2}e^2A^{a,\mu}K^{ab}_{\mu\nu}(\phi,W)A^{b,\nu} - m\Big(V^a_\mu - \frac{e}{m}J^a_\mu\Big)A^{a,\mu}\Big]\Big\}$$
$$\propto (\det K^{ab}_{\mu\nu})^{-1/2}\exp\Big\{i\int d^4x\Big[-\frac{1}{2}\Big(\frac{m}{e}\Big)^2\Big(V^{a,\mu} - \frac{e}{m}J^{a,\mu}\Big)(K^{-1})^{ab}_{\mu\nu}\Big(V^{b,\nu} - \frac{e}{m}J^{b,\nu}\Big)\Big]\Big\}\,. \tag{2.13}$$

In Eq. (2.13) $K^{ab}_{\mu\nu}$ is defined by

$$K^{ab}_{\mu\nu} \equiv \tfrac{1}{2}|\phi|^2\delta^{ab}g_{\mu\nu} - \frac{m}{e}\epsilon^{acb}\tilde{W}^c_{\mu\nu}\,, \tag{2.14}$$

the velocity vector V^a_μ of the hydrodynamics satisfying the continuity equation ($\partial^\mu V^a_\mu = 0$) is introduced by

$$V^a_\mu \equiv \partial^\nu \tilde{W}^a_{\nu\mu}\,, \tag{2.15}$$

and the current J^a_μ is given by

$$J^a_\mu = -i\phi^\dagger\tfrac{1}{2}\tau^a\overleftrightarrow{\partial}_\mu\phi \quad (\overleftrightarrow{\partial}_\mu \equiv \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\,. \tag{2.16}$$

Notice that J^a_μ vanishes under the gauge conditions (2.8). In the above integration over A^a_μ, we must define the inverse of the matrix K. We do it by the power-series expansion in m/e as

$$M^{ab}_{\mu\nu}K^{bc,\nu\lambda} = \delta^{ac}g_\mu{}^\lambda\,, \tag{2.17a}$$

and

$$M(\phi,W) = K(\phi,W)^{-1}$$
$$\equiv \Phi^{-1}\sum_{n=0}^{\infty}\Big(\frac{m}{e}\hat{W}\Phi^{-1}\Big)^n \tag{2.17b}$$
$$= \Phi^{-1} + \Big(\frac{m}{e}\Big)\Phi^{-1}\hat{W}\Phi^{-1} + \Big(\frac{m}{e}\Big)^2\Phi^{-1}\hat{W}\Phi^{-1}\hat{W}\Phi^{-1} + \cdots\,. \tag{2.17c}$$

Here we assume that ϕ has a nonvanishing vacuum expectation value so that Φ^{-1} is well defined by

$$(\Phi^{-1})^{ab}_{\mu\nu} \equiv \frac{2}{|\phi|^2}\delta^{ab}g_{\mu\nu}\,. \tag{2.18}$$

Also, the matrix W is introduced by

$$\hat{W}^{ab}_{\mu\nu} \equiv \epsilon^{acb}\tilde{W}^c_{\mu\nu}\,. \tag{2.19}$$

Now, from Eqs. (2.6), (2.9), (2.11), and (2.13), we obtain

$$Z_{\rm I} \propto Z^*_{\rm I} = \int \mathfrak{D}W^a_{\mu\nu}(x)\int\mathfrak{D}\phi_1(x)\prod_x\phi_1(x)^3 \times[\det M^{ab}_{\mu\nu}(\phi_1,W)]^{1/2} \times\exp\Big[i\int d^4x\,\mathcal{L}^*_{\rm I}(x)\Big]\,, \tag{2.20}$$

where

$$\mathcal{L}^*_{\rm I} = -\frac{1}{2}\Big(\frac{m}{e}\Big)^2 V^{a,\mu}M^{ab}_{\mu\nu}(\phi_1,W)V^{b,\nu} - \tfrac{1}{4}m^2(W^a_{\mu\nu})^2 + (\partial_\mu\phi_1)^2 - V_{\rm I}(\phi_1{}^2)\,. \tag{2.21}$$

Thus, we have proved that the non-Abelian Higgs-Kibble model I described by the Lagrangian density (2.1) gives the same partition function as the non-Abelian hydrodynamical model described by the Lagrangian density (2.21). We may say that the hydrodynamical model obtained above is *dually related* to the original non-Abelian Higgs-Kibble model described by Eq. (2.1), since the former is obtained from the latter with the help of the dual transformation (2.9). It should be stressed that the dually transformed non-Abelian Higgs-Kibble model, the relativistic hydrodynamical model described by the Lagrangian density (2.21), is defined by *a power-series expansion in terms of* $1/e$ as is seen in the definition of the matrix $M(\phi_1, W)$ [Eqs. (2.17b) and (2.17c)]. Therefore, this hydrodynamical model may play an important role in the future study of the strong-coupling region ($e \gg 1$) of the non-Abelian gauge theories with the help of the various dual relations between the Green's functions of these two models, which will be given in the next section.

