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# Phase diagrams of lattice gauge theories with Higgs fields

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We study the phase diagram of lattice gauge theories coupled to fixed-length scalar (Higgs) fields. We consider several gauge groups: $Z_2$, U(1), and SU($N$). We find that when the Higgs fields transform like the fundamental representation of the gauge group the Higgs and confining phases are smoothly connected, i.e., they are not separated by a phase boundary. When the Higgs fields transform like some representation other than the fundamental, a phase boundary may exist. This is the case for SU($N$) with all the Higgs fields in the adjoint representation and for U(1) with all the Higgs fields in the charge-$N$($N > 1$) representation. We present an argument due to Wegner that indicates the stability of the pure gauge transition. Another phase, free charge or Coulomb, is generally present. In this regime, the spectrum of the theory contains massless gauge bosons (for continuous groups) and finite-energy states that represent free charges.

## I. INTRODUCTION

### A. The problem of matter fields

The formulation of gauge theories on a lattice by Wilson and Polyakov[1] allows us to study these theories outside the realm of weak-coupling expansions. In particular, in the strong-coupling regime they are known to exhibit confinement of static sources.[1,2]

It is hoped that for non-Abelian groups in four space-time dimensions confinement persists for all couplings,[3] allowing one to make a continuum theory weakly coupled[4,5] at short distances, confining at large distances. The Abelian U(1) theory, that is, Polyakov's compact photodynamics,[6] in four dimensions is thought to be confining only down to a finite critical coupling $g_c$.[2,7-9] At $g_c$ a phase transition occurs, leading to a weak-coupling phase ($g<g_c$) characterized by the existence of massless photons and Coulomb-type forces between static sources. This picture (we hope) allows one to define a continuum theory where charged particles are free, such as conventional electrodynamics.

The above comments apply to pure gauge theories, possibly in the presence of static sources. Any attempt at realistic theory will necessarily include dynamic matter fields (e.g., quarks for quantum chromodynamics; Higgs scalars, leptons, and quarks for Weinberg-Salam theory; etc.). In certain regimes these matter fields can exert a dramatic effect on the behavior of the gauge theory. A prime example is the Higgs mechanism, where scalar fields interact with the gauge bosons rendering them massive and the forces they mediate short-ranged. It is important, therefore, to have some understanding of the combined matter-gauge system, in particular its phase diagram.

Dynamic matter fields immediately create a problem in classifying the phases of the theory. The criterion used for diagnosing confinement in the pure gauge theory, the energy between static sources, no longer works. Even if the energy starts increasing as the sources separate, it eventually becomes favorable to pop a particle-antiparticle pair out of the vacuum. This pair shields the gauge charge of the sources, and the energy stops growing. So even in a theory that "looks" very confining our signal fails.[10]

There are ways around this. In a noncompact Abelian theory one can introduce fractionally charged sources that cannot be shielded by integer-charged particles. For compact groups, however, charge is quantized,[11] and this trick is out. One can still imagine using matter fields in other than the fundamental representation,[12] more precisely, matter fields that cannot shield sources in the fundamental (e.g., fields in the adjoint representation). However, this still leaves open the question of the behavior of the theory when the matter fields carry the fundamental charge.

### B. The models

We shall restrict ourselves to lattice gauge theories coupled to scalar (Higgs) fields. To simplify the problem (without, we feel, throwing away any important physics) we freeze out the radial mode of the Higgs fields, working with fields with fixed norm $R$. Thus we shall be dealing with fields that are strictly compact.

The action of the model on a $d$-dimensional hypercubic lattice with finite lattice spacing (set to be one) reads

$$S[\phi(\vec{r});U_\mu(\vec{r})]=\frac{K}{2}\sum_{(\vec{r},\mu\nu)}\mathrm{Tr}[U_\mu(\vec{r})U_\nu(\vec{r}+\hat{e}_\mu)U_\mu^\dagger(\vec{r}+\hat{e}_\nu)U_\nu^\dagger(\vec{r})+\mathrm{h.c.}]+\frac{\beta}{2}\sum_{(\vec{r},\mu)}[\phi(\vec{r})\cdot D\{U_\mu(\vec{r})\}\cdot\phi^\dagger(\vec{r}+\hat{e}_\mu)+\mathrm{c.c.}]\,, \tag{1.1}$$

where

(i) $(\vec{r},\mu)$ labels the link with end points at the lattice sites $\vec{r}$ and $\vec{r}+\hat{e}_\mu$,

(ii) $(\vec{r},\mu\nu)$ labels the elementary plaquette defined by the links $(\vec{r},\mu)$ and $(\vec{r},\nu)$,

(iii) $\phi(\vec{r})$ is the Higgs field at site $\vec{r}$ and transforms like some $M$-dimensional irreducible representation of a compact gauge group $G$,

(iv) $U_\mu(\vec{r})$ are gauge group matrices residing at the link $(\vec{r},\mu)$,

(v) $D\{U_\mu(\vec{r})\}$ is an $M$-dimensional representation of $U_\mu(\vec{r})$.

The dimensionless coupling constants $\beta$ and $K$ are related to the gauge coupling constant $g$ and to the Higgs length $R$ through the relations $K=1/g^2$ and $\beta=R^2$.

The action (1.1) is invariant under arbitrary local gauge transformations $\{V(\vec{r})\}$ such that

$$\begin{aligned}U_\mu(\vec{r})\to U'_\mu(\vec{r})&=V(\vec{r})U_\mu(\vec{r})V^{-1}(\vec{r}+\hat{e}_\mu)\,,\\ \phi(\vec{r})\to\phi'(\vec{r})&=D(V(\vec{r}))\phi(\vec{r})\,,\end{aligned}\tag{1.2}$$

where $D(V)$ is the same matrix representation given above. For instance, if the gauge group $G$ is the Abelian group U(1) the model reads as follows:

$$\begin{aligned}\phi(\vec{r})&=\exp[i\theta(\vec{r})]\,,\quad 0\le\theta(\vec{r})<2\pi\,,\\ U_\mu(\vec{r})&=\exp[iA_\mu(\vec{r})]\,,\quad 0\le A_\mu(\vec{r})<2\pi\,.\end{aligned}\tag{1.3}$$

The action (1.1) takes the form

$$S_q[\theta(\vec{r});A_\mu(\vec{r})]=\beta\sum_{(\vec{r},\mu)}\cos[\Delta_\mu\theta(\vec{r})-qA_\mu(\vec{r})]+K\sum_{(\vec{r},\mu\nu)}\cos[F_{\mu\nu}(\vec{r})]\,. \tag{1.4}$$

The integer-valued parameter $q$ is the charge carried by the Higgs field. In Eq. (1.4) the field strength $F_{\mu\nu}(\vec{r})$ is defined by

$$F_{\mu\nu}(\vec{r})=\Delta_\mu A_\nu(\vec{r})-\Delta_\nu A_\mu(\vec{r})\,. \tag{1.5}$$

The gauge transformations for this Abelian example are

$$\begin{aligned}A_\mu(\vec{r})\to A'_\mu(\vec{r})&=A_\mu(\vec{r})+\Delta_\mu\alpha(\vec{r})\,,\\ \theta(\vec{r})\to\theta'(\vec{r})&=\theta(\vec{r})+q\alpha(\vec{r})\,,\end{aligned}\tag{1.6}$$

with $V(\vec{r})=\exp[i\alpha(\vec{r})]$.

### C. Limit models

The pure gauge theory and the Higgs model are recovered as limit situations of the model (1.1).

*(a) The Higgs-Heisenberg model* ($K=\infty$). When the gauge fields are frozen to pure gauge configurations ($K=\infty$) a family of models is obtained. These models have a global $G$ invariance and are generalized Heisenberg-$\sigma$ models.[13] Their action in an axial gauge $U_\mu(\vec{r})=I$ (identity of $G$) ($\mu=1$, for instance) is

$$S[\phi(\vec{r})]=\frac{\beta}{2}\sum_{(\vec{r},\mu)}[\phi(\vec{r})\cdot\phi^\dagger(\vec{r}+\hat{e}_\mu)+\mathrm{c.c.}]\,. \tag{1.7}$$

In particular, if $G=Z_2$ we get the Ising model, if $G=\mathrm{U}(1)$ we get the $XY$ model, and if $G=\mathrm{O}(N)$ we get the Heisenberg model. If the space-time dimensionality $d$ is high enough, two phases will be present.[14] When $\beta>\beta_c$ the global symmetry $G$ is *spontaneously* broken. The Higgs field develops a nonzero expectation value $\langle\phi\rangle$ and the correlation function (propagator) $\langle\phi(0)\cdot\phi^\dagger(\vec{r})\rangle$ has the asymptotic behavior

$$\langle\phi(0)\cdot\phi^\dagger(\vec{r})\rangle\underset{|\vec{r}|\to\infty}{\sim}\langle\phi\rangle^2\,\mathrm{const}\times\exp\left(\frac{\mathrm{const}}{|\vec{r}|^{d-2}}\right). \tag{1.8}$$

The behavior (1.8) is actually valid for *continuous symmetry groups* and it is obtained in the spin-wave (linearized) approximation. The $1/|\vec{r}|^{d-2}$ behavior is a consequence of the existence of Goldstone modes (spin waves) in the model. For discrete groups the excitations are always massive and the correlation function behaves like

$$\langle\phi(0)\cdot\phi^\dagger(\vec{r})\rangle\underset{|r|\to\infty}{\sim}\langle\phi\rangle^2+\mathrm{const}\times\exp\left(-\frac{|\vec{r}|}{\xi}\right), \tag{1.9}$$

where $\xi$ is the correlation length.

