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Inverse scattering transform as an operator method in quantum field theory

H. B. Thacker and David Wilkinson
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
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It is shown that the inverse scattering transform used to solve the classical nonlinear Schrödinger equation may be formulated as an operator method for solving the corresponding quantum field theory (δ-function many-body problem).

Much of the recent progress in quantum field theory has evolved from the study of nontrivial solutions to classical field equations. In general it is a difficult problem to determine the precise quantum-mechanical implications of such classical solutions, even when the latter are known exactly, and in most cases of interest, results must be based on semiclassical approximation techniques. To provide a base of experience in these matters, it is of interest to study a class of two-dimensional models for which the connection between classical and quantum phenomena may be analyzed by exact methods. These are the models whose classical field equations may be solved by the inverse-scattering-transform (IST) method.[1] In this paper we consider[2] the simplest of these models, which in its classical form is known as the nonlinear Schrödinger equation

$$i\partial_0\phi = -\partial_1{}^2\phi + 2c|\phi|^2\phi\,, \tag{1}$$

where ϕ is a complex scalar field and c is a constant, which in this paper we shall take to be positive. If we define the Poisson brackets any two functionals α and β as

$$\{\alpha,\beta\} = i\int dx\left[\frac{\delta\alpha}{\delta\phi(x)}\,\frac{\delta\beta}{\delta\phi^*(x)} - \frac{\delta\alpha}{\delta\phi^*(x)}\,\frac{\delta\beta}{\delta\phi(x)}\right], \tag{2}$$

then Eq. (1) may be written in Hamiltonian form $\partial_0\phi = \{H,\phi\}$ with the Hamiltonian H given by

$$H = \int dx(\partial_1\phi^*\partial_1\phi + c|\phi|^4)\,. \tag{3}$$

This classical theory is completely soluble by the IST method and possesses an infinite number of conserved quantities whose densities are polynomials in the fields ϕ and ϕ^* and their space derivatives.[3,4]

The quantum version of the theory is obtained by normal-ordering the Hamiltonian (placing all ϕ^*'s to the left of all ϕ's)

$$H = \int dx(\partial_1\phi^*\partial_1\phi + c\phi^*\phi^*\phi\phi)\,, \tag{4}$$

and considering ϕ and ϕ^* to have canonical commutation relations

$$[\phi(x),\phi^*(y)] = \delta(x-y)\,. \tag{5}$$

This nonrelativistic field theory is equivalent to the δ-function gas and the Hamiltonian is known to be diagonalized explicitly by Bethe ansatz wave functions.[5-10]

The purpose of this work is to make precise the connection between the solubility of the classical and quantum versions of the theory. Some progress in this direction was made by one of us in a previous paper,[11] where it was suggested that the Bethe eigenstates of the quantum theory were also eigenstates of suitably ordered counterparts of *all* the polynomial conservation laws of the classical theory. This was explicitly verified for the first four conserved quantities. However, an attempt to prove this result in general reveals that, due to short-distance singularities, the higher-order conservation laws are not unambiguously defined in the quantum theory without a cutoff prescription, and we were led instead to the considerations of this paper.

We shall first introduce the classical method of Zakharov and Shabat[3] and Zakharov and Manakov[4] in a language appropriate to the quantum version of the theory. This classical method is based on a mapping between the field configuration $\phi(x)$ at a given time and a set of scattering data associated with the Zakharov-Shabat eigenvalue problem

$$\left(i\frac{\partial}{\partial x} + \tfrac{1}{2}\xi\right)\Psi_1 = -\sqrt{c}\,\Psi_2\phi\,, \tag{6a}$$

$$\left(i\frac{\partial}{\partial x} - \tfrac{1}{2}\xi\right)\Psi_2 = \sqrt{c}\,\phi^*\Psi_1\,. \tag{6b}$$

For the classical theory the ordering of the factors on the right-hand side of (6) is immaterial. However, as we shall show in this paper, the eigenfunctions Ψ_1 and Ψ_2 of the linear problem (6) may also be considered as operators of the quantum theory, and in this case it is easy to see that the particular ordering in (6) leads to normal-ordering of Ψ_1 and Ψ_2.