Recently, Freedman[11] has found a non-Abelian version of the relativistic hydrodynamics of Kalb and Ramond[9] and of Nambu.[10] His Lagrangian is equivalent to

$$\mathcal{L}_{\rm Freedman} = -\frac{1}{2}\Big(\frac{m}{e}\Big)^2 V^{a,\mu}M^{ab}_{\mu\nu}(\phi_1,W)V^{b,\nu}\,, \tag{2.22}$$

under the restriction that ϕ_1 is constant. It is interesting to notice that the action corresponding to the Lagrangian density (2.22) is invariant under the following gauge transformation with parameters $\Lambda^b_\mu(x)$:

$$W^a_{\mu\nu} \to W^a_{\mu\nu} + \nabla^{ab}_\mu\Lambda^b_\nu - \nabla^{ab}_\nu\Lambda^b_\mu\,, \tag{2.23}$$

where

$$\nabla^{ab}_\mu = \delta^{ab}\partial_\mu + \Big(\frac{m}{e}\Big)\epsilon^{acb}P^c_\mu \tag{2.24}$$

and

$$P^c_\mu = M^{cd}_{\mu\nu}(\phi_1, W)V^{d,\nu} . \qquad (2.25)$$

This transformation is a non-Abelian version of the gauge transformation discussed by Kalb and Ramond[9] in the Abelian hydrodynamics,

$$W_{\mu\nu} \to W_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu .$$

In our Lagrangian (2.21), the symmetry associated with the transformation (2.23) is broken by the mass term so that it is not necessary to fix the gauge in our partiition function of the non-Abelian hydrodynamics. Furthermore, it is notable that the functional measure, which has appeared in Eq. (2.20) after the dual transformation from the parition function of the non-Abelian gauge model (2.6) with the correct functional measure, is just the necessary one for the non-Abelian relativistic hydrodynamical model defined by such a nonlinear Lagrangian (2.21). This correctness of the measure will be proved in the following.

As was discussed in the previous paper,[1,15] the partition function is originally defined by the functional integral over dynamical variables $q_i(x)$ and conjugate momenta $p_i(x)$,

$$Z_N \propto \int \mathfrak{D}q_i(x) \int \mathfrak{D}p_i(x) \exp\Big\{ i \int d^4x \times\Big[\sum_i p_i \dot{q}_i - \mathcal{H}_N(p,q)\Big]\Big\}, \qquad (2.26)$$

where $\mathcal{H}_N(p,q)$ is the Hamiltonian density corresponding to a nonlinear Lagrangian density such as

$$\mathcal{L}_N(q,\dot{q}) = \tfrac{1}{2}\sum_{i,j} \dot{q}_i f_{ij}(q)\dot{q}_j + \sum_i g_i(q)\dot{q}_i + h(q) . \qquad (2.27)$$

If we perform the integration over $p_i(x)$ completely, we obtain

$$Z_N \propto \int \mathfrak{D}\, q_i(x) \prod_x [\det f_{ij}(q)]^{1/2} \times \exp\Big[i\int d^4x\, \mathcal{L}_N(x)\Big] . \qquad (2.28)$$

In our case the matrix f is given by

$$(f)^{ab}_{ij} = -\left(\frac{m}{e}\right)^2 M^{ab}_{ij}(\phi_1, W) . \qquad (2.29)$$

This can be easily understood, if nondynamical variables $\vec{e}^{\,a}(x)$ and dynamical variables $\vec{b}^a(x)$ are introduced as[10]

$$e^{a,i} \equiv W^{a,0i} , \qquad (2.30a)$$

$$b^{a,i} \equiv \tfrac{1}{2}\epsilon_{0ijk} W^{a,jk} , \qquad (2.30b)$$

and if V^a is expressed in terms of $\vec{e}^a$ and $\vec{b}^a$, namely,

$$V^a_0 = V^{a,0} = \vec{\nabla}\cdot\vec{b}^a , \qquad (2.31a)$$

$$V^a_i = -V^{a,i} = [\dot{\vec{b}}^a + (\vec{\nabla}\times\vec{e}^a)]^i . \qquad (2.31b)$$

What we should prove becomes the following relation:

$$\phi_1(x)^3(\det M^{ab}_{\mu\nu})^{1/2} \propto (\det M^{ab}_{ij})^{1/2} , \qquad (2.32)$$

where we have omitted an unimportant constant factor. In order to prove Eq. (2.32), we prepare a useful formula in the following: If a regular matrix K and its inverse M are expressed as

$$K = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } M = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} , \qquad (2.33)$$

respectively, we have

$$\det D' = \det A \times \det M , \qquad (2.34)$$

where $A(A')$, $B(B')$, $C(C')$, and $D(D')$ are $r\times r$, $r\times s$, $s\times r$, and $s\times s$ matrices, respectively.