On the other hand, if $\beta<\beta_c$, the symmetry is restored. That is,

$$\langle\phi(0)\cdot\phi^\dagger(\vec{r})\rangle \underset{|r|\to\infty}{\sim} \text{const}\times\exp\left(-\frac{|\vec{r}|}{\xi}\right) \qquad (1.10)$$

for all the models. Note that even though the length of the Higgs field has been kept fixed ($R=\beta^{1/2}$), the symmetry here is normal (i.e., $\langle\phi\rangle=0$).

*(b) The pure gauge theory* ($\beta=0$). In this limit the Higgs fields decouple. The action now has the form

$$S_{\text{gauge}}[U_\mu(\vec{r})]=\frac{K}{2}\sum_{(\vec{r},\mu\nu)}\text{Tr}[U_\mu(\vec{r})U_\nu(\vec{r}+\hat{e}_\mu)\times U_\mu^\dagger(\vec{r}+\hat{e}_\nu)U_\nu^\dagger(\vec{r})+\text{H.c.}]\,. \qquad (1.11)$$

The model (1.11) has been studied by a number of authors with different techniques.[1-3,7-9,14,15,16] If the dimensionality is high enough[17] two phases are found.

If $K<K_c$ ($g>g_c$) we are in the strong-coupling regime. The behavior of the theory is characterized by the Wilson loop integral for sources in the fundamental representation

$$C_\Gamma=\left\langle\text{Tr}\left[\prod_{(\vec{r},\mu)\in\Gamma}U_\mu(\vec{r})\right]\right\rangle, \qquad (1.12)$$

where $\Gamma$ is a closed path of links. In the strong-coupling regime $C$ decays like

$$C_\Gamma\sim\exp(-\text{area of }\Gamma) \qquad (1.13)$$

for asymptotically large loops. The energy of two static fundamental sources $W(R)$ separated a distance $R$ during a time $T$ is given by

$$W(R)=-\frac{1}{T}\ln C_\Gamma\,. \qquad (1.14)$$

For a rectangular loop (1.13) and (1.14) give a linear potential (confinement).

In the weak-coupling phase $K>K_c$ ($g<g_c$) static fundamental sources are no longer confined. The Wilson loop obeys a perimeter law

$$C_\Gamma\sim\exp(-\text{perimeter of }\Gamma) \qquad (1.15)$$

and the force between the sources is weak. If the gauge group is discrete the force is exponentially damped (massive photon), while if the group is continuous the force is of a Coulomb type (massless photon).

### D. Phases of the theory

The phase diagram of the full theory depends crucially on whether the Higgs fields transform like the fundamental representation of $G$ or not. For instance, if the gauge group $G$ has a nontrivial center $C$ [like $Z_N$ for SU($N$)] it is possible to introduce Higgs fields that transform trivially under the center of $G$ (e.g., in the adjoint representation). If we introduce enough Higgs fields so that the gauge invariance $G/C$ is completely broken in the unitary gauge ($\phi$ = constant vector), a leftover local $C$ invariance will still survive even at $\beta=\infty$. In the SU($N$) example the result will be a $Z_N$ gauge theory.[18] In general, the Higgs fields may leave some subgroup of $G$ unbroken. Now a phase transition of the type discussed in (Sec. IB) may occur depending on which subgroup survives and on the space-time dimensionality. If this is the case we find that *three distinct phases* may generally occur:

*(a) A Higgs-mechanism-type phase.* Here the gauge bosons are massive. The force law is short-ranged and the Wilson loop exhibits a perimeter law ($\beta$ and $K$ large).

*(b) A free-charge or, for continuous groups, Coulomb phase.* Here, for continuous groups the gauge bosons are massless giving a Coulomb force between static sources. In general we shall see that in this phase there are finite-energy states that represent free charges ($\beta$ small, $K$ large).

*(c) A confinement phase.* In this regime the Wilson loop for fundamental sources has an area law. The gauge bosons are massive and there are no free charges in the spectrum.

When the Higgs fields are in the fundamental representation, however, the situation is drastically different. In this case, the unitary gauge completely breaks the gauge symmetry. If $\beta=\infty$ the gauge variables are locked at $U_\mu(\vec{r})$ = I (identity). Even if $\beta$ is finite but very large, not much can happen. Excitations are strongly suppressed and, in this limit, can be considered to be dilute. On the other hand, if $K\cong 0$, the theory represents a set of weakly coupled degrees of freedom living at the links of the lattice. Thus no phase between the Higgs ($\beta,K$ large) and confining ($\beta,K$ small) regimes can exist in this situation.

These arguments can be made precise. We shall show in the Appendix that, applying a result obtained by Osterwalder and Seiler,[19] the ground-state (vacuum) energy and all the Green's functions of the theory are analytic functions in a region of the ($\beta,K$) space that includes both the Higgs and confining regimes (Figs. 1 and 2). These two phases are *continuously connected*.

At first glance this result looks quite surprising. We should keep in mind, though, that most of what we know about Higgs fields and confinement comes

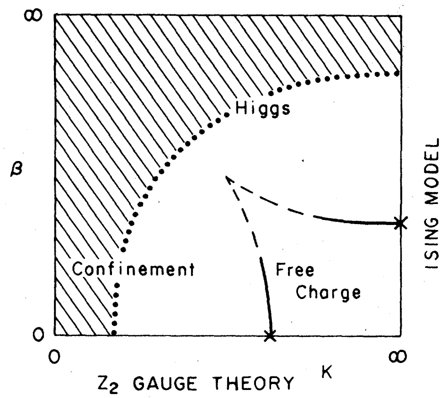


FIG. 1. The phase diagram for the $Z_2$ model ($d \geq 3$). The shaded region is where the bounds for analyticity hold. The full curves represent lines of second-order transitions given by (2.18). The broken lines are their extrapolation into the diagram. Notice that the analyticity region has a finite width at both the Higgs region ($K=\infty$) and confinement ($\beta=0$). Also note the curvature of the phase transition lines. The phases are described in the text.

from an approximate picture where one of the fields is either decoupled or frozen. In fact, all products of local operators that are candidates for distinguishing the two regimes turn out to have the same qualitative behavior in each (see Sec. II B). Furthermore, gauge-invariant operators

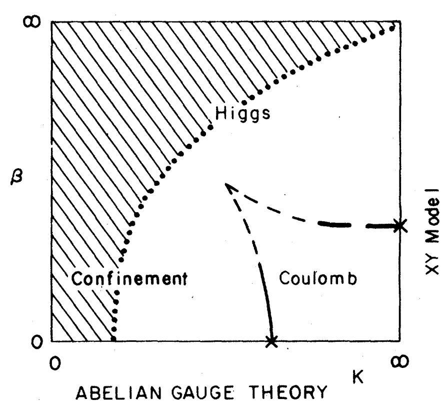


FIG. 2. Phase diagram for the Abelian model with Higgs fields in the fundamental representation ($d=4$). The broken line emerging from the $XY$ transition ($K=\infty$) is a line of first-order transitions. The full line that emerges from the pure gauge transition ($\beta=0$) is a line of transitions of the *same* order as the pure gauge critical point. Notice the curvature of the lines. The phases are described in the text.

such as

$$\phi(\vec{r})\, P\left\{\exp\left[\sum_{\Gamma(0,\vec{r})} A_\mu(\vec{x})dx_\mu\right]\right\}\phi^\dagger(0) \qquad (1.16)$$

create a "mesonlike" state in the strong-coupling regime, while in the Higgs regime they create a state with a massive photon (this is clearest in the unitary gauge). Thus the spectrum of the theory seems to be created by the same kind of operators in both regimes. For these and other reasons, Susskind has speculated that these two phases could be continuously connected.[20]

In general, if the Higgs fields are in the fundamental representation and all the gauge invariance has been broken no phase boundary will exist between the Higgs regime and confinement. Owing to analyticity we expect that the spectrum of the theory will evolve continuously from one regime to the other. Higgs fields and confinement are compatible phases. A theory can at the same time be confining and exhibit some sort of dynamical Higgs mechanism.

The pure gauge transition will be shown to be stable. The arguments are based on a study that Wegner[21] presented for the $Z_2$ model but that generalizes for any compact group and dimension (higher than the critical). A line of transitions emerging from the pure gauge critical point ($\beta=0, K=K_c$) is expected.

Generally, two phases will be present in this case:

(a) a Higgs-confinement phase,

(b) a free-charge or Coulomb (continuous groups) phase.

The two possible phase diagrams discussed above are, naturally, prototypes. They may change if, for instance, one of the pure transitions does not exist (generally the pure gauge transition). It is also possible to find more complicated situations depending on the structure of the Higgs sector.

Our analysis is done on a lattice with fixed, finite lattice spacing. The question of the continuum limit of these theories is still an open question. This problem has to be answered by means of a renormalization-group analysis.