We will discuss in particular the operator Jost function $\psi(x,\xi)$ defined as the solution of (6) with boundary condition

$$\psi(x,\xi) \underset{x\to-\infty}{\sim} e^{i\xi x/2}\begin{pmatrix}1\\0\end{pmatrix}. \tag{7}$$

It is convenient to write

$$\psi_1(x,\xi)=e^{i\xi x/2}A(x,\xi)\,,\qquad \psi_2(x,\xi)=-i\sqrt{c}\,e^{-i\xi x/2}B(x,\xi) \tag{8}$$

and define the scattering data

$$a(\xi)=\lim_{x\to\infty}A(x,\xi)\,, \tag{9a}$$

$$b(\xi)=\lim_{x\to\infty}B(x,\xi)\,. \tag{9b}$$

Both the Jost function $\psi(x,\xi)$ and the operator $a(\xi)$ are analytic in the lower half ξ plane. From Eqs. (6) and (7) the operators A and B satisfy a pair of coupled integral equations

$$A(x,\xi)=1+c\int dy\,\theta(y<x)e^{-i\xi y}B(y,\xi)\phi(y)\,, \tag{10a}$$

$$B(x,\xi)=\int dy\,\theta(y<x)e^{i\xi y}\phi^*(y)A(y,\xi)\,, \tag{10b}$$

where $\theta(y<x)\equiv\theta(x-y)$ is a step function. By iterating Eqs. (10), one may generate series expansions for A and B, and for the scattering data operators a and b, in terms of the canonical fields ϕ and ϕ^*. Recalling that normal-ordering in this model simply means grouping all ϕ^*'s to the left and all ϕ's to the right in each term, it is easily seen from (10) that the expressions obtained are normal-ordered. The series expansions for $a(\xi)$ and $b(\xi)$ are

$$a(\xi)=1+c\int dx_1dy_1\theta(x_1<y_1)e^{i\xi(x_1-y_1)}\phi^*(x_1)\phi(y_1)$$
$$+c^2\int dx_1dx_2dy_1dy_2\theta(x_1<y_1<x_2<y_2)e^{i\xi(x_1+x_2-y_1-y_2)}\phi^*(x_1)\phi^*(x_2)\phi(y_1)\phi(y_2)+\cdots\,, \tag{11a}$$

$$b(\xi)=\int dx_1e^{i\xi x_1}\phi^*(x_1)+c\int dx_1dx_2dy_1\theta(x_1<y_1<x_2)e^{i\xi(x_1+x_2-y_1)}\phi^*(x_1)\phi^*(x_2)\phi(y_1)+\cdots \tag{11b}$$

with an obvious notation for multiple step functions.

Before considering the quantities (11) as quantum operators we will collect here for comparison some of the results of the classical analysis of Zakharov and Manakov. In this paragraph only, $a(\xi)$ and $b(\xi)$ will be regarded as classical c-number quantities. By writing both $a(\xi)$ and $b(\xi)$ as a Wronskian of two solutions of the classical version of (6), one may compute the variational derivatives needed to evaluate the Poisson brackets defined in (2). In this way Zakharov and Manakov obtained

$$\{a(\xi),b(\xi')\}=\left(\frac{c}{\xi-\xi'-i\epsilon}\right)a(\xi)b(\xi')\,, \tag{12}$$

$$\{a^*(\xi),b(\xi')\}=\left(-\frac{c}{\xi-\xi'+i\epsilon}\right)a^*(\xi)b(\xi')\,, \tag{13}$$

$$\{a(\xi),a(\xi')\}=\{a(\xi),a^*(\xi')\}=0\,, \tag{14}$$

$$\{b(\xi),b(\xi')\}=0\,, \tag{15}$$

$$\{b(\xi),b^*(\xi')\}=2\pi i a^*(\xi)a(\xi)\delta(\xi-\xi')\,. \tag{16}$$

In addition, by transforming to action and angle variables it may be shown that $a(\xi)$ and $b(\xi)$ have simple Poisson brackets with the Hamiltonian:

$$\{H,a(\xi)\}=0\,, \tag{17}$$

$$\{H,b(\xi)\}=i\xi^2b(\xi)\,. \tag{18}$$

Equations (17) and (18) are the fundamental results of the classical theory. They show that the time evolution of the scattering data is extremely simple. The solution to the classical initial-value problem is completed by using the Gelfand-Levitan equation to recover the fields ϕ and ϕ^* at a later time from the scattering data.