Proof.

$$\det K = \det\begin{pmatrix} A & B - A\cdot A^{-1}B \\ C & D - C\cdot A^{-1}B \end{pmatrix}$$
$$= \det A \times \det(D - CA^{-1}B)$$
$$= \det A \times \det(D'^{-1}) .$$

If we take $D' = (M^{ab}_{ij})$ in Eq. (2.33), we have

$$\det A = \det(\tfrac{1}{2}|\phi|^2\delta^{ab}) \propto (\phi_1(x))^6 . \qquad (2.35)$$

Then Eq. (2.32) is derived from Eqs. (2.34) and (2.35).

Therefore, we have proved that the dual transformation from the partition function of the non-Abelian gauge model with the appropriate functional measure (determined by the Faddeev-Popov method[14]) gives the correct partition function of the relativistic hydrodynamical model with the measure inherent in the nonlinearity of this model.

B. Model II

Next we study the model II described by the Lagrangian density (2.2). The difference between models I and II lies in the type of representation of the group SU(2) for the Higgs scalars (the doublet representation in model I and the triplet representation in model II). Since $\pi_1(\mathrm{SU}(2)) = \{0\}$,[16] there is no nonvanishing topological quantum number in model I. It is, therefore, difficult to discuss the stability of a vortex solution in model I. On the other hand, the symmetry controlling model II is essentially O(3) which possesses a nonvanishing topological quantum number characterized by $\pi_1(O(3)) = Z_2 = \{0, 1\}$.[16] It is, therefore, much easier to discuss the stability of the vortex solution. In the following, we thus set the gauge condition so as to give a vortex solution. Namely, we start from the following partition function:

$$Z_{II} \propto \int \mathfrak{D}A_\mu^a(x) \int \mathfrak{D}|\phi(x)|\, |\phi(x)| \mathfrak{D}\phi^3(x) \int \mathfrak{D}|\psi(x)|\, |\psi(x)| \mathfrak{D}\psi^3(x) \int \mathfrak{D}\zeta(x)\mathfrak{D}\chi(x) \times \prod_x \Delta_{FP}^{II}(x)\, \delta(\phi^3(x))\delta(\psi^3(x))\delta(\chi(x) - n\theta(x)) \times \exp\left[i \int d^4x\, \mathfrak{L}_{II}(x)\right], \tag{2.36}$$

where $|\phi(x)|$, $|\psi(x)|$, $\xi(x)$, $\eta(x)$, $\zeta(x)$, and $\chi(x)$ are defined by

$$\phi^1(x) + i\phi^2(x) = |\phi(x)|\, e^{i\xi(x)}, \tag{2.37a}$$

$$\psi^1(x) + i\psi^2(x) = |\psi(x)|\, e^{i\eta(x)}, \tag{2.37b}$$

$$\xi(x) = \chi(x) + \zeta(x), \tag{2.37c}$$

and

$$\eta(x) = \chi(x) - \zeta(x). \tag{2.37d}$$

In Eq. (2.36), we fix $\chi(x)$ to be $n\theta(x)$ in a way similar to the Abelian case,[1] where $\theta(x)$ is the azimuthal angle of $\vec{x}$ around a given string S at the time x^0, and n is an integer. Since we have already presented in model I our prescription for the dual transformation in detail, we shall skip the complicated calculations in model II.

The Faddeev-Popov determinant[14] $\Delta_{FP}^{II}(x)$ is given by

$$\Delta_{FP}^{II}(x)\delta(\phi^3)\delta(\psi^3)\delta(\chi - n\theta) = |\phi^1\psi^2 - \psi^1\phi^2|\, \delta(\phi^3)\delta(\psi^3)\delta(\chi - n\theta). \tag{2.38}$$

The dual transformation (2.9) and the functional integrations over $A_\mu^a(x)$, $\phi^3(x)$, $\psi^3(x)$, and $\chi(x)$ lead to the following result:

$$Z_{II} \propto Z_{II}^* = \int \mathfrak{D}W_{\mu\nu}^a(x) \int \mathfrak{D}|\phi(x)| \int \mathfrak{D}|\psi(x)| \int \mathfrak{D}\zeta(x) |\phi(x)|\, |\psi(x)|\, |\vec{\phi}(x) \times \vec{\psi}(x)| (\det M_{\mu\nu}^{ab})^{1/2} \times \exp\left[i \int d^4x\, \mathfrak{L}_{II}^*(x)\right]. \tag{2.39}$$