The paper is organized as follows. In Sec. II we study the discrete $Z_2$ group. There we discuss most of the consequences of having the Higgs field in the fundamental representation since the calculations are much simpler. In Sec. III we consider the U(1) model (Abelian Higgs model). The results are generalized to non-Abelian groups [mainly SU($N$)] in Sec. IV, which also serves as a conclusion. In the Appendix we sketch the proof of analyticity referred to in the text.

## II. THE $Z_2$ MODEL

### A. The model

In this case the variables are of an Ising type:

$$\phi(\vec{r}) = \sigma(\vec{r}) = \pm 1\,,\qquad U_\mu(\vec{r}) = \pm 1\,. \tag{2.1}$$

The action (1.1) now looks like an Ising model coupled in a gauge-invariant manner to an Ising gauge theory,

$$S_{\text{Ising}}[\sigma(\vec{r}), U_\mu(\vec{r})] = \beta \sum_{(\vec{r},\mu)} \sigma(\vec{r})\sigma(\vec{r}+\hat{e}_\mu) + K \sum_{(\vec{r},\mu\nu)} \phi_{\mu\nu}(\vec{r})\,, \tag{2.2}$$

where the field strength $\phi_{\mu\nu}(\vec{r})$ through the plaquette $(\vec{r}, \mu\nu)$ is

$$\phi_{\mu\nu}(\vec{r}) = U_\mu(\vec{r})U_\nu(\vec{r}+\hat{e}_\mu)U_\mu(\vec{r}+\hat{e}_\nu)U_\nu(\vec{r})\,. \tag{2.3}$$

The action (2.2) is invariant under $Z_2$ gauge transformations

$$\sigma(\vec{r}) \to \sigma'(\vec{r}) = \sigma(\vec{r})s(\vec{r})\,,\qquad U_\mu(\vec{r}) \to U'_\mu(\vec{r}) = s(\vec{r})U_\mu(\vec{r})s(\vec{r}+\hat{e}_\mu)\,, \tag{2.4}$$

where $s(\vec{r}) = \pm 1$.

The generating functional (or partition function) for this model is defined by

$$Z_{\text{Ising}}(\beta, K) = \sum_{\{\sigma(\vec{r}), U_\mu(\vec{r})\}} \exp\{S_{\text{Ising}}[\sigma(\vec{r}), U_\mu(\vec{r})]\} \tag{2.5}$$

and the ground-state (vacuum) energy density by

$$\mathfrak{F}_{\text{Ising}}(\beta, K) = -\frac{1}{N}\ln Z_{\text{Ising}}(\beta, K)\,. \tag{2.6}$$

The limit models of this theory have been discussed in Sec. IC. The only distinctive characteristic of this model is the absence of massless modes (Goldstone and photons) in both limits because the symmetry is discrete. There are massless modes only at the phase transition (if it is second order).

### B. Higgs and confinement

(a) *Analysis of the order parameters.* First of all, it should be noticed that since the $Z_2$ group has only one nontrivial representation, the matter fields will be, by force, in the fundamental representation. Thus, the model will exhibit most of the general features produced by this situation, despite the simplicity of the $Z_2$ group.

What happens to the signatures of the pure phases when both fields are dynamical? Consider first the small-$K$ ("high-temperature") expansion of the Wilson loop in the pure gauge theory. For a square loop of linear dimensions $R$ and $T$ the result is

$$C_\Gamma \simeq (\tanh K)^{RT} + \cdots \simeq \exp(-\tau RT) \tag{2.7}$$

at the lowest order in $K$. Here $\tau = -\ln\tanh K$ is the "string tension" and $RT$ is the *area* of the square loop.

Now let $\beta$ be finite but small. In this case a kind of high-temperature expansion in $\beta$ can be performed. We find[22]

$$C_\Gamma \simeq (\tanh\beta)^{2(R+T)} + \cdots + (\tanh K)^{RT} + \cdots\,. \tag{2.8}$$

For a loop *asymptotically large* the "area" decaying term $(\tanh K)^{RT}$ is always exponentially smaller than the "perimeter" decaying term $(\tanh\beta)^{2(R+T)}$. Thus the long-distance behavior of the Wilson loop, for $\beta \neq 0$, is given by the perimeter law

$$C_\Gamma \simeq (\tanh\beta)^{2(R+T)} + \cdots \simeq \exp(-\lambda P)\,, \tag{2.9}$$

where $P = 2(R+T)$ is the perimeter of the loop and $\lambda = -\frac{1}{2}\ln\tanh\beta$. We see a sudden crossover from an area to a perimeter decay for any finite value of $\beta$. This perimeter dependence reflects the fact that a pair popped out of the vacuum can shield the external sources. In fact, perimeter behavior for all nonzero $\beta$ is a rigorous consequence of a Griffiths inequality. This crossover does not imply a singularity in the thermodynamic functions because the theory depends on two parameters, $\beta$ and $K$. In the pure gauge theory, however, there is only one parameter, $K$, and a change in the behavior of the loop signals a phase transition.

Now let us consider what has happened to the order parameter of the Ising model. The two-point function $\langle\sigma(\vec{0})\sigma(\vec{r})\rangle_{\beta,K}$ *is not gauge invariant,* so it vanishes identically for all values of the coupling $K$.[23] A possible way to make it gauge invariant is to insert a product of gauge variables along some path of links $\Gamma$ between $\vec{0}$ and $\vec{r}$.

The new operator reads

$$C_\Gamma(|\vec{r}|) \equiv \left\langle \sigma(\vec{0})\left(\prod_\Gamma U_\mu(\vec{r}\,')\right)\sigma(\vec{r})\right\rangle; \tag{2.10}$$

at $K = \infty$, the gauge variables can be set equal to one in a suitable gauge (see Sec. IC) and we obtain the correlation function of an Ising model,

$$\lim_{K\to\infty}\left\langle \sigma(0)\left(\prod_\Gamma U_\mu(\vec{r}\,')\right)\sigma(\vec{r})\right\rangle = \langle\sigma(0)\sigma(\vec{r})\rangle_{\text{Ising}}\,. \tag{2.11}$$

We can now compute $C_\Gamma(|\vec{r}|)$ when $K$ is large but finite by means of an excitation expansion valid when $\beta$ and $K$ are large. This expansion is the analog of the low-temperature expansion of the Ising model.

For very large $K$, the smallest excitation of the gauge fields ($d>2$) has $U_\mu=1$ at all the links of the lattice except one where $U_\mu=-1$. This flipped link variable gives field strength to all the plaquettes that share that link. In three dimensions this is a loop of field strength.[24]

In a dilute gas of excitations (or first cumulant approximation) $C_\Gamma(|\vec{r}|)$ has the behavior

$$\left\langle\sigma(0)\left(\prod_{\Gamma(\vec{0},\vec{r})} U_\mu(\vec{r}')\right)\sigma(\vec{r})\right\rangle \approx \exp\{-2|\vec{r}|\exp[-4K(d-1)-2\beta]\}. \quad (2.12)$$

We see that the gauge-invariant correlation function (2.10) decays exponentially for any finite value of $K$. It is important to note that the product of gauge variables is the source of the decay. Whenever one of the excitations crosses the string of gauge variables, the operator changes sign and the "gas of excitations" disorders the correlations.

However, it is in principle possible to write an operator that is invariant under gauge transformations but does not single out a given path as (2.10) does. The operator $\langle\sigma(0)\sigma(\vec{r})\rangle$ is not invariant, but its expectation value *in a fixed gauge* can nevertheless be nonzero.

A suitable gauge to study this operator is the "*minimal gauge*". It is defined as follows: Given a configuration of field strength $\{\Phi_{\mu\nu}(\vec{r})\}$, we choose $U_\mu(\vec{r})$ such that (i) it is consistent with the prescribed $\{\Phi_{\mu\nu}\}$ and (ii) it has a minimum number of links with $U_\mu=-1$. For certain configurations $\{\Phi_{\mu\nu}(\vec{r})\}$ it is possible to find more than one configuration $\{U_\mu\}$ that satisfies the minimal gauge prescription. This *gauge degeneracy* is not important if $K$ is very large but gets progressively worse as $K$ becomes smaller. This is a simple example of topological entropy[25] common in compact gauge theories.

Unlike $C_\Gamma(|\vec{r}|)$, (2.10), the correlation function in the minimal gauge does not develop an exponential decay in the dilute excitation limit. The important difference is that the string of $U_\mu$ variables is absent. The result to lowest order in $e^{-2K}$ is

$$\langle\sigma(0)\sigma(\vec{r})\rangle_{\text{minimal}} \approx \langle\sigma(0)\sigma(\vec{r})\rangle_{\text{Ising}}\left\{1-\left[\sum_{(\vec{r},\mu)}\left(\frac{\langle\sigma(0)\sigma(\vec{r})\rangle(\vec{r},\mu)}{\langle\sigma(0)\sigma(\vec{r})\rangle_{\text{Ising}}}-1\right)\right]\exp[-4K(d-1)-\beta W(\beta)]\right\}, \quad (2.13)$$

where $\langle\sigma(\vec{0})\sigma(\vec{r})\rangle(\vec{r},\mu)$ is the correlation function of an Ising model with a flipped bond at $(\vec{r},\mu)$ and $W(\beta)$ is the change in the free energy due to the flipped bond.[24] But the effects of a flipped bond are important only within a correlation length from the defect. Thus $\langle\sigma(\vec{0})\sigma(\vec{r})\rangle(\vec{r},\mu)$ can be different from $\langle\sigma(\vec{0})\sigma(\vec{r})\rangle_{\text{Ising}}$ if the flipped bond is close to $\vec{0}$ or $\vec{R}$ and (2.13) is stable as $R\to\infty$. Then, to lowest order, it is possible to find a nonzero value of $\langle\sigma\rangle$ given by

$$\langle\sigma\rangle \approx \langle\sigma\rangle_{\text{Ising}}\{1-\tfrac{1}{2}d\exp[-4K(d-1)-\beta W(\beta)]\}. \quad (2.14)$$

Therefore we expect some sort of long-range order in the system. This operator is able to distinguish between the Higgs and disordered regimes, since (2.14) is valid when $K$ is very large. However, we think the gauge degeneracies will spoil this long-range order and this operator will fail to distinguish between the Higgs and confining regimes.

