PHYSICAL REVIEW D VOLUME 19, NUMBER 12 15 JUNE 1979

Static model of the quark potential

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(Received 18 December 1978)

We present a semiclassical method for calculating the potential energy of a heavy quark-antiquark pair. Our method preserves the operator charge structure of the quark and antiquark. The operator structure of the gluon fields is approximately maintained by truncating the gluon degrees of freedom to a minimal set, a set which preserves the operator charge structure of the quark-antiquark-gluon system. The energy of this truncated system is determined using a variational principle. The potential thus determined accurately reproduces the results of renormalization-group improved perturbation theory up to and including effects of at least order $\alpha^4 \ln\alpha$.

I. INTRODUCTION

Perhaps the simplest manifestation of the forces which bind quarks is to be found in the structure of heavy quark bound states, for which dynamics are relatively unclouded by the intricacies of the relativistic bound-state problem. Fortunately, nature has provided us with such states in the ψ/J (Refs. 1–6) and Υ (Refs. 7–9) families of particles. Early attempts to fit the ψ/J spectrum provided an adequate description using a combination of Coulomb and linear potentials.[10-12] More recently, attempts have been made to systematically reconstruct a phenomenological potential using the inverse-scattering method.[13]

A challenge to our understanding of these bound states is the problem of relating phenomenological potentials to quantum chromodynamics (QCD), the candidate for a fundamental theory of strong interactions. Unlike the corresponding calculation in an Abelian gauge theory, the determination of such a potential in QCD is nontrivial. In an Abelian theory, the interaction between massive point charges is simple. It is governed by the Coulomb potential, and this potential is the energy of the unique static classical field configuration in the presence of stationary charged sources of given separation. Neither of these statements is known to hold in the non-Abelian case.

Perturbative analysis of the quark-antiquark Bethe-Salpeter equation in QCD in the limit of large quark mass[14-17] indicates that a description of the interaction in terms of an effective potential is appropriate in the color-singlet channel. In this channel the potential can be written in the form

$$V(t) = -[\alpha(t)/t]F(\alpha(t))T_2, \tag{1.1}$$

where t is the momentum transfer, $T_2 = (N^2-1)/2N$ is the quadratic Casimir operator of SU(N), and $\alpha(t)$ is the running QCD coupling constant determined from the Gell-Mann–Low equations.[18-20] For small coupling, F is of the form

$$F(\alpha) \cong 1 + b\alpha^2, \tag{1.2}$$

so that at short distances (corresponding to small α) the potential is Coulombic. At large distances a confining potential can arise only by virtue of the dependence of α on the momentum scale and the nontrivial dependence of $F(\alpha)$ on α. Evaluation of such a confining potential must necessarily be nonperturbative.

This paper attempts to address the question of what we can learn about the quark-antiquark force from classical and semi-classical analysis of Yang-Mills theory. Analogy with the Abelian case suggests finding static solutions of the classical Yang-Mills equations in the presence of stationary external quark charges. However, the non-Abelian nature of the theory complicates the classical problem and prevents a straightforward interpretation of classical solutions in a quantum context.

Since the classical theory has no ultraviolet divergences and thus no need of renormalization, the classical interaction energy of two charges separated by distance r must be of the form

$$V_{\rm cl}(r) = -(\alpha/r)TF_{\rm cl}(\alpha, T), \tag{1.3}$$

where T represents the classical color dependence of the interaction. The dependence of V on r is necessarily Coulombic on dimensional grounds. The essential content of the theory is contained in the function $F_{\rm cl}(\alpha)$. Ultimately our semiclassical analysis will be aimed at determining $F_{\rm cl}$ as an approximation to the $F(\alpha)$ occurring in Eq. (1.1). This treatment sheds no light on the scale dependence of α arising from the renormalization group

in the full quantum theory. Such a nonperturbative approximation to $F(\alpha)$, when combined with conventional perturbative renormalization-group analysis of $\alpha(t)$, will perhaps allow a reasonable extension of (1.1) from the short-distance Coulomb region to intermediate-distance scales. We shall see that this semiclassical approximation agrees very well with QCD perturbation theory, at least to order $\alpha^3 \ln\alpha$.

It is clear that a purely classical description of quark charges does not in general offer an adequate approximation. In the quantum theory the charge operators of a quark or antiquark satisfy

$$[Q_a, Q_b] = i f_{abc} Q_c , \tag{1.4}$$

whereas classical Yang-Mills charges are simply c-number (N^2-1)-component vectors in the adjoint representation of SU(N). Classical treatment of the charge operators would be a good approximation only in a limit in which quarks lie in a very large representation of SU(N). As we are interested rather in the case in which the quarks lie in the smallest representation of SU(N), the fundamental representation, our analysis must be semiclassical, at least insofar as the commutation relations (1.4) must be maintained. This point has been noted before by Adler[21] and by the authors[22] and provides part of the motivation for the introduction of Adler's "algebraic chromodynamics."

The semiclassical model we shall introduce here keeps not only these quantum effects due to quark charges, but also some of the effects due to gluons. The necessity of retaining these effects reflects the color-charge structure of the quantum theory. The quark-antiquark pair must be allowed to make transitions between the singlet and (N^2-1)-plet quark states, accompanied by the emission or absorption of a (N^2-1)-plet gluon. We treat quark and antiquark in the large-mass limit as stationary point charges. Our approximation involves truncating the infinite-dimensional Hilbert space of transverse gluon states in Coulomb-gauge QCD to a space of gluon states in which all gluons share a single spatial wave function. This wave function is determined by a self-consistent variational principle applied to the expectation value of the Hamiltonian in the resulting state. The semiclassical model of the quark-antiquark state thus obtained is very similar to the static model of nuclear physics[23-26]; our truncation of the QCD Hamiltonian to a single set of gluon modes is the analog of the Tomonaga approximation in the theory of P-wave pion-nucleon interactions.[27]

The organization of this paper is as follows. In Sec. II we briefly review the classical two-charge problem in non-Abelian gauge theory. We discuss the problems and ambiguities associated with static solutions of the equations of motion and describe the resolution of these ambiguities in perturbation theory.

Section III concerns the quantum-mechanical quark-antiquark system. We introduce the radiation-gauge QCD Hamiltonian. We apply the Tomonaga approximation to this Hamiltonian. We then proceed to a mean-field approximation to this truncated Hamiltonian and arrive at an effective model Hamiltonian describing a simple, tractable system of color charges interacting with N^2-1 harmonic oscillators. We describe the diagonalization of this effective Hamiltonian in the singlet sector.

In Sec. IV we compare the results of our semiclassical approximation to known results in QCD perturbation theory. We see that our model reproduces perturbation theory up to terms of order $\alpha^3 \ln\alpha$ in $F(\alpha)$.

Section V is devoted to some concluding remarks on the implications of our results for physical systems such as the ψ/J and Υ. In a later paper we shall present detailed numerical analysis of the effective Hamiltonian.