The integrand of Eq. (2.39) is supposed to be expressed in terms of the three independent variables $|\phi(x)|$, $|\psi(x)|$, and $\zeta(x)$ with the help of Eqs. (2.37a)–(2.37d). The Lagrangian density $\mathfrak{L}_{II}^*$, dually related to $\mathfrak{L}_{II}$, is given by

$$\mathfrak{L}_{II}^* = -\frac{1}{2}\left(\frac{m}{e}\right)^2 \left(V^{a,\mu} - \frac{e}{m} j^{a,\mu}\right) M_{\mu\nu}^{ab}(\phi, \psi, W) \left(V^{b,\nu} - \frac{e}{m} j^{b,\nu}\right) - \tfrac{1}{4} m^2 (W_{\mu\nu}^a)^2 + \tfrac{1}{2}(\partial_\mu \phi^a)^2 + \tfrac{1}{2}(\partial_\mu \psi^a)^2 - V_{II}(\phi, \psi). \tag{2.40}$$

The matrix M is defined by the power-series expansion in m/e, as in Eqs. (2.17a), (2.17b), and (2.17c) with the help of the matrix $\hat{W}$ of Eq. (2.19). The matrices K and Φ are given by

$$K_{\mu\nu}^{ab} = \Phi_{\mu\nu}^{ab} - \frac{m}{e} \hat{W}_{\mu\nu}^{ab} \tag{2.41}$$

and

$$\Phi_{\mu\nu}^{ab} = g_{\mu\nu}[(\phi)^2\delta^{ab} - \phi^a\phi^b + (\psi)^2\delta^{ab} - \psi^a\psi^b], \tag{2.42}$$

respectively. It should be stressed that in order to give a well-defined Φ^{-1}, it is necessary to introduce two Higgs scalars into the triplet representation and to require that these two scalars ϕ and ψ have nonvanishing vacuum expectation values and be not parallel or antiparallel to each other. This can be easily understood from the explicit form of Φ^{-1},

$$(\Phi^{-1})^{ab}_{\mu\nu}=\frac{g_{\mu\nu}}{(\phi)^2+(\psi)^2}\left[\delta^{ab}+\frac{(\phi)^2\phi^a\phi^b+(\psi)^2\psi^a\psi^b+(\vec{\phi}\cdot\vec{\psi})(\phi^a\psi^b+\psi^a\phi^b)}{(\vec{\phi}\times\vec{\psi})^2}\right]. \tag{2.43}$$

The current $j^a_\mu(x)$ in Eq. (2.34), given by

$$j^a_\mu(x)=\epsilon^{abc}(\phi^b\partial_\mu\phi^c+\psi^b\partial_\mu\psi^c), \tag{2.44}$$

has the only nonvanishing component

$$j^3_\mu(x)=(|\phi|^2+|\psi|^2)\partial_\mu(n\theta)+(|\phi|^2-|\psi|^2)\partial_\mu\zeta \tag{2.45}$$

under the gauge-fixing conditions given in Eq. (2.36). The term proportional to $\partial_\mu(n\theta)$ plays an important role for producing a vorticity source $\omega^a_{\mu\nu}(x)$ in our non-Abelian hydrodynamics. The vorticity source $\omega^a_{\mu\nu}(x)$ is defined as the coefficient of the term linearly proportional to $W^a_{\mu\nu}$, namely, $\frac{1}{2}(4\pi m/e)\times W^a_{\mu\nu}(x)\omega^{a,\mu\nu}(x)$, in the Lagrangian density $\mathcal{L}^*$ of the hydrodynamics. In our case, $\omega^3_{\mu\nu}(x)$ is the only nonvanishing component of the vorticity source. It consists of a regular part $\omega'_{\mu\nu}(x)$ and a singular part $\omega''_{\mu\nu}(x)$, namely

$$\omega^3_{\mu\nu}(x)=\omega'_{\mu\nu}(x)+\omega''_{\mu\nu}(x), \tag{2.46a}$$

$$\omega'_{\mu\nu}(x)=\frac{1}{2\pi}\epsilon_{\mu\nu\lambda\rho}\partial^\lambda\left(\frac{|\phi|^2-|\psi|^2}{|\phi|^2+|\psi|^2}\right)\partial^\rho\zeta, \tag{2.46b}$$

and

$$\omega''_{\mu\nu}(x)=\frac{1}{4\pi}\epsilon_{\mu\nu\lambda\rho}(\partial^\lambda\partial^\rho-\partial^\rho\partial^\lambda)n\theta(x) \tag{2.46c}$$

$$=n\int\int d\tau\,d\sigma\,\delta^{(4)}(x-y(\tau,\sigma))\left|\frac{\partial(y_\mu,y_\nu)}{\partial(\tau,\sigma)}\right|. \tag{2.46d}$$

In Eq. (2.46d) we have used $y^\mu(\tau,\sigma)$ as a parametrization of the world sheet of the string S, where τ and σ denote the timelike and spacelike parameters, respectively. (See Ref. 17.)