(b) *The Higgs and confining regimes belong to the same phase.* We now want to show that the Higgs and confining regimes belong to the same phase. That is, we need to show that there is no phase boundary separating these regimes. Following the lines of the Introduction, we first notice that if such a phase boundary is really present, the vacuum energy (free energy), as well as all the possible Green's functions, should exhibit a line of singularities. The strategy is thus to show that $\mathfrak{F}(\beta,K)$ and all the Green's functions are analytic functions in a strip of the $(\beta,K)$ plane that includes both confinement ($K<K_c$, $\beta$ small) and Higgs field ($\beta>\beta_c$, $K$ large).

In the particular case of *discrete* gauge groups, this result can be shown by transforming the model (2.2) into a lattice gas. In the unitary gauge the action of the model is

$$S[U_\mu(\vec{r})]=\beta\sum_{(\vec{r},\mu)}U_\mu(\vec{r})+K\sum_{(\vec{r},\mu\nu)}\Phi_{\mu\nu}(\vec{r}), \quad (2.15)$$

and it turns into a lattice gas (with degrees of freedom on the links) by setting the occupation number of the link $n_\mu(\vec{r})$ equal to

$$n_\mu(\vec{r}) = \frac{1 - U_\mu(\vec{r})}{2}. \tag{2.16}$$

When $\beta$ is large, the configurations of gauge fields that contribute the most to the partition function are those with the fewest links with $U_\mu = -1$. Thus the lattice gas is very dilute in this regime. On the other hand, if $K$ is small, the gas is not dilute, but the interaction energy is very small. The system is a set of weakly interacting degrees of freedom. These comments can be stated formally through the construction of a set of Kirkwood-Salzburg equations.[13,26] Gallavotti and Miracle-Solé[26,27] have proved a theorem on the analytic properties of the free energy and correlation functions of lattice gases that, with minor changes, applies to our case. The theorem, when applied to our model, establishes the analyticity of the free energy and all the Green's functions in the strip of interest. This result also follows from the more general proof discussed in the Appendix. There is no phase boundary between the Higgs and confining regimes. It is also interesting to note (see Appendix) that the analyticity region has a *finite* width in each of these regimes. Then there are no transitions "off the axis." This proof indicates that it is not possible to construct a test to distinguish between these regimes. It is usually assumed that the existence of such a test would imply a nonanalytic behavior along some line (phase boundary) between these regimes. We have just shown, however, that this is not the case. There are, however, certinaly quantitative differences between the Higgs and confining regimes, just as there are between liquid and gas.

### C. Stability of the transitions of the pure models

(a) *Stability of the transition of the pure gauge theory*. In subsection B we have studied some analytic properties of the vacuum energy. We found that there is a domain in the $(\beta, K)$ plane where $\mathfrak{F}(\beta, K)$ is analytic. What about the rest of the diagram?

In (2.15) we wrote down the action for the model in the unitary gauge $[\sigma(\vec{r}) = 1,\ \text{all}\ \vec{r}]$. Formally (2.15) is analogous to the action of an Ising model in a uniform magnetic field, $h$:

$$S'[\sigma(\vec{r})] = \beta \sum_{(\vec{r},\mu)} \sigma(\vec{r})\sigma(\vec{r}+\hat{e}_\mu) + h \sum_{(\vec{r})} \sigma(\vec{r}). \tag{2.17}$$

The Ising model has a *global* $Z_2$ invariance and $h$ is a symmetry-breaking field. When $h = 0$ and $\beta > \beta_c$ (the Ising critical point), the global symmetry is spontaneously broken and the local order parameter $\langle \sigma(\vec{r}) \rangle$ is nonzero. At the critical point, the fluctuations of the order parameter become long-ranged and the spin-spin correlation $\langle \sigma(\vec{0})\sigma(\vec{r}) \rangle$ decays as a power of the distance $|\vec{r}|$. However, if there is a symmetry-breaking field acting on the system, the connected part of the correlation function becomes short-ranged for all $\beta$. Quantities such as the susceptibility $\chi(\beta, h)$ that in the absence of a symmetry-breaking field are singular at the critical point become analytic functions of $\beta$ and $h$ as soon as the magnetic field is turned on. Thus a symmetry-breaking field has destroyed the transition.

But in the case of a gauge theory, we have a local symmetry, and a local symmetry is never spontaneously broken.[23] Thus, gauge-noninvariant operators, such as $U_\mu(\vec{r})$ or $U_\mu(\vec{0}) \cdot U_\mu(\vec{r})$, have a zero expectation value for all values of the coupling constant $K$ no matter what boundary conditions are imposed. Therefore, even though the coupling $\beta$ to the matter fields formally breaks the local invariance, it is not coupled to an order parameter, i.e., to a field with some sort of long-range order. We conclude that the physics of this term should be very different from that of a symmetry-breaking field in a model with a global symmetry. Wegner[21] has analyzed this model, and for reasons explained above he concludes that the transition of the pure gauge theory should be stable. Thus, he predicts the existence of a line of phase transitions starting at the pure gauge critical point ($\beta = 0$, $K = K_c$).

We now present a slightly different version of Wegner's arguments. Consider the behavior of the model when $\beta$ is small but finite. In order to understand the effect of the matter fields, we shall integrate them out and construct an effective action $S_{\text{eff}}[U_\mu]$ for the gauge fields:

$$\exp\{S_{\text{eff}}[U_\mu(\vec{r})]\} = \sum_{[\sigma(\vec{r})]} \exp\Big[\beta \sum_{(\vec{r},\mu)} \sigma(\vec{r}) U_\mu(\vec{r}) \sigma(\vec{r}+\hat{e}_\mu) + K \sum_{(\vec{r},\mu\nu)} \Phi_{\mu\nu}(\vec{r})\Big]. \tag{2.18}$$

If $\beta$ is small it is possible to expand $S_{\text{eff}}$ in a power series in $\beta$ [this is in fact equivalent to computing the free energy of an Ising model in a fixed distribution of bonds $\{U_\mu(\vec{r})\}$ by means of the high-temperature expansion]:

$$\exp\{S_{\rm eff}[U_\mu(\vec r)]\}=2^N(\cosh\beta)^{Nd}\sum_\Gamma(\tanh\beta)^{L(\Gamma)}\left[\prod_{(\vec r,\mu)\,\Gamma}U_\mu(\vec r)\right]\exp\left[K\sum_{(\vec r,\mu\nu)}\Phi_{\mu\nu}(\vec r)\right],\tag{2.19}$$

where $\Gamma$ is a closed path of links of the lattice and $L(\Gamma)$ is the length of that path.

The leading contribution to $S_{\rm eff}$ will be given by the smallest loops, i.e., the elementary plaquettes. Thus, to lowest orders in $\beta$, $S_{\rm eff}$ will have the form

$$S_{\rm eff}[U_\mu]=[K+\tanh^{-1}(\tanh\beta)^4]\sum_{(\vec r,\mu\nu)}\Phi_{\mu\nu}(\vec r)+(\text{larger-plaquettes interactions})\,.\tag{2.20}$$

Thus, at lowest orders in $\beta$, the effect of matter fields is a finite renormalization of the coupling $K$ into an effective coupling $K_{\rm eff}$ given by

$$K_{\rm eff}\cong K+\beta^4\,.\tag{2.21}$$

The higher-order terms will produce interactions involving many plaquettes. However, there will be interactions only between linked plaquettes (i.e., plaquettes that share at least one vertex) and, at lowest order, they contribute to the effective action with a coupling of the order of $(\tanh\beta)^n$ where $n$ is the length of the loop that encloses the group of plaquettes.[28] Interactions at long distances will be exponentially damped by a factor $e^{-n|\ln\beta|}$ ($\beta$ small). Scaling arguments (assuming a second-order transition) suggest that such finite-ranged interactions are unable to destabilize the system. Hence, in the neighborhood of $\beta=0$, the system can be approximated by a pure gauge theory with an effective coupling $K_{\rm eff}$ given above. Thus the curve in the $(\beta,K)$ plane given by the equation[21]

$$K_c=K+\beta^4\tag{2.21a}$$

represents a line of second-order transitions starting at the pure gauge critical point $(0,K_c)$. Notice that as $\beta$ *increases* the coupling $K$ *decreases*. Thus for finite $\beta$, the coupling $g^2$ ($g^2=1/K$) necessary to confine the matter fields is shifted to *stronger* values.