We begin our consideration of the quantum theory by deriving the commutator of the normal-ordered Hamiltonian (4) with the operators $a(\xi)$ and $b(\xi)$ in (11). It is useful to note that all the Poisson-bracket relations (12)–(18) may be explicitly verified order by order, using the series expansions of the various quantities; the transition to the quantum theory is accomplished by paying careful attention to the ordering. The crucial observation here is that the spatial ordering of the integration variables (due to the multiple step functions) in (11) leads to a great simplification in the problem of reordering operator expressions. For a generic term in (11a) or (11b) a field ϕ^* at the point x_i will have a nonvanishing commutator only with the fields ϕ at

y_{i-1} and y_i. Similarly a field ϕ at y_i has a nonvanishing commutator only with the fields ϕ^* at x_i and x_{i+1}. To compute the commutators of $a(\xi)$ and $b(\xi)$ with the Hamiltonian we simply commute H with each term in the series (11a) or (11b) in turn, and note that the result in each case may be brought to normal-ordered form without encountering any nonzero commutators. Thus the calculation of the commutator of H with $a(\xi)$ or $b(\xi)$ is identical in structure to the calculation of the corresponding Poisson brackets, and it leads to similar results:

$$[H, a(\xi)] = 0\,, \tag{19}$$

$$[H, b(\xi)] = \xi^2 b(\xi)\,. \tag{20}$$

Equation (20) shows that eigenstates of H may be constructed by repeated application of the operator b to the vacuum state $|0\rangle$. Here $|0\rangle$ is the state with no particles defined by $\phi(x)|0\rangle = 0$. Thus we are led to consider the states[2]

$$|k_1 \cdots k_n\rangle \equiv b(k_1) \cdots b(k_n)|0\rangle \tag{21}$$

with the property

$$H|k_1 \cdots k_n\rangle = \left(\sum_{i=1}^{n} k_i^{\,2}\right)|k_1 \cdots k_n\rangle\,. \tag{22}$$

These states have definite particle number n and for $n \leq 3$ we have verified explicitly that they are identical to the known n-particle Bethe wave functions given by

$$\int \left[\sum_{i=1}^{n} dx_i e^{ik_i x_i}\right] \left\{ \prod_{1 \leq j < i \leq n} \left(1 - \frac{ic}{k_i - k_j}\epsilon(x_i - x_j)\right)\right\} \phi^*(x_1) \cdots \phi^*(x_n)|0\rangle\,, \tag{23}$$

a result which we believe to hold for all n.

The result (19) shows that $a(\xi)$ and H may be diagonalized together and suggests the possibility that $a(\xi)$ is already diagonal on the states $|k_1 \cdots k_n\rangle$. This is verified by the calculations of the Appendix, where it is shown that the operators a and b satisfy the following commutation relations among themselves:

$$[a(\xi), b(\xi')] = -\frac{ic}{\xi - \xi' - i\epsilon} b(\xi')a(\xi)\,, \tag{24}$$

$$[a^*(\xi), b(\xi')] = +\frac{ic}{\xi - \xi' + i\epsilon} b(\xi')a(\xi)\,, \tag{25}$$

$$[a(\xi), a(\xi')] = [a(\xi), a^*(\xi')] = 0\,, \tag{26}$$

$$[b(\xi), b(\xi')] = 0\,. \tag{27}$$

From (24)–(26) we see that all the $a(\xi)$, $a^*(\xi)$ commute, and are simultaneously diagonalized by the states (21) with

$$a(\xi)|k_1 \cdots k_n\rangle = \left[\prod_{i=1}^{n}\left(1 - \frac{ic}{\xi - k_i - i\epsilon}\right)\right]|k_1 \cdots k_n\rangle\,, \tag{28}$$

$$a^*(\xi)|k_1 \cdots k_n\rangle = \left[\prod_{i=1}^{n}\left(1 + \frac{ic}{\xi - k_i + i\epsilon}\right)\right]|k_1 \cdots k_n\rangle\,. \tag{29}$$

In addition, the relation (27) shows that the states $|k_1 \cdots k_n\rangle$ are symmetric in $k_1 \cdots k_n$ as indeed the Bethe ansatz states (23) are.

It is interesting to compare the commutators (24)–(27) with the corresponding Poisson brackets (12) to (15). We see that they are identical in form, except for the fact that the ordering of the two operators on the right-hand side of (24)–(26) must be as shown. On the other hand, the Poisson bracket $\{b(\xi), b^*(\xi')\}$, Eq. (16), has no comparably simple operator analog. It can be shown by explicit calculation from the series (11b) and its Hermitian conjugate that the commutator $[b(\xi), b^*(\xi')]$ *does not vanish* even if $\xi \neq \xi'$. This illustrates the danger in making any general statements about the correspondence between Poisson brackets and commutators.