It is also possible to prove that the functional measure in Eq. (2.39) is the only necessary one for the nonlinear Lagrangian density (2.40). The matrix f for model II is

$$f=\begin{pmatrix}|\phi|^2+|\psi|^2-(|\phi|^2-|\psi|^2)^2M^{33}_{00} & \left(\frac{m}{e}\right)(|\phi|^2-|\psi|^2)(M^{3b}_{0j})\\ \left(\frac{m}{e}\right)(|\phi|^2-|\psi|^2)(M^{a3}_{i0}) & -\left(\frac{m}{e}\right)^2(M^{ab}_{ij})\end{pmatrix}. \tag{2.47}$$

We must now prove the following relation:

$$|\phi|^2|\psi|^2|\vec{\phi}\times\vec{\psi}|^2\det(M^{ab}_{\mu\nu})\propto\det f$$
$$\propto-(|\phi|^2+|\psi|^2)\det(M^{ab}_{ij})+(|\phi|^2-|\psi|^2)^2\det\begin{pmatrix}M^{33}_{00} & (M^{3b}_{0j})\\ (M^{a3}_{i0}) & (M^{ab}_{ij})\end{pmatrix}. \tag{2.48}$$

This equation can be easily checked by the useful formula Eqs. (2.33) and (2.34). Thus we conclude that the functional measure in Eq. (2.39) derived automatically from the dual transformation coincides with that in Eq. (2.28) with $\mathcal{L}_N=\mathcal{L}^*_{\rm II}$.

III. THE DUAL RELATIONS AND THE VORTICITY SOLUTION

In this section we shall discuss several relations between the original non-Abelian gauge theories described by the Lagrangian densities $\mathcal{L}_{\rm I}$ and $\mathcal{L}_{\rm II}$ and the non-Abelian hydrodynamics described by the Lagrangian densities $\mathcal{L}^*_{\rm I}$ and $\mathcal{L}^*_{\rm II}$. For this purpose, it is useful to introduce an external source $J^a_{\mu\nu}$ through the replacement

$$F^a_{\mu\nu}(x)\to F^a_{\mu\nu}(x)+J^a_{\mu\nu}(x), \tag{3.1}$$

in a way similar to the Abelian case.[1] The dual transformation (2.9) with the replacement (3.1) leads us to the formula

$$Z[J^a_{\mu\nu}] \propto Z^*[J^a_{\mu\nu}], \tag{3.2}$$

where $Z[J^a_{\mu\nu}]$ and $Z^*[J^a_{\mu\nu}]$ are obtained from Z and Z^* by replacing $\mathcal{L}$ and $\mathcal{L}^*$ by $\mathcal{L}[J^a_{\mu\nu}]$ and $\mathcal{L}^*[J^a_{\mu\nu}]$, respectively. The definitions of $\mathcal{L}[J^a_{\mu\nu}]$ and $\mathcal{L}^*[J^a_{\mu\nu}]$ are as follows:

$$\mathcal{L}[J^a_{\mu\nu}] = \mathcal{L} - \tfrac{1}{2} F^a_{\mu\nu}(x) J^{a,\mu\nu}(x) - \tfrac{1}{4} J^a_{\mu\nu}(x) J^{a,\mu\nu}(x), \tag{3.3a}$$

and

$$\mathcal{L}^*[J^a_{\mu\nu}] = \mathcal{L}^* + \tfrac{1}{2} m \tilde{W}^a_{\mu\nu}(x) J^{a,\mu\nu}(x). \tag{3.3b}$$

From Eqs. (3.2), (3.3a), and (3.3b), we obtain the following relations between the Green's functions of the two models dually related to each other:

$$im\langle \tilde{W}^a_{\mu\nu}(x)\rangle_* = (-i)\langle F^a_{\mu\nu}(x)\rangle, \tag{3.4}$$

$$(im)^2\langle T^* \tilde{W}^{a_1}_{\mu_1\nu_1}(x_1)\tilde{W}^{a_2}_{\mu_2\nu_2}(x_2)\rangle_* = (-i)^2\langle T^* F^{a_1}_{\mu_1\nu_1}(x_1) F^{a_2}_{\mu_2\nu_2}(x_2)\rangle + (-i)\delta_{(1)(2)}, \tag{3.5}$$

$$(im)^3\langle T^* \tilde{W}^{a_1}_{\mu_1\nu_1}(x_1)\tilde{W}^{a_2}_{\mu_2\nu_2}(x_2)\tilde{W}^{a_3}_{\mu_3\nu_3}(x_3)\rangle_* = (-i)^3\langle T^* F^{a_1}_{\mu_1\nu_1}(x_1) F^{a_2}_{\mu_2\nu_2}(x_2) F^{a_3}_{\mu_3\nu_3}(x_3)\rangle + (-i)^2[\delta_{(1)(2)}\langle F^{a_3}_{\mu_3\nu_3}(x_3)\rangle + \delta_{(2)(3)}\langle F^{a_1}_{\mu_1\nu_1}(x_1)\rangle + \delta_{(3)(1)}\langle F^{a_2}_{\mu_2\nu_2}(x_2)\rangle], \tag{3.6}$$