II. THE CLASSICAL POTENTIAL

Let us consider the classical system of stationary pointlike charges interacting with an SU(N) Yang-Mills field. By a stationary point charge we mean a point charge the spatial position of which is time independent. The Yang-Mills equations appropriate to the description of this system are

$$D_\mu F^{\mu\nu} = J^\nu , \tag{2.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu . \tag{2.2}$$

In these equations the covariant derivative is

$$D_\mu U \equiv \partial_\mu U + g A_\mu \times U , \tag{2.3}$$

with the cross-product of any two vectors U_a and V_a in the adjoint representation of SU(N) defined to be

$$(U \times V)_a \equiv f_{abc} U_b V_c . \tag{2.4}$$

For an assembly of k stationary point charges located at the positions $\vec{r}_1, \ldots, \vec{r}_k$ the corresponding color current J^ν is of the form

$$J^\nu(\vec{r}, t) = \delta^{\nu 0} \sum_{\alpha=1}^{k} e Q^{(\alpha)}(t) \delta^{(3)}(\vec{r} - \vec{r}_\alpha) . \tag{2.5}$$

We have written the charges as $eQ^{(\alpha)}(t)$, where

$Q^{(\alpha)}$ is a dimensionless SU(N) adjoint vector. In the quantum theory these dimensionless charge vectors satisfy SU(N) commutation relations and form the basis of a charge algebra.

The parameters e and g, and the fields A, have dimensions which are easily determined by inspection of the classical Hamiltonian

$$H=\tfrac{1}{4}\sum_{a=1}^{N}\int d^3x F_a^{\mu\nu}(\vec{\mathbf{x}},t)F_a^{\mu\nu}(\vec{\mathbf{x}},t)\,. \tag{2.6}$$

The dimensions of e, g, and A^μ are, respectively, (energy × length)$^{1/2}$, (energy × length)$^{-1/2}$, and (energy/length)$^{1/2}$. In the quantum theory, gauge invariance relates particle and gluon charges and requires that e be equal to $g\hbar$. In the classical theory, there is no such requirement. The dimensionless parameter which characterizes the strength of interactions in the classical theory is eg.

Our objective is to determine the lowest-energy, nonradiating configuration of gluon fields in the presence of specified charges. In the Abelian theory, these configurations are simply the Coulomb-field solution appropriate to the charge distribution. Several complications, however, arise in the non-Abelian theory.

A primary problem is that the values of the external charges are gauge dependent. Under a gauge transformation $G(x)\in$ SU(N), the charges transform as

$$Q_a^{(\alpha)}(t)\to D_{ab}(G(\vec{\mathbf{r}}_\alpha))Q_b^{(\alpha)}(t)\,, \tag{2.7}$$

where $D_{ab}(G)$ is the matrix representation of G in the adjoint representation. Although each charge vector has N^2-1 degrees of freedom, this equation implies that only $N-1$ of these are gauge invariant. These $N-1$ correspond to the $N-1$ independent eigenvalues of the traceless matrix $Q_a^{(\alpha)}\tau_a$, where τ_a is a generator of SU(N).

The problem of gauge invariance is manifest in the extended current conservation condition

$$D_\mu J^\mu=0\,. \tag{2.8}$$

For stationary charges, this equation becomes

$$\frac{dQ^\alpha(t)}{dt}-gA^0(\vec{\mathbf{r}}_\alpha,t)\times Q^\alpha(t)=0\,, \tag{2.9}$$

and the N^2-N gauge-dependent degrees of freedom of Q^α process in time.

Whether or not these gauge-dependent degrees of freedom, combined with a complete specification of gauge, correspond to distinct field configurations is an open question. In perturbation theory, this correspondence is the case. It is not known, however, whether these perturbative solutions converge to a well-behaved, unique solution of the Yang-Mills equations.

In addition to ambiguities arising from gauge degrees of freedom, we must consider those ambiguities arising dynamically from the Coulomb instability. Mandula has shown that for a sufficiently large charge, $eg/4\pi>\frac{3}{2}$ in SU(2), the Coulomb solution describing a single charge is unstable.[28] The lowest-energy configuration for $eg/4\pi>\frac{3}{2}$ may prove to be a static solution involving nontrivial chromomagnetic fields, or perhaps a time-dependent finite-energy screening solution such as that discussed by Sikivie and Weiss.[29]

We will consider the problem of finding static perturbative solutions to the Yang-Mills equations in the presence of two charges. Static field configurations are those for which all gauge-invariant quantities are time independent. A gauge can be found in which such static configurations are represented by time-independent potentials.[30] As we shall soon see, these time-independent potentials are such that A^0 is finite and, in general, nonzero at spatial infinity. After requiring that the fields be static, we still have the freedom to make a time-independent gauge transformation, and can therefore impose an arbitrary gauge condition on the spatial components of the vector potential.

The Yang-Mills equations are

$$-\nabla^2A^0=J^0+g\nabla_j(A_j\times A^0)+gA_j\times\nabla_jA^0+g^2A_j\times(A_j\times A^0)\,, \tag{2.10a}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathbf{A}})=gA^0\times\vec{\nabla}A^0+g\nabla^k(A^k\times\vec{\mathbf{A}})+gA^k\times\nabla^k\vec{\mathbf{A}}-gA^k\times\vec{\nabla}A^k+g^2A^0\times(\vec{\mathbf{A}}\times A^0)+g^2A^k\times(A^k\times\vec{\mathbf{A}})\,, \tag{2.10b}$$

and the extended current conservation condition is

$$A^0(\vec{\mathbf{r}}_\alpha)\times Q^{(\alpha)}=0\,. \tag{2.10c}$$

Assuming no background field, the perturbative solution begins in order e. In this order, the fields are of the Abelian Coulomb form

$$\vec{\mathbf{A}}^{(1)}=0\,, \tag{2.11a}$$

$$A^0_{(1)}=\frac{e}{4\pi}\left[Q^{(1)}\left(\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}_1|}-\frac{1}{|\vec{\mathbf{r}}_1-\vec{\mathbf{r}}_2|}\right)+Q^{(2)}\left(\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}_2|}-\frac{1}{|\vec{\mathbf{r}}_1-\vec{\mathbf{r}}_2|}\right)\right]. \tag{2.11b}$$

In this order, the field energy

$$E=\tfrac{1}{2}\int d^3\vec{\mathbf{r}}(\vec{\mathbf{E}}^2+\vec{\mathbf{B}}^2)=E_0+\frac{e^2}{4\pi|r_1-r_2|}Q^{(1)}\cdot Q^{(2)}\,. \tag{2.12}$$

The quantity E_0 is the divergent self-energy

of the point sources. The interaction energy is the familiar Coulomb energy.

We should note that this leading-order contribution to the fields depends on all N^2-1 parameters which specify each charge. If the perturbation series converged, we would expect that solutions which differ in leading order would be independent, except for equivalence under overall global SU(N) gauge rotations.

The leading nonzero contribution to $\vec{A}$ is of order e^2g. Using Eq. (2.10b), we find

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}^{(3)})=\vec{J}^{(3)}\,, \tag{2.13}$$

where

$$\begin{aligned}\vec{J}^{(3)}&=gA^0_{(1)}\times\vec{\nabla}A^0_{(1)}\\&=\frac{e^2g}{16\pi^2}Q^{(1)}\times Q^{(2)}\left(\frac{1}{|\vec{r}-\vec{r}_1|}-\frac{1}{|\vec{r}_1-\vec{r}_2|}\right)\\&\quad\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}_2|}-\frac{1}{|\vec{r}_1-\vec{r}_2|}\right).\end{aligned} \tag{2.14}$$

The current $\vec{J}^{(3)}$ is transverse. We observe that this current generates a nonzero chromomagnetic field whenever the charges are not parallel, $Q^{(1)}\times Q^{(2)}\neq 0$.