(b) *Stability of the Ising transition.* The singularity of the pure Ising model is stable against fluctuations of the gauge fields. In the particular case of $d=3$, this result follows immediately from a duality transformation.[21] The model (2.2) is self-dual in three dimensions. The duality transformation maps a model with couplings $(\beta,K)$ onto a model with couplings $(\beta^*,K^*)$, where

$$\exp(-2\beta^*)=\tanh K\,,\qquad \exp(-2K^*)=\tanh\beta\,.\tag{2.22}$$

Note that the duality transformation not only exchanges large with small couplings, but matter and gauge couplings as well. In particular, the pure gauge model is dual ($d=3$) to the three-dimensional Ising model.[15,21] The line of transitions given by (2.21a) has as its dual image another line of transitions starting at the $d=3$ Ising critical point. To lowest order, in the large-$K$ regime, the effect of the gauge fields is a finite renormalization of the Ising coupling $\beta$ (Ref. 21),

$$\beta_{\rm eff}\cong\beta-\sinh 2\beta\exp(-8K)\,.\tag{2.23}$$

In higher orders other effects appear. In the previous sections we pointed out that larger-plaquettes couplings are in fact generated. The dual image of those couplings, by simple topological arguments, can be seen to be many spin interactions. But the important fact is that the interactions that are generated only involve an *even* number of spins and are finite ranged. Interactions of these types do not break the global symmetry of the Ising model. They can change the value of the critical point [as in (2.23)] but are unable to destabilize it, at least if $K$ is large enough.

The stability of the Ising critical point is not a special feature of three dimensions. The duality argument can be generalized to any dimension. The difference is that if $d\neq 3$ the model is no longer self-dual. Wegner[21] has studied the dual transformation of this model in any dimension. The dual model is, in general, a higher gauge theory. The link interaction dualizes into an interaction on a hypercube [$(d-1)$-dimensional simplex] and the plaquette term into an interaction on a $(d-2)$-dimensional simplex. In four dimensions, for instance, links go into cubes and plaquettes into plaquettes. The couplings are related by the usual duality relations. In any event, the arguments formulated about the stability of pure gauge theories also generalize to the higher gauge models. Hence the stability of the transition near the Ising regime ($K=\infty$) follows from the stability of the transition of the higher gauge theory. The result (2.23) is then essentially valid in any dimension. The only change is that the small parameter $\exp(-8K)$ is now $\exp[-4K(d-1)]$.

(c) *Spectrum of the theory.* The results of the previous subsections, summarized in Fig. 1, suggest the idea that there is a closed region of the phase diagram separated from the Higgs and confining regimes. In the case of a continuous gauge group there is a simple test for such a region: the existence of a massless photon (see Sec. III). The discreteness of the $Z_2$ group rules out this possibility. In this theory, the photon is always massive, except at the phase transition points.

In order to understand the physics of these regions, we find it useful to discuss the qualitative nature of the spectrum in each of them. We introduce here the Hamiltonian formulation of this theory. The Hamiltonian of this model can be constructed by means of the transfer matrix formalism. Using the methods of Ref. 8 we find that this theory on a $d$-dimensional space-time lattice is equivalent to a $(d-1)$-dimensional quantum-mechanical system with Hamiltonian

$$\begin{aligned} H = &- \sum_{(\vec{r})} \sigma_1(\vec{r}) - \sum_{(\vec{r},\mu)} \tau_1^\mu(\vec{r}) \\ &- \lambda \sum_{(\vec{r},\mu)} \sigma_3(\vec{r})\tau_3^\mu(\vec{r})\sigma_3(\vec{r}+\hat{e}_\mu) \\ &- \omega \sum_{(\vec{r},\mu\nu)} \tau_3^\mu(\vec{r})\tau_3^\nu(\vec{r}+\hat{e}_\mu)\tau_3^\mu(\vec{r}+\hat{e}_\nu)\tau_3^\nu(\vec{r}), \end{aligned} \tag{2.24}$$

where the $\sigma$'s and the $\tau$'s are two sets of Pauli matrices residing on sites and links, respectively. The $\sigma$'s represent the Higgs fields and the $\tau$'s the gauge fields. The states of the theory are subject to the constraint of gauge invariance. If $|\psi\rangle$ is a physical state, it must satisfy

$$\sigma_1(\vec{r})\left[\prod_\mu \tau_1^\mu(\vec{r})\right]|\psi\rangle = |\psi\rangle \tag{2.25}$$

at each vertex $\vec{r}$ of the lattice. Here $(\vec{r},\mu)$ labels all the links that emerge from site $\vec{r}$. The new (renormalized) couplings $\lambda$ and $\omega$ play a role analogous to that of $\beta$ and $K$ (see Ref. 8). In the confinement regime ($\lambda$ and $\omega$ small) the (perturbative) spectrum is made up of (a) box excitations, which are created by the operator $\Pi_\Gamma \tau_3^\mu(\vec{r})$ ($\Gamma$ is a closed path of links), and (b) mesons, whose creation operators are

$$\sigma_3(\vec{r})\left(\prod_{\Gamma(\vec{r},\vec{R})} \tau_3^\mu(\vec{r})\right)\sigma_3(\vec{R})$$

[$(\vec{r},\vec{R})$ is a path of links that joins $\vec{r}$ and $\vec{R}$]. In the Higgs regime the (perturbative) spectrum is made of (a) Higgs excitations, whose creation operator is $\sigma_1(\vec{r})$, and (b) monopole strings. In two space dimensions the monopoles are created in pairs by the operator $\Pi_{\Gamma(\vec{r},\vec{R})}\tau_1^\mu(\vec{r})$, where $\Gamma(\vec{r},\vec{R})$ is a set of links in a direction perpendicular to a path between the points $\vec{r}$ and $\vec{R}$, located on the sites of the dual lattice. In $3+1$ dimensions, we find instead that the monopoles arrange in closed loops. The operators that create the perturbative spectrum on the confining and Higgs regimes share a common property: They are gauge-invariant local operators. Thus there is no room in these regimes for states that represent free charges. This is not the case of the $\lambda$-small, $\omega$-large regime (large $K$, small $\beta$ in Fig. 1), the free-charge regime. In addition to states that are created by gauge-invariant local operators, there are other states that represent free charges. In this regime the operator $\sigma_1(\vec{r})$ measures the charge residing at site $\vec{r}$ and, in this limit, is a conserved quantity. Here the unperturbed Hamiltonian is

$$\begin{aligned} H_0 = - \sum_{(\vec{r})} \sigma_1(\vec{r}) - \omega \sum_{(\vec{r},\mu\nu)} &\tau_3^\mu(\vec{r})\tau_3^\nu(\vec{r}+\hat{e}_\mu) \\ &\times \tau_3^\mu(\vec{r}+\hat{e}_\nu)\,\tau_3^\nu(\vec{r}). \end{aligned} \tag{2.26}$$

Let $|\psi(\vec{x})\rangle$ represent a free charge residing at $\vec{x}$. It is defined by

$$\begin{aligned} &\sigma_1(\vec{y})|\psi(\vec{x})\rangle = |\psi(\vec{x})\rangle, \quad \vec{x}\neq\vec{y}, \\ &\sigma_1(\vec{x})|\psi(\vec{x})\rangle = -|\psi(\vec{x})\rangle, \\ &\tau_3^\mu(\vec{y})|\psi(\vec{x})\rangle = |\psi(\vec{x})\rangle \text{ all } (\vec{y},\mu). \end{aligned} \tag{2.27}$$

Notice that this state is not gauge invariant. It is possible to construct a gauge-invariant state out of it by considering a (normalized) linear superposition of $|\psi(\vec{x})\rangle$ with all the states that are obtained by gauge transformations of it. This superposition is a gauge singlet and obeys (2.27), i.e., represents a free charge. This state can be shown to have finite energy and, after symmetrizing under space translations, it is stable (i.e., its energy changes smoothly). States such as this one cannot be created by a gauge-invariant local operator.

Therefore, it seems plausible that there are states in the free-charge regime that may not exist in the Higgs-confinement phase. The qualitative differences in their spectrum lead us to speculate that the lines of phase transitions depicted in Fig. 1 may meet at some point in the $\beta, K(\lambda,\omega)$ plane separating the Higgs-confinement phase from a free-charge phase.

### D. Summary

In summary, we argue that this model ($d \geqslant 3$) has the following phase diagram (Fig. 2):

(a) Higgs-confinement phase,

(b) free-charge phase,

with a line of transitions separating both regimes. In three dimensions the model is self-dual and the line of transitions is symmetric under duality. However, the dual transformation does not map one phase into the other.

In two dimensions the free-charge phase does not exist. The model is dual to an Ising model in a magnetic field $H$ with the correspondence

$$\begin{aligned} \tanh\beta_I &= \exp(-2\beta)\,, \\ \tanh H &= \exp(-2K)\,, \end{aligned} \qquad (2.28)$$

where $\beta_I$ is the inverse temperature of the dual Ising model. It is a well-established fact that this dual model has no singularities other than the pure Ising transition ($H=0$). Then the free-charge phase cannot exist. The Higgs-confinement phase is the only phase.