and, in general,

$$\begin{aligned}(im)^n\langle T^* \tilde{W}^{a_1}_{\mu_1\nu_1}(x_1)\tilde{W}^{a_2}_{\mu_2\nu_2}(x_2)\cdots\tilde{W}^{a_n}_{\mu_n\nu_n}(x_n)\rangle_* &= (-i)^n\langle T^* F^{a_1}_{\mu_1\nu_1}(x_1) F^{a_2}_{\mu_2\nu_2}(x_2)\cdots F^{a_n}_{\mu_n\nu_n}(x_n)\rangle \\ &\quad + (-i)^{n-1}[\delta_{(1)(2)}\langle T^* F^{a_3}_{\mu_3\nu_3}(x_3)\cdots F^{a_n}_{\mu_n\nu_n}(x_n)\rangle + \cdots] \\ &\quad + (-i)^{n-2}[\delta_{(1)(2)}\delta_{(3)(4)}\langle T^* F^{a_5}_{\mu_5\nu_5}(x_5)\cdots F^{a_n}_{\mu_n\nu_n}(x_n)\rangle + \cdots] \\ &\quad + \cdots,\end{aligned} \tag{3.7}$$

where we have used the notation

$$\delta_{(i)(j)} \equiv \delta^{a_i a_j}(g_{\mu_i\mu_j}g_{\nu_i\nu_j} - g_{\mu_i\nu_j}g_{\mu_j\nu_i}) \times \delta^{(4)}(x_i - x_j). \tag{3.8}$$

These equations are easily obtained by the successive functional differentiation of $\ln Z$ with respect to $J^a_{\mu\nu}(x)$, namely,

$$\frac{\delta^n \ln Z[J^a_{\mu\nu}(x)]}{\delta J^{a_1}_{\mu_1\nu_1}(x_1)\cdots\delta J^{a_n}_{\mu_n\nu_n}(x_n)} = \frac{\delta^n \ln Z^*[J^a_{\mu\nu}(x)]}{\delta J^{a_1}_{\mu_1\nu_1}(x_1)\cdots\delta J^{a_n}_{\mu_n\nu_n}(x_n)}. \tag{3.9}$$

In our models, the original local SU(2) symmetry is broken spontaneously by the Higgs scalars so that the Green's functions are defined at the spontaneously broken vacuum. Therefore, for the definition of these Green's functions, it is necessary to change the variables of the generating functional from the external sources of the Higgs scalars to the vacuum expectation values of the Higgs scalars by using the Legendre transformation.[18] This well-known step can be easily performed in the two models dually related to each other since the Higgs scalars are common in both of these two models. We shall skip this step here. It is very interesting to discuss the possible renormalizability of the non-Abelian hydrodynamics described by the Lagrangian densities $\mathcal{L}_{\rm I}^*$ [Eq. (2.21)] and $\mathcal{L}_{\rm II}^*$ [Eq. (2.40)] by using the dual relations (3.4)–(3.7) for the Green's functions and the renormalization prescription of the non-Abelian Higgs-Kibble models described by $\mathcal{L}_{\rm I}$ [Eq. (2.1)] and $\mathcal{L}_{\rm II}$ [Eq. (2.2)]. We leave this problem to a future study. In the following, however, we shall discuss a simple correspondence between the classical solutions of the two models described by $\mathcal{L}_{\rm II}$ and $\mathcal{L}_{\rm II}^*$. As pointed out by Nielsen and Olesen,[13] in the non-Abelian Higgs-Kibble model described by $\mathcal{L}_{\rm II}$ there exists a vortex solution, which is static and axially symmetric with respect to the x^3 axis, and the magnetic field with a third isospin index F^3_{12} is squeezed along the x^3 axis. The decomposition of (3.4) into "electric" and "magnetic" component shows

$$\langle m\, \vec{e}^{\,a}(x)\rangle_* = -\langle \vec{\rm H}^a(x)\rangle \tag{3.10a}$$

and

$$\langle m\vec{b}^a(x)\rangle_* = \langle\vec{E}^a(x)\rangle\,, \tag{3.10b}$$

where

$$E^{a,i}(x) \equiv F^{a,0i}(x) \tag{3.11a}$$

and

$$H^{a,i}(x) \equiv \tfrac{1}{2}\epsilon_{0ijk}F^{a,jk}(x)\,. \tag{3.11b}$$

From Eqs. (3.10a) and (3.10b), we find that the electric component $\vec{e}^a$ and the magnetic component $\vec{b}^a$ of the non-Abelian hydrodynamics correspond to the magnetic field $-\vec{H}^a$ and the electric field $\vec{E}^a$ of the non-Abelian Higgs-Kibble model, respectively. Therefore, there must exist in the non-Abelian hydrodynamics a vortex solution which is static and axially symmetric with respect to the x^3 axis along which the electric component $W^3_{03} = -(\vec{e}^3)_3$ is squeezed. In the following, we shall search for such a classical solution. The field equations of the hydrodynamics in model II described by the Lagrangian density $\mathcal{L}^*_{II}$ are