The inversion of $\nabla\times(\nabla\times A^{(3)})=J^{(3)}$ to find $A^{(3)}$ is ambiguous in that we may add to it a solution of the homogeneous equation. To eliminate this ambiguity and to facilitate comparison of the classical results with the results of Sec. III as well as with the results of QCD computed in the radiation gauge, we impose the radiation-gauge condition $\vec{\nabla}\cdot\vec{A}=0$. To the extent that we are working in perturbation theory with small fields, we are insensitive to the Gribov-Mandelstam ambiguities.[31,32]

The contribution of $\vec{A}^{(3)}$ leads to a correction of order e^3g^2 to A^0. Using Eq. (2.10a), we find

$$-\nabla^2A^0_{(5)}=g\nabla^jA^j_{(3)}\times A^0_{(1)}+gA^j_{(3)}\times\nabla^jA^0_{(1)}\,. \tag{2.15}$$

Again, the constant term in $A^0_{(5)}$ must be determined so that (2.10c) is satisfied.

The vector potential [Eq. (2.14)] and the contribution to A^0 [Eq. (2.15)] give an order-e^4g^2 contribution to the energy of the form

$$E^{(6)}=-e^4g^2\mathcal{K}(|r_1-r_2|)(Q^{(1)}\times Q^{(2)})^2\,. \tag{2.16}$$

The coefficient $\mathcal{K}(|r_1-r_2|)$ may be written as

$$\mathcal{K}=\tfrac{1}{2}\int d^3\vec{r}\,d^3\vec{r}\,'\vec{J}_i(\vec{r})\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\vec{J}_i(\vec{r}\,')\,, \tag{2.17}$$

where

$$\begin{aligned}\vec{J}(\vec{r})\equiv\frac{1}{4\pi}&\left(\frac{1}{|r-r_1|}-\frac{1}{|r_1-r_2|}\right)\\&\times\vec{\nabla}\left(\frac{1}{|r-r_2|}-\frac{1}{|r_1-r_2|}\right).\end{aligned} \tag{2.18}$$

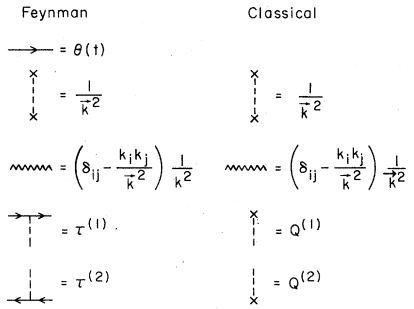

FIG. 1. Elements of Feynman and classical tree-graph perturbation theory.

The function $\mathcal{K}$ is encountered in evaluating the "H-graph" contribution to the quark-antiquark potential in QCD. In this feature, the QCD H graph reflects a classical effect. It is interesting to note that in QCD this graph represents the dominant contribution to the coefficient of the g^6 color-singlet potential by at least two orders of magnitude.[33]

Indeed, there is a general parallelism between the classical perturbation expansion of the energy and the corresponding expansion in QCD. If we introduce $\hbar$ in the QCD perturbation theory and introduce e's rather than g's at charge-Coulomb vertices, we see that the QCD expansion for the potential will consist of terms of the form $e^2(eg)^n(g^2\hbar)^m$. The terms which correspond to the classical perturbation expansion are those which survive the formal limit $\hbar\to 0$ with e and g fixed. Topologically, these are graphs containing no gluon loops (see Figs. 1 and 2).

Despite the similarities between some impor-

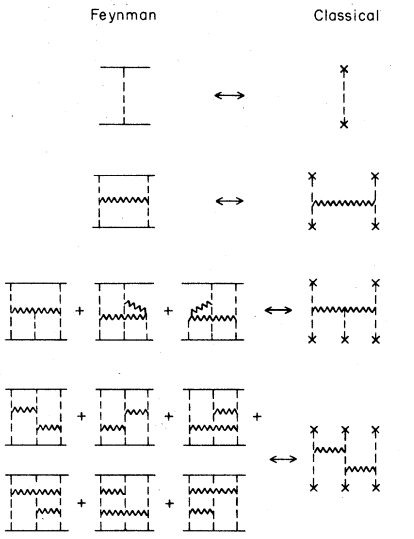

FIG. 2. Correspondence between Feynman graphs and classical tree graphs.

tant sets of Feynman diagrams and classical tree graphs, there is a fundamental obstacle to straightforward application of classical results to the quantum problem. This obstacle is posed by the charge structure of the two theories, a point which has been discussed at length elsewhere.[22] In the quantum theory the charge of a particle is an operator in a space of color states, and satisfies the equal-time algebra

$$[Q_a^{(\alpha)}, Q_b^{(\beta)}] = i\delta_{\alpha\beta} f_{abc} Q_c^{(\alpha)} . \tag{2.19}$$

In small representations (e.g., the fundamental representation appropriate to quarks), the commutator of two charges is of the same order of magnitude as the charges themselves and cannot be reasonably neglected.

The approximation discussed in Sec. III is inspired by the parallels between classical and quantum chromodynamics. We shall retain, however, the essentially quantum nature of the quark charge operators. Our approximation represents a truncation of the full set of states of quarks and gluons to a set of states appropriate to the description of heavy quarks, and of gluons in a single spatial wave function. These states are similar to the gluon coherent states which are appropriate to the description of classical gluon configurations.

III. THE SEMICLASSICAL APPROXIMATION

We now turn to the problem of developing a semiclassical approximation to the quark-antiquark interaction which takes into account the charge structure of QCD. In the infinite-quark-mass limit, quark recoil and spin effects are completely suppressed. In this limit, we may therefore introduce one-dimensional fermionic operators $q_a(t)$ and $\bar{q}_a(t)$, $a = 1, \ldots, N$, which describe a quark and antiquark at the fixed positions r_q and $r_{\bar{q}}$.[15,33] The Lagrangian for this system is

$$L = \int d^3\vec{x} \left[+\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + q^\dagger \left(\frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial t} - g\tau \circ A^0(\vec{r}_q, t) \right) q \right.$$
$$\left. + \bar{q}^\dagger \left(\frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial t} + g\tau^* \circ A^0(\vec{r}_{\bar{q}}, t) \right) \bar{q} \right], \tag{3.1}$$

where the matrices τ are the matrix generators of SU(N) in the fundamental representation.