## III. THE ABELIAN HIGGS MODEL [U(1)]

We want to discuss the Abelian Higgs [U(1)] model. We shall show that most of the results exhibited in Sec. II are not related to the discreteness of the $Z_2$ group. In particular, the connection between the Higgs and confining regimes when the Higgs fields transform like the fundamental representation of the gauge group persists.

The action for this model was given in Eq. (1.4). If the matter field $\phi(\vec{r})$ carries $q$ units of charge, we have

$$S_q[\theta(\vec{r});A_\mu(\vec{r})]=\beta\sum_{(\vec{r},\,\mu)}\cos[\Delta_\mu\theta(\vec{r})-qA_\mu(\vec{r})] + K\sum_{(\vec{r},\,\mu\nu)}\cos[F_{\mu\nu}(\vec{r})]\,, \qquad (1.4')$$

with the notation defined in the Introduction.

This model has been analyzed by several authors.[12,29-31] After recognizing the failure of the Wilson loop as a test for confinement if $q$ is one, they argue that only the confining properties of fractional test charges can be meaningful. But if the fields are compact, as they are in (1.4), the only consistent way of introducing fractional test charges is to let the charge of the Higgs fields $q$ be larger than one. Then we may ask: What is the force between static sources with $q=1$? However, incrementing the charge of the Higgs field is a drastic change in the theory. The problem of Higgs fields with the fundamental charge remains.

The general properties of the limit models—the $XY$ model ($K=\infty$) and Abelian gauge theory ($\beta=0$)—have already been discussed in the Introduction. In contrast to what happens in the $Z_2$ model, the U(1) model has Goldstone modes ("spin waves") and massless photons for certain values of the couplings. In particular, the mass of the photon will provide a simple signature for the Coulomb phase.

### A. Matter fields with the fundamental charge ($q=1$)

For simplicity we shall assume that the dimension $d$ is high enough so that both the Abelian gauge theory and the $XY$ model have transitions. This means that $d\geq 4$. The analyticity results, however, are valid for $d\geq 2$.

(a) *The Higgs and confining regimes*. The Wilson loop and the gauge-invariant correlation function behave as in the $Z_2$ model. The Wilson loop decays like the perimeter for all $\beta\neq 0$ and fails as a signature of confinement.

The gauge-invariant correlation function decays exponentially even close to the ordered phase of the $XY$ model. This result can be seen by means of a free-field approximation valid for large $K$ and $\beta$. Here too we fail to find order parameters to distinguish between the Higgs and confining regimes. Again, we will show that this failure is due to the fact that there is no phase boundary between them. In the unitary gauge [$\theta(\vec{r})=0$] the action (1.4) reads

$$S_{\text{fund}}[A_\mu(\vec{r})]=\beta\sum_{(\vec{r},\,\mu)}\cos A_\mu(\vec{r}) + K\sum_{(\vec{r},\,\mu\nu)}\cos F_{\mu\nu}(\vec{r})\,. \qquad (3.1)$$

This model has the required form for the theorem discussed in the Appendix to apply. There is a strip in the ($\beta,K$) plane where the vacuum energy and all the Green's functions are analytic. However, since U(1) is a continuous group, the strip collapses into a point *at* the limit $K=\infty$. The reason is that in the region where $\beta$ and $K$ are large ($d\geq 2$) the partition function is dominated by the low-lying excitations of the linearized theory, i.e., massive photons. Their mass is $m^2=\beta/K$. So as $K$ increases the mass gets smaller. The cluster-expansion technqiues used in the Appendix rely on localized excitations and naturally do not apply for $m^2$ small. Nevertheless, in the entire neighborhood of $\beta=K=\infty$, ordinary perturbation theory in the broken phase is expected to be well behaved. It is easy to check that the topological excitations[29] of this model do not destabilize this expansion ($d>2$).

We conclude that since the vacuum energy is analytic in $\beta$ and $K$ in that strip, there are no transitions. There is no phase boundary separating the Higgs and confining regimes.

Notice also that the strip has a finite width in the strong-coupling regime ($g^2 > g_c^2$). Thus the strong-coupling expansion is convergent and there is no transition "off the axis."

(b) *The pure gauge transition is stable.* Wegner's arguments can be generalized for an arbitrary (compact) gauge group. Indeed, these arguments do not depend even on the statistics of the matter fields (bosons or fermions). Rather, they are a consequence of local gauge invariance.

As in the $Z_2$ model, it is possible to integrate out the matter fields. The result is a model whose effective action $S_{eff}[A_\mu]$ is determined by

$$\exp\{S_{eff}[A_\mu]\} = \int \mathcal{D}\theta(\vec{r}) \exp\left\{ K \sum_{(\vec{r},\mu\nu)} \cos F_{\mu\nu}(\vec{r}) + \beta \sum_{(\vec{r},\mu)} \cos[\Delta_\mu \theta(\vec{r}) - A_\mu(\vec{r})\right\}. \tag{3.2}$$

The arguments of Sec. II give here the result that at lowest order in $\beta$ the effect of the matter fields is a finite renormalization of the gauge coupling constant

$$K_{eff} \simeq K + \beta^4/8 . \tag{3.3}$$

Naturally, there are higher corrections that involve larger plaquette interactions (which are exponentially damped) and interaction with higher symmetries such as $\cos pF_{\mu\nu}$ ($p$ integer). As in the discrete case, we can also argue that these additional operators do not destroy the transition (because the critical behavior of this model is not as well understood, these arguments are not as solid as in the $Z_2$ case). So we also expect to have a line of transitions starting at the pure gauge critical point. For this analysis to hold, however, it is essential to have in the pure gauge theory a transition at *finite* coupling ($d \geq 4$). In three dimensions Polyakov[6,9] has shown that the transition occurs at $K=\infty$. In this case our analysis says that it *stays* at $K=\infty$ to all orders in $\beta$.

(c) *The XY transition.* For $d \geq 4$ the pure gauge theory has a phase where there are massless photons (large $K$). It seems reasonable to analyze the stability of the $XY$ transition in this case by treating the gauge field in the noncompact free-field approximation. This type of model has been studied by Coleman and Weinberg,[32] Halperin *et al.*,[33] and Peskin.[34] They find that the transition becomes first order. The mass of the photon has a finite jump across the phase boundary.

The mass of the photon provides a natural way of distinguishing between the Higgs-confinement phase and the Coulomb phase. A simple way to study it is to consider the *connected field-strength correlation function* $C(|\vec{r}|)$:

$$C(|\vec{r}|) = \langle \exp[i(F_{\mu\nu}(0) - F_{\mu\nu}(\vec{r}))] \rangle - \langle \exp[iF_{\mu\nu}(0)] \rangle^2 . \tag{3.4}$$

If $C(|\vec{r}|)$ decays like $\exp(-\mu r)$, the photon is massive. Conversely, if $C(\vec{r})$ decays like $1/|r|^\lambda$ the photon is massless.

The quantity

$$W_{\mu\nu,\alpha\beta}(\vec{r}) = -\ln \langle \exp[i(F_{\mu\nu}(0) - F_{\alpha\beta}(\vec{r}))] \rangle \tag{3.5}$$

is the energy of two small static loops (i.e., dipoles) at a distance $r$. The effective potential between the dipoles is given by $-C(\vec{r})$ where $|\vec{r}|$ is much larger than their size. It depends on their relative separation and orientation.

Let us consider the behavior of $C(\vec{r})$ (face-to-face loops) in the different regimes of the theory.

(i) *Coulomb phase* ($K > K_c$, $\beta$ *small*). In this regime, we find massless photons. Indeed, $C(\vec{r})$ is not directly sensitive to the matter fields. They only enter to lowest order through the $\beta$ dependence of the effective gauge coupling. In the free-field approximation we get the result

$$W_{int}(\vec{r}) \cong -C(\vec{r}) = \frac{1}{K_{eff} r^d} . \tag{3.6}$$

The minus sign in (3.6) shows that oppositely oriented dipoles attract each other. This is clearly the magnetostatic interactions between two loops of current. We conclude that for $K$ large and $\beta$ small there is a long-range static force between the loops. There is a massless photon in this phase and it stays massless to all orders in $\beta$. We call this regime the *Coulomb phase*.

(ii) *Confinement regime* ($K$, $\beta$ *small*). In this regime the photon is massive. A strong-coupling expansion shows that $C(r)$ behaves like

$$C(\vec{r}) \approx \exp(-4|\ln K_{eff}|\,|\vec{r}|) \tag{3.7}$$

for two face-to-face loops.

Notice that the effect of the matter fields is only a coupling-constant renormalization. The reason is that the effective action (3.2) does not have interactions between *disconnected* loops. Then higher orders in $\beta$ cannot destabilize the low-order results.

(iii) *Higgs regime* ($K$, $\beta$ *large*). In the Higgs regime we also find a massive photon. Here the mass of the photon comes from the Higgs mechanism. Again in the free-massive-field (linearized) approximation ($m^2 = \beta/K$) we find

$$C(r) \approx \frac{m^{(d-1)/2}}{K} \frac{\exp(-mr)}{r^{(d+1)/2}} . \tag{3.8}$$

In the Higgs regime the photon is massive and the force between dipoles exponentially damped.