$$-\left(\frac{m}{e}\right)^2\left[\partial_\mu \hat{P}^a_\nu - \partial_\nu \hat{P}^a_\mu + \left(\frac{m}{e}\right)\epsilon^{abc}\hat{P}^b_\mu \hat{P}^c_\nu\right] - m^2\tilde{W}^a_{\mu\nu} = 0\,, \tag{3.12a}$$

$$\Box|\phi| - |\phi|(\partial_\mu\xi)^2 + \frac{\delta V_{II}}{\delta|\phi|} - \left(\frac{m}{e}\right)2|\phi|(\partial^\mu\xi)\hat{P}^3_\mu - \frac{1}{2}\left(\frac{m}{e}\right)^2\hat{P}^a_\mu\frac{\delta\hat{\Phi}^{ab}}{\delta|\phi|}\hat{P}^{b,\mu} = 0\,, \tag{3.12b}$$

$$\Box|\psi| - |\psi|(\partial_\mu\eta)^2 + \frac{\delta V_{II}}{\delta|\psi|} - \left(\frac{m}{e}\right)2|\psi|(\partial^\mu\eta)\hat{P}^3_\mu - \frac{1}{2}\left(\frac{m}{e}\right)^2\hat{P}^a_\mu\frac{\delta\hat{\Phi}^{ab}}{\delta|\psi|}\hat{P}^{b,\mu} = 0\,, \tag{3.12c}$$

and

$$\partial^\mu\left[|\phi|^2\partial_\mu\xi - |\psi|^2\partial_\mu\eta + \left(\frac{m}{e}\right)(|\phi|^2 - |\psi|^2)\hat{P}^3_\mu\right] + \frac{\delta V_{II}}{\delta\zeta} - \frac{1}{2}\left(\frac{m}{e}\right)^2\hat{P}^a_\mu\frac{\delta\hat{\Phi}^{ab}}{\delta\zeta}\hat{P}^{b,\mu} = 0\,, \tag{3.12d}$$

where

$$\hat{P}^a_\mu \equiv M^{ab}_{\mu\nu}(\phi,\psi,W)\left(V^{b,\nu} - \frac{e}{m}j^{b,\nu}\right), \tag{3.13}$$

$$\Phi^{ab}_{\mu\nu} = g_{\mu\nu}\hat{\Phi}^{ab}\,. \tag{3.14}$$

These equations may possess the solutions of a type

$$W^{1,2}_{\mu\nu} \equiv 0 \text{ and } W^3_{\mu\nu} \neq 0\,. \tag{3.15}$$

This can be easily checked by the help of the simple relations

$$M^{a3}_{\mu\nu} = (\Phi^{-1})^{a3}_{\mu\nu} = \frac{1}{|\phi|^2 + |\psi|^2}\delta^{a3}g_{\mu\nu}\,, \tag{3.16}$$

$$\hat{P}^{1,2}_\mu \equiv 0\,, \tag{3.17a}$$

and

$$\hat{P}^3_\mu = \frac{1}{|\phi|^2 + |\psi|^2}\left(V^3_\mu - \frac{e}{m}j^3_\mu\right), \tag{3.17b}$$

where we have assumed that Eq. (3.15) and the gauge-fixing conditions imposed in the definition of the partition function Z_{II} [Eq. (2.36)] hold. For simplicity, we put the string S [see the explanation below Eq. (2.31)] on the x^3 axis, where the only component of the singular vorticity source $\omega''_{\mu\nu}$ [Eq. (2.46d)] has a nonvanishing value,

$$\omega''^{03}(x) = n\delta(x^1)\delta(x^2) = n\frac{1}{2\pi r}\delta(r) \tag{3.18}$$

with $r \equiv [(x^1)^2 + (x^2)^2]^{1/2}$.

Now, in order to find a static and axially symmetric (with respect to the x^3 axis) solution we can make the following assumptions in addition to (3.15):

$$e^{3,3} = e^{3,3}(r),\quad |\phi| = |\phi|(r),\quad |\psi| = |\psi|(r),\quad \zeta = \zeta(r)\,, \tag{3.19a}$$

$$\vec{b}^3 = \vec{0}\,. \tag{3.19b}$$

Then the field equations (3.12a)–(3.12d) reduce to

$$-\left(\frac{m}{e}\right)^2\frac{1}{r}\frac{d}{dr}\left(\frac{1}{|\phi|^2 + |\psi|^2}r\frac{d}{dr}e^{3,3}\right) + m^2e^{3,3} + 2\left(\frac{m}{e}\right)n\pi\frac{1}{2\pi r}\delta(r) = 0\,, \tag{3.20a}$$