The quark charge density is expressed as

$$\rho_a(\vec{r}, t) = Q_a(t)\delta^{(3)}(\vec{r} - \vec{r}_q) + \bar{Q}_a(t)\delta^{(3)}(\vec{r} - \vec{r}_{\bar{q}}) , \tag{3.2}$$

where

$$Q_a(t) \equiv q^\dagger(t)\tau_a q(t) \tag{3.3a}$$

and

$$\bar{Q}_a(t) \equiv -\bar{q}^\dagger(t)\tau_a^* \bar{q}(t) . \tag{3.3b}$$

The quark and antiquark number operators $q^\dagger q$ and $\bar{q}^\dagger \bar{q}$ are conserved. We shall work exclusively in the sector of the theory where $q^\dagger q$ and $\bar{q}^\dagger \bar{q}$ have eigenvalue 1. In this sector, the charges Q_a and $\bar{Q}_a$ satisfy the algebraic relations

$$Q_a Q_b = \frac{1}{2}\left(\frac{1}{N}\delta_{ab} + d_{abc} Q_c + i f_{abc} Q_c \right) , \tag{3.4a}$$

$$\bar{Q}_a \bar{Q}_b = \frac{1}{2}\left(\frac{1}{N}\delta_{ab} - d_{abc} \bar{Q}_c + i f_{abc} \bar{Q}_c \right) . \tag{3.4b}$$

It is convenient to work with the Hamiltonian form of the theory in the radiation gauge,

$$\vec{\nabla} \cdot \vec{A}_a = 0 . \tag{3.5}$$

In this gauge, the separation between the independent degrees of freedom of the transverse gluon field and the constrained nature of the longitudinal electric field is particularly clear.

The radiation-gauge Hamiltonian is[15]

$$H = \int d^3\vec{x} \tfrac{1}{2} [\vec{E}_\perp^{\,2}(\vec{x}) + \vec{B}^2(\vec{x}) + \vec{E}_L^{\,2}(\vec{x})] , \tag{3.6}$$

where the chromomagnetic field is $\vec{B}$, the transverse chromoelectric field is $\vec{E}_\perp$, and the longitudinal electric field is defined in terms of the independent degress of freedom by

$$\vec{E}_L^a(\vec{r}, t) = -\vec{\nabla} \int d^3\vec{r}\,' \sum_b G_{ab}(\vec{r}, \vec{r}\,'; t|\vec{A}) J_b^0(\vec{r}\,' ; t). \tag{3.7}$$

The charge density J^0 generated by quarks and transverse quanta is

$$J_a^0(\vec{r}, t) = \rho_{\text{quark}}^a(\vec{r}, t) + g f^{abc} \vec{E}_{\text{tr}}^b(\vec{r}, t) \cdot \vec{A}^c(\vec{r}, t) . \tag{3.8}$$

The Green's function $G_{ab}(\vec{r}, \vec{r}\,'; t|\vec{A})$ satisfies

$$[-\nabla^2 \delta_{aa'} - g f_{aca'} \vec{A}^c(\vec{r}, t) \cdot \vec{\nabla}] G_{a'b}(\vec{r}, \vec{r}\,'; t|\vec{A})$$
$$= \delta_{ab} \delta^{(3)}(\vec{r} - \vec{r}\,') . \tag{3.9}$$

The operators E_{tr} and A satisfy the canonical radiation-gauge commutation relations

$$[E_{i,a}^{\text{tr}}(\vec{r}, t), E_{j,b}^{\text{tr}}(\vec{r}\,', t)]$$
$$= [A_{i,a}(\vec{r}, t), A_{j,b}(\vec{r}\,', t)] = 0 ,$$
$$[E_{i,a}^{\text{tr}}(\vec{r}, t), A_{j,b}(\vec{r}\,', t)] \tag{3.10}$$
$$= i\delta_{ab} \left(\delta^{ij} + \nabla^i \frac{1}{-\nabla^2} \nabla^j \right) \delta^{(3)}(\vec{r} - \vec{r}\,') .$$

Mandelstam and Gribov have pointed out difficulties in properly defining a quantized Yang-Mills theory in the radiation gauge.[31,32] These difficulties stem from the nonuniqueness of the construction of G in the solution of Eq. (3.9). This nonuniqueness is known to arise if $\vec{A}$ is a sufficiently singular function of $\vec{r}$. In the analysis

we shall perform, the vector potentials we consider are sufficiently nonsingular that the Mandelstam-Gribov ambiguities would appear to give no problem. Whether or not these ambiguities appear outside the analysis we present is a question we shall not address. It seems possible to adopt our techniques to other gauges if the radiation gauge is singular.

A representation of $\vec{A}$ and $\vec{E}_{tr}$ in terms of plane-wave creation and annihilation operators is

$$\vec{A}_a(\vec{r},t)=\int\frac{d^3k}{(2\pi)^3 2k}[e^{ik\cdot x}\vec{a}_a(\vec{k})+e^{-ik\cdot x}\vec{a}_a^{\dagger}(\vec{k})],$$
$$\vec{E}_a^{tr}(\vec{r},t)=\frac{i}{2}\int\frac{d^3k}{(2\pi)^3}[e^{ik\cdot x}\vec{a}_a(\vec{k})-e^{-ik\cdot x}\vec{a}_a^{\dagger}(\vec{k})]. \tag{3.11}$$

The operators $\vec{a}_a(\vec{k})$ are transverse, with

$$\vec{k}\cdot\vec{a}_a(\vec{k})=0, \tag{3.12}$$

and satisfy the commutation relations

$$[a_a^i(\vec{k}),a_b^j(\vec{k}')]=0,$$
$$[a_a^i(\vec{k}),a_b^{j\dagger}(\vec{k}')]=\delta^{ab}\left(\delta^{ij}-\frac{k^ik^j}{k^2}\right)(2\pi)^3 \times 2k\delta^{(3)}(\vec{k}-\vec{k}'). \tag{3.13}$$

This plane-wave oscillator basis is not, however, the most profitable basis in which to begin making approximations. From analysis of the classical external field problem, we expect that a coherent chromoelectric and chromomagnetic field will develop in the presence of charges. This observation suggests we transform to a different basis which is better tailored to suit such a situation. We therefore write

$$a_a^i(\vec{k})=\sum_n\psi_n^i(\vec{k})a_n^a. \tag{3.14}$$

The wave functions $\psi_n^i(\vec{k})$ are transverse,

$$\vec{k}\cdot\psi_n(\vec{k})=0, \tag{3.15}$$

and complete,

$$\sum_n\psi_n^i(\vec{k})\psi_n^{j*}(\vec{k})=\left(\delta^{ij}-\frac{k^ik^j}{k^2}\right)(2\pi)^3 \times 2k\delta^{(3)}(\vec{k}-\vec{k}'). \tag{3.16}$$

This completeness condition guarantees the orthonormality condition

$$\int\frac{d^3k}{(2\pi)^3 2k}\psi_n^*(\vec{k})\cdot\psi_m(\vec{k})=\delta_{nm} \tag{3.17}$$

and the commutation relations

$$[a_n^a,a_m^b]=0,\quad [a_n^a,a_m^{b\dagger}]=\delta^{ab}\delta^{nm}. \tag{3.18}$$

We can now insert Eq. (3.14) into Eq. (3.6) and attempt to diagonalize the Hamiltonian. A full diagonalization would, of course, be extremely difficult, because of the nonlinear couplings of the infinite set of coherent modes. We shall proceed by truncating the Hamiltonian to the sector which involves only one family of excitations with wave function $\psi_0^i(\vec{k})$. A truncation to such a small number of modes could directly offer a good approximation if modes of high excitation are relatively unimportant in the dynamics of the $q\bar{q}$ interaction. We know, on the contrary, that highly excited modes control the high-momentum (short-distance) structure of the field theory. This short-distance structure is important for converting an expansion in α into an expansion in a scale-dependent charge, $\alpha(R)$. For the two-charge problem, we attempt to take into account the effect of these high modes by expanding in a coupling appropriate to the momentum scale given by R. The dependence of $\alpha(R)$ on R may be estimated by solving the Gell-Mann–Low equation. Put another way, we use the renormalization group to minimize the effect of highly excited coherent modes, and thus allow for a consistent truncation to a small number of coherent modes. In Sec. IV we shall compare our truncated calculation to renormalization-group improved perturbation theory. We shall find good agreement.