In summary, in agreement with the results of

(a) and (b), if $d \geq 4$, we find two phases (Fig. 2). For $K$ large and $\beta$ small there is a *Coulomb phase*. Here the photon is massless and the forces are long-ranged. As in the $Z_2$ model it is also possible to find states in the spectrum that behave like free charges. In the Higgs-confinement phase the photon is massive and the forces short-ranged. The only states in the spectrum are created by gauge-invariant local operators.

### B. Matter fields with multiple charge

The situation is completely different if the matter fields carry more than one unit of charge. The introduction of the matter fields in some higher representation generates a phase boundary (i.e., singularities) between the Higgs and confinement regimes that does not exist otherwise.

The reason is that if the matter fields carry $q$ units of charge at the limit $\beta=\infty$ the system is nontrivial. If we write the action in the unitary gauge we get

$$S_q[A_\mu]=\beta\sum_{(\vec{r},\mu)}\cos[qA_\mu(\vec{r})] + K\sum_{(\vec{r},\mu\nu)}\cos[F_{\mu\nu}(\vec{r})]\,. \tag{3.9}$$

If $\beta=\infty$ the only configurations of $A_\mu$ fields that survive are those such that

$$A_\mu(\vec{r})=\frac{2n_\mu(\vec{r})\pi}{q},\quad n_\mu(\vec{r})\text{ integer}. \tag{3.10}$$

The constrained model ($\beta=\infty$) is just a $Z_q$ gauge theory. The Wilson loop for sources in the fundamental representation provides a test for confinement of this $Z_q$ gauge charge. For $K$ small, the Wilson loop decays like the area. In this regime we get confinement of static sources with the fundamental charge. This phase exists for all values of $\beta$ and $K$ small (see Fig. 3). On the other hand, if $K$ is large enough, the Wilson loop has a perimeter law: Static fundamental sources are not confined.[35]

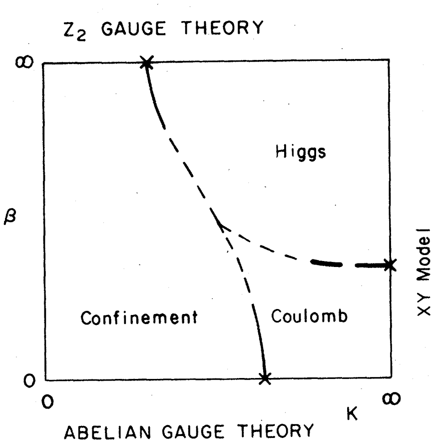


FIG. 3. Phase diagram of the Abelian Higgs model for Higgs fields with two units of charge. The difference with Fig. 2 is that there is a phase with confinement (in the Wilson sense) of static sources in the fundamental representation.

There is still the transition associated with the massive or massless character of the photon. This transition has already been discussed in the model with $q=1$ [Eq. (3.1)] and the same arguments are valid for $q\neq 1$.

In summary, when $q\neq 1$ three phases are expected to occur ($d\geq 4$) (Fig. 3):

(a) Confinement of static sources with the fundamental charge ($K>K_c$, all $\beta$). The spectrum is made of gauge and $Z_q$ gauge charge neutral states. The gauge boson is massive.

(b) Higgs phase ($k>K_c$, $\beta>\beta_c$). The gauge boson is still massive but $Z_q$ gauge charge is not confined. Static sources in the fundamental representation are free, with an exponentially damped force law.

(c) Coulomb phase ($K>K_c$, $\beta>\beta_c$). The gauge boson is massless and the static sources in the fundamental representation are free with a Coulombic force law. There is no confinement of gauge charge. There are states in the spectrum that represent free charges and have finite energy (such as the Ising case).

### C. Three dimensions

We have pointed out above that the argument on the stability of Abelian gauge theory does not apply in $d=3$ where there is no transition at finite coupling. Indeed, the stability argument shows that the transition occurs *at* $K_c=\infty$ to all orders in $\beta$. We have no evidence for a Coulomb phase in $d=3$. Nevertheless, the analyticity arguments apply here too. So, for charge-one Higgs field, the Higgs and confining regimes still belong to the same phase. The situation *might* be analogous to the $Z_2$ model in $d=2$. However, we cannot rule out the existence of a "pocket" of Coulomb phase. Another possibility is a line of transitions terminating at an interior point of the diagram. When the Higgs fields carry $q$ units of charge, we still expect a phase boundary between the Higgs and confining regimes to occur.

## IV. CONCLUSIONS: NON-ABELIAN GROUPS

In the previous sections we have seen certain general features of the phase diagram that are the same for $Z_2$ and U(1) gauge groups. We want to show now that these features persist for the more general case of a compact non-Abelian group.

Let us begin with the case in which the Higgs fields transform like the fundamental representation of the gauge group $G$. In order to be definite let us consider $G$ to be SU($N$). In this case the gauge fields $U_\mu(\vec{r})$ will be $N \times N$ SU($N$) matrices.

There are a variety of ways to introduce Higgs fields. One possibility is to let the Higgs field $\Phi(\vec{r})$ be an $N$-component complex vector transforming like the fundamental representation of SU($N$). However, one Higgs field is not enough, in general, to break completely the gauge symmetry. Thus we shall add as many Higgs fields as necessary to totally break down the local symmetry. This scheme has the unwanted (for us) feature of generating pseudo-Goldstone bosons.

Another possible way is to introduce Higgs fields $\Phi(\vec{r})$ that behave like a group element, namely SU($N$) matrices. By going to the unitary gauge $\Phi(\vec{r})=I$, where $I$ is the identity matrix, the gauge symmetry is completely broken. No Higgs degrees of freedom are left. We shall choose this scheme.[36]

The action of the non-Abelian model reads

$$S[\Phi(\vec{r}); U_\mu(\vec{r})] = \frac{\beta}{2}\sum_{(\vec{r},\mu)} \mathrm{Tr}[\Phi(\vec{r})U_\mu^\dagger(\vec{r})\Phi^\dagger(\vec{r}+\hat{e}_\mu) + \mathrm{H.c.}] + \frac{K}{2}\sum_{(\vec{r},\mu\nu)} \mathrm{Tr}[U_\mu(\vec{r})U_\nu(\vec{r}+\hat{e}_\mu)U_\mu^\dagger(\vec{r}+\hat{e}_\nu) \times U_\nu^\dagger(\vec{r}) + \mathrm{H.c.}], \tag{4.1}$$

where $\Phi(\vec{r})$ and $U_\mu(\vec{r})$ are SU($N$) matrices.

The analytic properties of the model (4.1) can be examined by the same methods of the previous sections. In the proof sketched in the Appendix, we show that the region of the $(\beta, K)$ plane where the vacuum energy is analytic extends to the whole strip of interest. [The only difficulty, as in the U(1) case, arises in the vicinity of $\beta=K=\infty$, where the strip shrinks into a point. Here too conventional continuum perturbation theory should be well behaved if there is no transition at $\beta=K=\infty$ ($d>2$).] Thus if the Higgs fields transform like the fundamental representation of SU($N$), the Higgs and confining regimes belong to the same phase of the theory.

In addition, all the Green's functions, i.e., the products of local operators, are analytic functions of the coupling constants in that strip. This means that the spectrum evolves smoothly in the whole strip. The type of excitations is the same although the energies will generally be different. The Higgs-confinement phase is characterized by a completely massive spectrum. We expect all the states in the spectrum to be created by gauge-invariant local operators.

The pure gauge transition should be stable. Wegner's arguments generalize to any compact group, since it is only a consequence of the gauge-invariant nature of the interactions.

If $d>4$, there is a second phase ($K$ large, $\beta$ small) characterized by a massless gauge boson. The forces are of a Coulomb type. The gauge boson is massless and stays massless to all orders in $\beta$. As in the $Z_2$ theory it is possible, within the framework of perturbation theory, to find finite-energy states in the spectrum that represent a free charge. Thus (for $d>4$) the phase diagram is like that shown in Fig. 2.

The situation is different if the Higgs fields transform like some higher representation of SU($N$), for instance, the adjoint. Even if there are enough Higgs fields to break completely the local continuous symmetry, a discrete $Z_N$ local symmetry will survive. As in the U(1) case, the $\beta\to\infty$ limit is just a $Z_N$ gauge theory.[18] In this case, for $d>2$, we expect a phase boundary to separate the Higgs and confining regimes.[35] The Wilson loop for sources in the fundamental representation will be a good criterion for differentiating between these two phases. If $d>2$, a transition from a phase where $Z_N$ gauge charge is confined to another phase where it is unconfined will occur.

As in the U(1) case, when the Higgs fields are not in the fundamental representation, we expect three distinct phases: confinement, a Higgs phase, and a Coulomb phase.

This situation is depicted in Fig. 3. In four dimensions, the pure non-Abelian gauge theory (we hope) has a phase transition *at* $K_c=\infty$. As in the U(1) case in $d=3$, we find that the transition stay at $K_c=\infty$ to all orders in $\beta$. Hence there is no evidence for a Coulomb (or free) phase here either, although we cannot rule out the existence of a "pocket" of Coulomb phase. The arguments of Coleman and Weinberg[32] suggest here, also, that the pure matter transition persists and becomes first order. It is possible that this line terminates at some interior point of the diagram for Higgs fields in the fundamental representation. For Higgs fields in the adjoint representation a two-region phase diagram (Higgs and confinement) is likely to occur except for the case described in Ref. 35.