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}|\phi|\right) + \frac{\partial V_{II}}{\partial|\phi|} + \frac{|\phi|}{(|\phi|^2 + |\psi|^2)^2}\left[\left(\frac{m}{e}\frac{d}{dr}e^{3,3}\right)^2 + \left(-2|\psi|^2\frac{d}{dr}\zeta\right)^2\right] = 0\,, \tag{3.20b}$$

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}|\psi|\right) + \frac{\partial V_{II}}{\partial|\psi|} + \frac{|\psi|}{(|\phi|^2 + |\psi|^2)^2}\left[\left(\frac{m}{e}\frac{d}{dr}e^{3,3}\right)^2 + \left(2|\phi|^2\frac{d}{dr}\zeta\right)^2\right] = 0\,, \tag{3.20c}$$

and

$$-\frac{1}{r}\frac{d}{dr}\left(\frac{4|\phi|^2|\psi|^2}{|\phi|^2 + |\psi|^2}r\frac{d}{dr}\zeta\right) + \frac{\partial V_{II}}{\partial\zeta} = 0\,. \tag{3.20d}$$

For the sake of simplicity, we further assume that the coefficient e_2 in the potential $V_{II}(\phi,\psi)$ [Eq. (2.5)] vanishes. Then Eqs. (3.20a)–(3.20d) are decoupled with each other at least for very large r. Under this assumption, we obtain the following solution for very large r:

$$e^{3,3}(r) \underset{r\to\infty}{\simeq} aK_0(e(C+D)^{1/2}r)\,, \qquad (3.21a)$$

$$|\phi|(r) \underset{r\to\infty}{\simeq} \sqrt{C} + bK_0(2\sqrt{c_2}\,r)\,, \qquad (3.21b)$$

$$|\psi|(r) \underset{r\to\infty}{\simeq} \sqrt{D} + cK_0(2\sqrt{d_2}\,r)\,, \qquad (3.21c)$$

$$\zeta(r) \underset{r\to\infty}{\simeq} \frac{\pi}{4} + dK_0([2e_4(C+D)]^{1/2}r)\,, \qquad (3.21d)$$

where the modified Bessel function $K_0(\kappa r)$ satisfies

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}K_0(\kappa r)\right] - \kappa^2 K_0(\kappa r) = 0 \qquad (3.22a)$$

with the asymptotic behavior

$$K_0(\kappa r) \underset{r\to\infty}{\simeq} \left(\frac{\pi}{2\kappa r}\right)^{1/2} e^{-\kappa r}\,, \qquad (3.22b)$$

and the constants C and D are given by

$$C = \frac{c_2}{2c_4} \text{ and } D = \frac{d_2}{2d_4}\,. \qquad (3.23)$$

This solution represents a circulation flow of fluid around the x^3 axis with a third isospin index. The rapid stream is localized in the neighborhood of the x^3 axis. This can be seen in the behavior of the velocity vector having the only nonvanishing component V_θ^3 (θ indicates the angle direction around the x^3 axis.):

$$V_\theta^3 = -\frac{d}{dr}e^{3,3}(r)$$

$$\underset{r\to\infty}{\simeq} ae(C+D)^{1/2}K_1(e(C+D)^{1/2}r) \qquad (3.24a)$$

$$\underset{r\to\infty}{\simeq} ae(C+D)^{1/2}\left[\frac{\pi}{2e(C+D)^{1/2}r}\right]^{1/2} e^{-e(C+D)^{1/2}r}\,. \qquad (3.24b)$$

Corresponding to the quantization of the magnetic flux in the non-Abelian vortex solution of Nielsen and Olesen, the electric flux quantization rule holds in the solution of the non-Abelian hydrodynamics. The field equation (3.12a) with Eqs. (3.13) and (2.45) leads to

$$\left(\frac{m}{e}\right)^2 \oint_C \frac{1}{|\phi|^2+|\psi|^2}\vec{V}^3 d\vec{s} - m^2\int_D e^{3,3}dS = \begin{cases} 0 & \text{for } (0,0)\notin D \\ \dfrac{2\pi m}{e}n & \text{for } (0,0)\in D \end{cases} \qquad (3.25)$$

where C denotes the boundary of a domain D. Taking D in Eq. (3.25) to be a very large domain D_∞ including the origin, and considering the asymptotic behavior (3.24b), we obtain a flux quantization rule in the non-Abelian hydrodynamics,

$$-m\int_{D_\infty} e^{3,3}dS = \frac{2\pi}{e}n\,. \qquad (3.26)$$

Here n is defined by modulo 2 from a topological consideration: The fundamental group characterizing the string of the original SU(2) gauge model II with the two Higgs scalars in the triplet representation is $\pi_1(O(3)) = Z_2 = \{0,1\}$, an additive group of integers defined by modulo 2.[16]

In conclusion, we stress again that the method developed in this paper to obtain the dual correspondence between the non-Abelian gauge theories and the non-Abelian hydrodynamics may become a very useful tool for a future study of the non-Abelian gauge theories in the strong-coupling region.

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