The procedure we employ was first devised by Tomonaga in the context of the static model of p-wave pion-nucleon interactions. It yields a variational estimate of the ground-state energy of the normal-ordered Hamiltonian. In the implementation of the Tomonaga approximation, we first consider a Hamiltonian $\vec{H}$ obtained from the original Hamiltonian by truncation to the sector of states involving only gluon excitations in $n=0$ modes of the basis ψ_n^i. We next diagonalize this Hamiltonian and find its lowest energy $\mathcal{E}_0$ as a functional of the wave function ψ_0^i. We then compute the optimal ψ_0^i by minimizing $\mathcal{E}_0[\psi_0^i]$ over the space of normalized, transverse ψ_0^i's. This procedure yields a variational estimate of the minimum energy of the system of gluons and quarks.

We note that this procedure preserves the charge structure of the theory. The truncated Hamiltonian depends on the quark and antiquark charge operators and one set of N^2-1 gluon creation and annihilation operators, $a^a\equiv a_{n=0}^a$, $a=1,\ldots,N^2-1$. The total charge

$$Q_T=ia\times a^{\dagger}+Q+\bar{Q} \tag{3.19}$$

is conserved.

Before proceeding to an explicit calculation, we make one further approximation. The Hamiltonian of Eq. (3.7) in the Tomonaga approximation still describes a complicated, nonlinear system. The nonlinearities arise from the nonpolynomial

dependence of $\vec{E}_L$ on $\vec{A}$. We have not succeeded in diagonalizing this nonpolynomial Hamiltonian, even in the $n=0$ sector.

The truncated Hamiltonian can, however, be diagonalized in a mean-field approximation. The mean-field approximation we employ linearizes the E_L^2 contribution to H with respect to $E_{\rm tr}$ and A. For a system of a quark at position $\vec{r}_1$ and an antiquark at $\vec{r}_2$,

$$\tfrac{1}{2}\vec{E}_L^{\,2} \cong \frac{g^2}{4\pi|\vec{r}_1-\vec{r}_2|}Q\cdot\bar{Q} - g^3\int d^3r' Q\times\bar{Q}\cdot\vec{A}(\vec{r}',t)\cdot\left(\frac{1}{4\pi|\vec{r}_1-\vec{r}'|}\vec{\nabla}'\frac{1}{4\pi|\vec{r}'-\vec{r}_2|}\right). \tag{3.20}$$

This mean-field approximation retains those contributions to H which shift local gauge-invariant operators from their values in the absence of sources to nonzero average values. The approximation ignores distortions of gluon wave propagation due to interactions with quarks, and ignores interactions of transverse gluons among themselves. The region of validity of this approximation is briefly discussed in Sec. IV.

The second term in Eq. (3.20) for E_L^2 may be interpreted as a chromomagnetic contribution to the energy generated by a current source $\vec{J}$. Since only the transverse part of this current contributes to the integral, we can write

$$\tfrac{1}{2}\vec{E}_L^{\,2} \cong \frac{g^2 Q\cdot\bar{Q}}{4\pi|\vec{r}_1-\vec{r}_2|} - g^3(Q\times\bar{Q})_a\int d^3\vec{r}\,\vec{A}_a(\vec{r},t)\cdot\vec{J}(\vec{r}), \tag{3.21}$$

where

$$\vec{\nabla}\cdot\vec{J}=0, \tag{3.22}$$

and, explicitly,

$$\vec{J}(\vec{r}) = \left(\frac{1}{4\pi|\vec{r}-\vec{r}_1|}-\frac{1}{4\pi|\vec{r}_1-\vec{r}_2|}\right)\times\vec{\nabla}\left(\frac{1}{4\pi|\vec{r}-\vec{r}_2|}-\frac{1}{4\pi|\vec{r}_1-\vec{r}_2|}\right). \tag{3.23}$$

The current generates chromoelectric and chromomagnetic fields, and induces a cloud of transverse virtual gluon radiation. This current is represented by the graph of Fig. 3.

The Hamiltonian in the Tomonaga mean-field approximation is

$$\tilde{H} = \mathcal{E}_0 a^\dagger\cdot a + \beta Q\cdot\bar{Q} + Q\times\bar{Q}\cdot(a\gamma + a^\dagger\gamma^*). \tag{3.24}$$

The parameters $\mathcal{E}_0$, β, and γ are functionals of $\psi_0^i(\vec{k})$:

$$\mathcal{E}_0 = \int\frac{d^3k}{(2\pi)^3 2}\vec{\psi}_0^*(\vec{k})\cdot\vec{\psi}_0(\vec{k}), \tag{3.25}$$

$$\beta = \frac{g^2}{4\pi|\vec{r}_1-\vec{r}_2|}, \tag{3.26}$$

and

$$\gamma = -g^3\int\frac{d^3k}{(2\pi)^3 2k}\vec{J}(\vec{k})\cdot\vec{\psi}_0(\vec{k}). \tag{3.27}$$

The truncated Hamiltonian $\tilde{H}$ describes a complicated system. We know of no analytic solution to the equations describing this system. The numerical diagonalization of $\tilde{H}$, together with the numerical solution to the Tomonaga mean-field Hamiltonian for color-singlet states, comprises the subject of a later paper. In the Appendix, we briefly discuss the procedure by which $\tilde{H}$ is diagonalized for singlet states. The procedure described is applied to the perturbative calculation of the quark-antiquark potential in Sec. IV.

The wave function $\psi_0^i(\vec{k})$ is determined by the minimization of the smallest eigenvalue of $\tilde{H}$, subject to the renormalization and transversality constraints on $\psi_0^i(\vec{k})$. The minimization becomes

$$0 = \frac{\delta}{\delta\psi_0^i(\vec{k})}\left\{\mathcal{E}^{\min}[\psi_0^i] + \lambda\int\frac{d^3k}{(2\pi)^3 2k}\psi_0^i(\vec{k})^*\cdot\psi_0^i(\vec{k})\right\}, \tag{3.28}$$

where λ is the Lagrange multiplier which imposes the normalization constraint. The variation of $\mathcal{E}^{\min}$ is determined by first-order perturbation theory in $\tilde{H}$ about its state $|\Omega\rangle$. We obtain

$$\psi(\vec{k}) = \psi^*(\vec{k}) = \frac{g^3}{2}\frac{\langle\Omega|(a+a^\dagger)\circ(Q\times\bar{Q})|\Omega\rangle}{\langle\Omega|a^\dagger\circ a|\Omega\rangle}\cdot\frac{J(\vec{k})}{k+\Lambda}, \tag{3.29}$$

with the parameter Λ defined as

$$\Lambda \equiv \langle\Omega|a^\dagger\cdot a|\Omega\rangle^{-1}\lambda. \tag{3.30}$$

The transversality condition is satisfied by Eq. (3.29).