*Note added.* When this work was near completion, we received a report from de Angelis, de Falco, Guerra, and Marra [Salerno report, 1978 (unpublished)] where a similar analyticity result is proved. Also, T. Banks and E. Rabinovici have found similar results for the U(1) model independently. E. F. wishes to thank them for interesting discussions about their work.

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## APPENDIX

We want to show here that there are no phase transitions separating the Higgs and confining regimes when the Higgs fields transform like the fundamental representation of the gauge group and break—in the sense of the unitary gauge—the local invariance completely. Here we give a sketch of the proof of Osterwalder and Seiler[19] (OS) specialized to the case of fixed-length Higgs fields.

We study actions of the form

$$S=K\sum_{(\vec{r},\mu\nu)}[\chi(U_\mu(\vec{r})U_\nu(\vec{r}+\hat{e}_\mu)U_\mu^{-1}(\vec{r}+\hat{e}_\nu)U_\nu^{-1}(\vec{r}))-D] + \beta\sum_{(\vec{r},\mu)}H(U_\mu(\vec{r}))\,, \tag{A1}$$

where $\chi(U)=\frac{1}{2}(\mathrm{Tr}\,U+\mathrm{c.c.})$, $D$ is the dimension of the representation of $U$, i.e., $\chi(I)=D$, $H(U)$ is the Higgs part of the action in the unitary gauge. We assume that all the Higgs degrees of freedom are gone in this gauge so that $H$ only depends on $U$. Up to an (infinite) constant the action (A1) is the same as (1.1).

We want to study expectations of gauge-invariant operators $\mathfrak{F}$ which in this gauge are just a finite collection of $U$'s,

$$\langle\mathfrak{F}\rangle=\frac{1}{Z}\int\prod_{(\vec{r},\mu)}dU_\mu(\vec{r})\exp[S(U)]\mathfrak{F}\,,\qquad Z=\int\prod_{(\vec{r},\mu)}dU_\mu(\vec{r})\exp[S(U)]\,. \tag{A2}$$

We define a new measure by absorbing the Higgs part of the action

$$d\mu=\frac{\prod_{(\vec{r},\mu)}dU_\mu(\vec{r})\exp\left[\beta\sum_{(\vec{r},\mu)}H(U_\mu(\vec{r}))\right]}{\int\prod_{(\vec{r},\mu)}dU_\mu(\vec{r})\exp\left[\beta\sum_{(\vec{r},\mu)}H(U_\mu(\vec{r}))\right]} \tag{A3}$$

which satisfies $\int d\mu=1$.

For Higgs fields in the fundamental representation—assuming all gauge invariance is broken—$H(U)$ has one maximum at $U=I$, and it is the only one. For $\beta$ large $U\simeq I$ will be strongly favored by the measure $d\mu$. When $U=I$ the gauge part of $S$ reaches its maximum value, zero. We exploit this by defining

$$\exp\{K[\chi(U_\mu(\vec{r})U_\nu(\vec{r}+\hat{e}_\mu)U_\mu^{-1}(\vec{r}+\hat{e}_\nu)U_\nu^{-1}(\vec{r}))-D]\}=1+\rho_{\mu\nu}(\vec{r})\,. \tag{A4}$$

For $K$ small, $\rho_{\mu\nu}(\vec{r})$ is near zero. For $\beta$ large the $U$'s favored by $d\mu$ make $\rho_{\mu\nu}(\vec{r})$ small. So the strategy is to expand in powers of $\rho_{\mu\nu}(\vec{r})$. Rewriting

$$\langle\mathfrak{F}\rangle=\frac{\int d\mu\left(\prod_{(\vec{r},\mu\nu)}[1+\rho_{\mu\nu}(\vec{r})]\right)\mathfrak{F}}{\int d\mu\prod_{(\vec{r},\mu\nu)}[1+\rho_{\mu\nu}(\vec{r})]}\,, \tag{A5}$$

we expand the product, getting a sequence of terms corresponding to larger and larger blocks of plaquettes. We end up with an expansion [OS Eq. (4.17)]

$$\langle\mathfrak{F}\rangle=\sum_{Q(Q_0)}\int d\mu\,\mathfrak{F}\prod_{(\vec{r},\mu\nu)\in Q}\rho_{\mu\nu}(\vec{r})\,\frac{\mathfrak{Z}\{\text{without all plaquettes in } Q\cup Q_0\}}{\mathfrak{Z}}\,, \tag{A6}$$

where $Q$ denotes sets of plaquettes, $Q_0$ is the set of plaquettes where the operator $\mathfrak{F}$ lives, and $Q(Q_0)$ is the set of plaquettes connected to $Q_0$. We want to show this cluster expansion converges as the volume tends to infinity. We need the following:

(i) The number of possible sets $Q$ connected to $Q_0$ containing $n$ plaquettes grows at most exponentially with $n$ (with constants independent of $\beta$ and $K$). See OS lemma (3.4).

(ii) $\mathfrak{Z}\{\text{without all plaquettes in } Q\cup Q_0\}/\mathfrak{Z}$ grows at most exponentially with the number of plaquettes in $Q\cup Q_0$. The disconnected diagrams are controlled by this estimate. This is proved using (iii). See OS lemma (3.2).

(iii) If $n$ is the number of plaquettes in $Q$ then

$$\left|\int d\mu\, \mathcal{F} \prod_{(\vec{r},\mu\nu)\in Q} \rho_{\mu\nu}(\vec{r})\right| < \text{const} \times (\text{const})^n . \tag{A7}$$

If conditions (i)–(iii) hold for suitable constants we can bound the series (A6) by a geometric one, proving uniform convergence. We now examine for what values of $\beta$ and $K$ condition (iii), the crucial one, holds. By Hölder's inequality

$$\left|\int d\mu\, \mathcal{F} \prod_{(\vec{r},\mu\nu)\in Q} \rho_{\mu\nu}(\vec{r})\right| < \text{const} \prod_{(\vec{r},\mu\nu)\in Q} \left|\int d\mu\, |\rho_{\mu\nu}(\vec{r})|^p\right|^{1/p} , \tag{A8}$$

where $p$ is an integer independent of $\beta$ and $K$. So (A7) holds if

$$\left|\int d\mu\, |\rho_{\mu\nu}(\vec{r})|^p\right|^{1/p} < \text{const}. \tag{A9}$$

For a $Z_2$ theory we can directly compute this bound. For an action $S = K\sum_P (UUUU - 1) + \beta \sum_L U$ ($P$ and $L$ are plaquettes and links respectively) (A9) becomes

$$\left[\frac{4(e^{2\beta}+e^{-2\beta})\,|e^{-2K}-1|^p}{(e^{\beta}+e^{-\beta})^4}\right]^{1/p} < \text{const}. \tag{A10}$$

This yields a region of convergence like the one shown in the shaded area of Fig. 1. Notice that this bound gives a finite width in both the Higgs and confining regimes.

For continuous groups it is convenient to do a little further analysis. Equation (A9) holds if [see OS lemmas (4.2) and (4.4)]

$$K \int d\mu\, |\chi(U_0) - D|^{2p\,1/2p} < \text{const}, \tag{A11}$$

where $U_0$ is any link variable. This further analysis deteriorates the quality of the bound (A9). For $Z_2$ models (A11) no longer yields a finite-width region in $\beta$ as $K \to \infty$. For continuous models the deterioration is minor. As discussed in the text we do not expect a finite width in $\beta$ as $K \to \infty$ for continuous groups (see Secs. III and IV).

For a U(1) model this yields the explicit bound

$$K\left(\frac{\int_0^{2\pi} d\theta \exp(\beta\cos\theta)\,|\cos\theta - 1|^{2p}}{\int_0^{2\pi} d\theta \exp(\beta\cos\theta)}\right)^{1/2p} < \text{const}. \tag{A12}$$

For $K$ small the bound holds. For $\beta$ large we do a quadratic expansion of the cosines and the Gaussian integrals give the result

$$K\left(\frac{1}{\beta}\right) < \text{const} \tag{A13}$$

for $\beta$ large enough. This yields a convergence region like that of Fig. 2. Similar results clearly hold for non-Abelian groups with appropriate Higgs couplings.

The convergence of such an expansion implies the following:

(i) Analyticity of $\langle \mathcal{F} \rangle$ in $K$ and $\beta$, because the series converges uniformly and the terms are each analytic. This implies that the free energy is analytic.

(ii) Exponential clustering. If $\mathcal{F}$ consists of two local operators $\mathcal{F}_1$ and $\mathcal{F}_2$ separated by a distance $R$, then

$$\langle \mathcal{F}_1 \mathcal{F}_2 \rangle = \langle \mathcal{F}_1 \rangle \langle \mathcal{F}_2 \rangle \leq \text{const} \times \exp(-\text{const}\, R). \tag{A14}$$

$\langle \mathcal{F}_1 \mathcal{F}_2 \rangle - \langle \mathcal{F}_1 \rangle \langle \mathcal{F}_2 \rangle$ only gets contributions in the cluster expansion (A6) from terms containing a path of plaquettes connecting $\mathcal{F}_1$ to $\mathcal{F}_2$. These terms contain, at least, $R$ factors of $\rho_{\mu\nu}(\vec{r})$ and so the bound (A14) holds.

For further details of these proofs we refer to the work of Osterwalder and Seiler[19] and references therein.

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