The functional form of $\psi_0^i(\vec{k})$ is completely specified by the known current $J(\vec{k})$ and Λ. The parameter Λ must be chosen so that ψ_0^i has norm

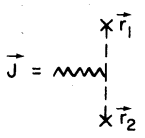

FIG. 3. Classical current $\vec{J}$.

1. Because of the implicit dependence of ψ, and therefore Λ, on the ground state $|\Omega\rangle$, the solution of the system of equations (3.29) and (3.30) is not straightforward. For each g there is some Λ which solves the system of equations. The parameter $\Lambda(g)$ may be determined numerically. In Sec. IV we discuss the solution to this system of equations in perturbation theory.

IV. PERTURBATION THEORY

In this section, we compute the energy of our effective quark-antiquark Hamiltonian to order $g^8 \ln g^2$ in perturbation theory. The result of our computation agrees with the result calculated from corresponding QCD Feynman graphs.

It is very difficult to disentangle analytically the relationships between the parameters $\mathcal{E}_0$, β, γ, and Λ, and g and R as given by Eqs. (3.25)–(3.27) and (3.29). This difficulty arises from the implicit dependence of $\mathcal{E}_0$, β, γ, and Λ on the ground-state wave function of our effective Hamiltonian.

The structure of this problem can be clarified by a few algebraic manipulations. First, we scale the R dependence out of all dimensional quantities. We can then measure $\mathcal{E}_0$, β, γ, and Λ in units of $1/R$, which units correspond to setting $R=1$ in all equations for these parameters.

Next, we define integral functions of Λ as

$$C_n(\Lambda) \equiv \int \frac{d^3k}{(2\pi)^3 2k} \frac{1}{(k+\Lambda)^n} \vec{J}(\vec{k})^2 . \tag{4.1}$$

For small Λ, $C_1(\Lambda)$ becomes a constant, while $C_2(\Lambda)$ is proportional to $\ln\Lambda$.

We further define the ratio of ground-state expectation values which occur in Eq. (3.29) as

$$P \equiv \frac{\langle\Omega|(a+a^\dagger)\cdot(Q\times\bar{Q})|\Omega\rangle}{2\langle\Omega|a^\dagger\cdot a|\Omega\rangle} . \tag{4.2}$$

The normalization condition on ψ, which implicitly defines Λ, becomes

$$g^6 C_2(\Lambda) P^2 = 1 . \tag{4.3}$$

Finally, we write a rescaled effective Hamiltonian $\mathcal{H}$ as

$$\mathcal{H} \equiv \tilde{H}/\mathcal{E}_0 = a^\dagger\cdot a + \bar{\beta} Q\cdot\bar{Q} + \bar{\gamma} Q\times\bar{Q}\cdot(a+a^\dagger) . \tag{4.4}$$

If we combine Eqs. (3.25)–(3.27) with Eq. (4.3) we have

$$\mathcal{E}_0 = \frac{C_1 - \Lambda C_2}{C_2} , \tag{4.5}$$

$$\bar{\beta} = \frac{\beta}{\mathcal{E}_0} = \frac{g^2}{4\pi}\frac{C_2}{C_1 - \Lambda C_2} , \tag{4.6}$$

and

$$\bar{\gamma} = \frac{\gamma}{\mathcal{E}_0} = -g^3 \frac{C_1 C_2^{1/2}}{C_1 - \Lambda C_2} . \tag{4.7}$$

We see from Eqs. (4.5)–(4.7) that for any g, Λ determines $\mathcal{E}_0$, $\bar{\beta}$, and $\bar{\gamma}$. Given $\bar{\beta}$ and $\bar{\gamma}$, we can determine the ground-state wave function of $\mathcal{H}$, from which we can then determine the ratio P. The parameter Λ is then obtained as a function of g by the normalization condition of Eq. (4.3). The explicit determination of Λ analytically for arbitrary g^2 appears to be an intractable problem. We shall present a detailed numerical analysis in a later paper. We consider only perturbation theory for small g^2 in this paper.

We begin the perturbative analysis by observing that $\bar{\beta}$ is of order g^2, and $\bar{\gamma}$ is of order g^3 for small g. The Hamiltonian $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I , \tag{4.8}$$

where

$$\mathcal{H}_0 = a^\dagger\cdot a + \bar{\beta} Q\cdot\bar{Q} \tag{4.9}$$

and

$$\mathcal{H}_I = \bar{\gamma} Q\times\bar{Q}\cdot(a^\dagger + a) . \tag{4.10}$$

We shall find the ground-state wave function and energy in perturbation theory in $\mathcal{H}_I$.

The structure of $\mathcal{H}$ and the eigenstates of $\mathcal{H}_0$ are discussed in some detail in the Appendix. The unperturbed color-singlet ground state is denoted by $|0\rangle$,

$$(Q+\bar{Q})_a|0\rangle = 0 , \quad a_a|0\rangle = 0 . \tag{4.11}$$

In first-order perturbation theory, this state mixes only with the first radial excitation

$$|1\rangle = \left[\frac{8}{N(N^2-1)}\right]^{1/2} a^\dagger\cdot(Q\times\bar{Q})|0\rangle , \tag{4.12}$$

and the ground-state wave function is

$$|\psi\rangle = |0\rangle - \frac{\bar{\gamma}}{2}\left[\frac{N(N^2-1)}{2}\right]^{1/2}\frac{1}{1+\frac{1}{2}N\bar{\beta}}|1\rangle + O(\bar{\gamma}^2) . \tag{4.13}$$

The first-order correction to the ground-state energy appears in second-order perturbation theory. We find

$$\epsilon = -\frac{N^2-1}{2N}\left[\bar{\beta} + \frac{\bar{\gamma}^2 N^2}{4(1+\frac{1}{2}N\bar{\beta})} + O(\bar{\gamma}^4)\right] . \tag{4.14}$$

The quantity P, defined in Eq. (4.7), is

$$P = \frac{1+\frac{1}{2}N\bar{\beta}}{\bar{\gamma}}[1+O(\bar{\gamma}^2)] . \tag{4.15}$$

We can now solve for Λ as a function of g^2. The normalization condition of Eq. (4.3) is

$$1 = g^6\frac{(1+\frac{1}{2}N\bar{\beta})^2}{\bar{\gamma}^2} C_2(\Lambda)[1+O(\bar{\gamma}^2)] . \tag{4.16}$$

Using Eqs. (4.6) and (4.7) to express $\bar{\beta}$ and $\bar{\gamma}$ in

terms of Λ, C_1, and C_2, we find

$$1=\left[1+\left(\frac{Ng^2}{8\pi}-\Lambda\right)\frac{C_2(\Lambda)}{C_1(\Lambda)}\right]^2[1+O(g^6)]. \quad (4.17)$$

We have, therefore,

$$\Lambda=Ng^2/8\pi+O(g^6). \quad (4.18)$$

The energy corresponding to the scaled energy ϵ is

$$E=-\frac{1}{R}\frac{N^2-1}{2N}\left[\frac{g^2}{4\pi}+\frac{N^2g^6}{4}C_1(\Lambda)\frac{Ng^2}{8\pi}+O(g^{10})\right]. \quad (4.19)$$

The expansion of C_1 for small Λ is performed in Ref. 33. We find that

$$C_1(\Lambda)=\frac{3}{2(4\pi)^3}\left(1-\frac{\pi^2}{12}\right)+\frac{2}{3\pi(4\pi)^3}\Lambda\ln\Lambda+O(\Lambda). \quad (4.20)$$

Equation (4.19) becomes, therefore,

$$E=-\frac{T_2}{R}\alpha_s\left[1+\tfrac{3}{8}C_2{}^2\alpha_s{}^2\left(1-\frac{\pi^2}{12}\right)+\frac{1}{12\pi}C_2{}^3\alpha_s{}^3\ln\alpha_s+O(\alpha_s{}^3)\right], \quad (4.21)$$

where

$$T_2=(N^2-1)/2N \quad (4.22)$$

and

$$C_2=N, \quad (4.23)$$

T_2 and C_2 being the quadratic Casimir operators of the fundamental and adjoint representations of SU(N).

The first term in Eq. (4.21), of order g^2, is the Coulomb interaction in a color-singlet state. The correction term

$$\Delta E=-\frac{g^6}{4R}T_2C_2{}^2C_1\left(\frac{N}{2}\alpha_s\right) \quad (4.24)$$

is the sum of all H graphs containing arbitrary numbers of Coulomb exchanges (Fig. 4). This fact is most easily seen by evaluating the Feynman graphs for the four-point amplitude at zero external momentum in a mixed momentum–relative-time basis. The bare quark-antiquark pair propagator over relative time t is $\Theta(t)$ (Fig. 5). The effect of summing Coulomb exchanges is to produce a phase proportional to the

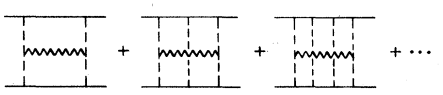

FIG. 4. H graph and its modifications arising from Coulomb exchange.

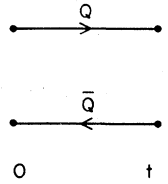

FIG. 5. Quark-antiquark pair propagator.

Coulomb interaction energy, a phase which modifies the pair propagator

$$\Theta(t)\to\Theta(t)e^{-i(\alpha_s/R)Q\cdot\bar{Q}t}. \quad (4.25)$$

The sum of these Coulomb exchanges is represented graphically in Fig. 6.

The time-dependent part of the integral representation for the sum of graphs shown in Fig. 4 arises from the transverse gauge field propagator, and from the ratio of phase factors between the singlet and N-plet states given by Eq. (4.25). The integrals over t and the gluon energy give

$$\int\frac{dk_0}{2\pi}\int_0^\infty dt\,\frac{e^{-i(N\alpha_s/2)t-ik_0t}}{k_0{}^2-\vec{k}^2+i\epsilon}=\frac{1}{k(k+\frac{1}{2}N\alpha_s)}. \quad (4.26)$$

When this result is combined with all relevant factors from the Coulomb gauge Feynman rules, we find precisely the result of Eq. (4.19), with C_1 expressed in the integral form given by Eq. (4.1).

We have seen that in perturbation theory the Tomonaga mean-field approximation has left intact the Coulomb interaction and the entire sum of Coulomb modified H graphs. That this should be so is not surprising. Our approximations have dropped only two of those effects which are included in QCD perturbation theory. We have truncated the longitudinal electric field $\vec{E}_L$ to a linear term in $\vec{A}$. This term gives the H graph. The Tomonaga approximation excises all fluctuations of the gluon field save those in the single mode with wave function $\vec{\psi}(\vec{k})$. The H graph includes only a one-gluon intermediate state, and the operator which creates this state is of the Tomonaga form. The Coulomb modifications of the H graph are also included in the truncation of $\vec{E}_L$ and involve a one-gluon intermediate state.

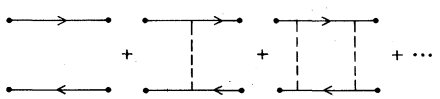

FIG. 6. Modifications of the pair propagator arising from Coulomb exchanges.

V. SUMMARY AND CONCLUSIONS

The object of this paper has been to develop an approximation to the description of the interaction of heavy quarks and antiquarks in QCD, an approximation stressing the classical aspects of the Yang-Mills theory while retaining the operator charge structure of the quantum theory. Our efforts have been motivated by the observation that the dominant $O(g^6)$ H-graph contribution to the color-singlet quark-antiquark potential arises from a classical effect, and by the hope that information about the classical quark-antiquark interaction will be applicable to the description of the quantum interaction. Our result is an approximation of the interacting quark, antiquark, and gluon system as a simple quantum-mechanical system in which all gluons share a single classical wave function. The validity of this approximation depends, however, on a variety of assumptions and simplifications which we shall now review.

We have described quarks in the infinite mass limit as stationary, pointlike objects which are entirely characterized by a (N^2-1)-component charge operator. This description is appropriate to the lowest-order contribution to the QCD interaction in inverse powers of the quark mass M_Q. Whether the $1/M_Q$ expansion is valid in QCD for nonsinglet states is unclear.[15,34] The only state we consider is a color singlet.

In our analysis, we have not addressed the interesting problems of including quark kinetic energies and spin-dependent forces. These contributions appear in the lowest-order $1/M_Q{}^2$ corrections to our results. Our procedure is based on an essentially variational analysis of the normal-ordered QCD Hamiltonian. The solution to our equations gives an approximation to the difference between the energy of the lowest state in the presence of a quark and antiquark and the energy of the vacuum.

To calculate this energy difference, we consider QCD in a Hamiltonian framework. The interpretation of this Hamiltonian appears simplest in a physical gauge, such as the Coulomb gauge which we have used in this paper. (We have not fully addressed the problems of properly fixing the gauge and of resolving potential problems arising from Gribov-Mandelstam ambiguities.) In the Coulomb gauge, the Hamiltonian is a complicated nonpolynomial function of the transverse gluon field. Normal ordering of the Coulomb operator $\int \frac{1}{2}E_L{}^2$ is related to renormalization of the theory. For example, the dominant contribution to the β function in one loop comes from the normal-ordering term shown in Fig. 7. The

FIG. 7. Normal-ordering correction to $\mathcal{H}$.

proper implementation of the normal ordering of the Coulomb-gauge Hamiltonian, and properly interpreting the Hamiltonian as a finite operator to all orders in g, appear to be very complicated problems.

In this paper we have sidestepped these problems by considering only the mean-field approximation to the Hamiltonian, and by assuming that a running coupling constant $\alpha(R)$ can be defined at a momentum scale of $O(1/R)$, chosen so that the finite normal-ordering terms such as that of Fig. 7 can be ignored. Although we can easily imagine going beyond the mean-field approximation, in practice it seems difficult to achieve a better approximation than $\alpha \to \alpha(R)$ and dropping of finite normal-ordering terms.

Finally, we note that there appears to be no obvious way in which the Tomonaga approximation might include the effects of a complicated phase structure for the vacuum. In the mean-field approximation, the gluon wave function contains contributions from low-momentum gluons. We expect, however, that in a confining theory these low-momentum contributions will be drastically modified because of a nontrivial vacuum phase structure. If these low-frequency gluons give important contributions to the quark-antiquark potential, then our model would not accurately represent the true structure of the quark-antiquark system.

Despite these reservations and our dependence on numerous assumptions and simplications, it is interesting how accurately our approximation reproduces the results of QCD perturbation theory. It is particularly reassuring that our method is sufficiently sensitive to treat accurately the nonanalyticity in the coupling reflected by $g^8 \ln g^2$.

In a forthcoming paper, we will present the results of a numerical diagonalization of H and the solution of the coupling constraints of Eq. (4.3) and Eqs. (4.5)–(4.7). Our hope is that these results, combined with the scale dependence of $\alpha(R)$, will give a reasonable approximation to the quark-antiquark interaction for intermediate coupling strength.

ACKNOWLEDGMENTS

We gratefully acknowledge some useful conversations with S. Adler, M. Dine, F. Feinberg, M. Kalb, J. Mandula, P. Sikivie, C. Sommerfield, and M. Weinstein. In particular, we especially thank J. Mandula and M. Weinstein for their enlightening suggestions. We also thank Alice McLerran for her aid in editing this manuscript. This work was supported by the Department of Energy under contracts No. EY-76-C-03-0515 and No. EY-76-C-02-3069.

APPENDIX

In this appendix, we discuss some properties of the scaled Hamiltonian of Eq. (4.4),

$$\mathcal{H} = a^\dagger \cdot a + \bar{\beta} Q \cdot \bar{Q} + \bar{\gamma}(Q \times \bar{Q}) \cdot (a + a^\dagger) . \tag{A1}$$

This Hamiltonian is invariant under global SU(N) color rotations generated by the operator

$$Q_T = ia \times a^\dagger + Q + \bar{Q} . \tag{A2}$$

It is also symmetric under charge conjugation:

$$Q \to \bar{Q} , \quad a \to -a , \quad a^\dagger \to -a^\dagger . \tag{A3}$$

The ground state of this system is expected to be a color-singlet state. In the color-singlet sector, $\mathcal{H}$ may be written in a simple and natural basis. We construct this basis by operating on the state $|0\rangle$. This state is the ground state of $\mathcal{H}$ for $\bar{\gamma} = 0$, and is formed using a product of ground-state harmonic oscillators and the singlet state of the total quark charge:

$$a|0\rangle = 0 , \quad (Q + \bar{Q})|0\rangle = 0 . \tag{A4}$$

The set of color-singlet states is generated from $|0\rangle$ by operating with arbitrary integer powers of $a^\dagger \cdot Q \times \bar{Q}$. These radially excited states have even and odd occupation numbers and can be written as

$$|2K\rangle = \left[\frac{(N^2-3)!!}{(2K)!!\,(2K+N^2-3)!!} \right]^{1/2} (a^{\dagger 2})^K |0\rangle , \tag{A5a}$$

$$|2K+1\rangle = \left[\frac{(N^2-3)!!}{(2K)!!\,(2K+N^2-1)!!} \right]^{1/2} \times (a^{\dagger 2})^K \left(\frac{8}{N} \right)^{1/2} a^\dagger \cdot (Q \times \bar{Q})|0\rangle . \tag{A5b}$$

The orthonormality of these states is easily verified by use of the identity

$$(Q \times \bar{Q})_a (Q \times \bar{Q})_b |\Omega\rangle = (N/8)\,\delta_{ab} |\Omega\rangle , \tag{A6}$$

where $|\Omega\rangle$ is any singlet state of $Q + \bar{Q}$.

In the basis given by Eq. (A5), the first two terms of $\mathcal{H}$ are diagonal. The interaction term proportional to $\bar{\gamma}$ mixes only states of adjacent occupation number. The nonvanishing matrix elements of $\mathcal{H}$ are

$$\langle 2K|\mathcal{H}|2K\rangle = 2K - \bar{\beta}\frac{N^2-1}{2N} , \tag{A7a}$$

$$\langle 2K+1|\mathcal{H}|2K+1\rangle = 2K+1+\bar{\beta}\frac{1}{2N} , \tag{A7b}$$

$$\langle 2K+1|\mathcal{H}|2K\rangle = \frac{\bar{\gamma}}{2}\left[N\left(K + \frac{N^2-1}{2} \right) \right]^{1/2} , \tag{A7c}$$

$$\langle 2K-1|\mathcal{H}|2k\rangle = \frac{\bar{\gamma}}{2}\sqrt{NK} . \tag{A7d}$$

Equations (A7) form the basis for the perturbative diagonalization of $\mathcal{H}$ discussed in Sec. IV.

Although in general $\mathcal{H}$ must be diagonalized numerically, for a special value of $\bar{\beta}$ the ground state may be exactly determined. To find this state, we define an operator Z as

$$Z = (8/N)^{1/2} a \cdot (Q \times \bar{Q}) . \tag{A8}$$

We easily compute

$$Z^\dagger Z|2K+1\rangle = 2K|2K\rangle \tag{A9a}$$

and

$$Z^\dagger Z|2K+1\rangle = (2K+N^2-1)|2K+1\rangle , \tag{A9b}$$

so that, operating on these states,

$$Z^\dagger Z = a^\dagger \cdot a + \frac{N^2-2}{N}(Q+\bar{Q})^2 . \tag{A10}$$

The Hamiltonian $\mathcal{H}$ may therefore be written as

$$\mathcal{H} = \left[Z^\dagger + \frac{\bar{\gamma}}{2}\left(\frac{N}{2} \right)^{1/2} \right]\left[Z + \frac{\bar{\gamma}}{2}\left(\frac{N}{2} \right)^{1/2} \right] + Q \cdot \bar{Q}\left[\bar{\beta} - \frac{2(N^2-2)}{N} \right] - \left[\frac{(N^2-1)(N^2-2)}{N^2} + \frac{\bar{\gamma}^2 N}{8} \right] . \tag{A11}$$

If

$$\bar{\beta} = \frac{2(N^2-2)}{N} , \tag{A12}$$

this Hamiltonian is

$$\mathcal{H} = \left[Z + \frac{\bar{\gamma}}{2}(N/2)^{1/2} \right]^\dagger \left[Z + \frac{\bar{\gamma}}{2}(N/2)^{1/2} \right] - \frac{(N^2-1)(N^2-2)}{N^2} - \frac{\bar{\gamma}^2 N}{8} . \tag{A13}$$

The product term in this equation is a nonnegative operator whose minimum eigenvalue corresponds to the state $|\psi\rangle$ such that

$$\left[Z + \frac{\bar{\gamma}}{2}(N/2)^{1/2} \right]|\psi\rangle = 0 . \tag{A14}$$

The ground-state energy is

$$E_0 = -\frac{(N^2-2)(N^2-1)}{N^2} - \frac{\bar{\gamma}^2 N}{8} . \tag{A15}$$

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