Using the results (21) and (27), we can construct another operator of particular interest,

$$R(\xi) = b(\xi)[a(\xi)]^{-1}\,. \tag{30}$$

Classically, the quantity $b(\xi)/a(\xi)$ is just the reflection coefficient for the Zakharov-Shabat scattering problem. The operator $R(\xi)$ can be used to construct the normalized in and out scattering states. [Note that the states created by $b(\xi)$ are not properly normalized, as can easily be verified by taking inner products.] If we choose the k_i variables in a specified order, e.g.,

$$k_1 < k_2 < \cdots < k_n\,, \tag{31}$$

then we find that

$$|\Phi(k_1, \ldots, k_n)\rangle_{\text{in}} = R(k_1) \cdots R(k_n)|0\rangle \tag{32}$$

is a normalized in state, and

$$|\Phi(k_1, \ldots, k_n)\rangle_{\text{out}} = R(k_n) \cdots R(k_1)|0\rangle \tag{33}$$

is a normalized out state. These are the states which evolve from free plane waves by the unitary Moeller wave operators $U(0, \mp\infty)$. They can be written in a form like Eq. (23) with the factor in curly brackets replaced by

$$\prod_{1 \leq j < i \leq n} [\theta(x_j - x_i) + \theta(x_i - x_j)e^{2i\Delta(k_i - k_j)}] \tag{34}$$

for the in states (32), and by

$$\prod_{1\leq j<i\leq n}[\theta(x_j - x_i)e^{-2i\Delta(k_i-k_j)} + \theta(x_i - x_j)] \tag{35}$$

for the out states (33). Here $\Delta(k_i - k_j)$ is the two-body phase shift given by

$$e^{2i\Delta(k_i-k_j)} = \frac{k_i - k_j - ic}{k_i - k_j + ic}\,. \tag{36}$$

The R operators have a very simple commutation relation which illustrates their relevance to the two-body scattering process,

$$R(k')R(k) = e^{2i\Delta(k-k')}R(k)R(k')\,. \tag{37}$$

Thus, R is an explicit realization of the operators introduced by Zamolodchikov in the S-matrix analysis of certain relativistic theories.[12] Although it is nonrelativistic and hence simpler, the nonlinear Schrödinger model is in many respects a paradigm for this class of theories which includes the sine-Gordon (equivalently massive Thirring) model,[13] the O(N) nonlinear σ model, and the Gross-Neveu model. The operator R is also interesting because, in the classical theory, it is the reflection coefficient $b(\xi)/a(\xi)$ which enters as the kernel of the Gelfand-Levitan equation for the inverse problem associated with the eigenvalue equation (6). In this paper we have considered the operator analog of the direct problem in which the scattering data $a(\xi)$ and $b(\xi)$ are expressed as functionals of the fields $\phi(x)$ and $\phi^*(x)$. It is amusing that we have been led in a natural way to consider the operator analog of the kernel of the classical inverse problem. This suggests that the Gelfand-Levitan analysis (i.e., expressing the field in terms of the scattering data) may also have an operator analog. This question together with applications of these methods to other theories is currently under investigation.

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APPENDIX

In this Appendix we prove the commutation relations between the operators a, a^*, b, b^* given in (24)–(27). The proof will follow closely the classical Poisson-bracket derivation of Zakharov and Manokov, but with careful attention paid to the ordering of the various operators. We first introduce the definition and properties of the normal-ordered operator Jost functions of the eigenvalue equation (6). In addition to the operator $\psi(x,\xi)$ with asymptotic behavior

$$\psi \sim \begin{pmatrix}1\\0\end{pmatrix} e^{i\xi x/2} \text{ as } x\to-\infty\,, \tag{A1}$$

we shall need the operator solution $\chi(x,\xi)$ with the property

$$\chi(x,\xi) \sim \begin{pmatrix}0\\1\end{pmatrix} e^{-i\xi x/2} \text{ as } x\to+\infty\,. \tag{A2}$$

Like ψ, χ is analytic in the lower half ξ plane. Writing the differential equation (6) as an integral equation, we find that the nonvanishing commutators of the Jost functions with the elementary fields ϕ, ϕ^* are given by

$$\begin{aligned}
[\psi_1, \phi^*] &= \frac{i\sqrt{c}}{2}\psi_2\,,\\
[\psi_2, \phi] &= \frac{i\sqrt{c}}{2}\psi_1\,,\\
[\chi_1, \phi^*] &= -\frac{i\sqrt{c}}{2}\chi_2\,,\\
[\chi_2, \phi] &= -\frac{i\sqrt{c}}{2}\chi_1\,,
\end{aligned} \tag{A3}$$

where all the fields are evaluated at the same point x, and we have made the symmetric choice $\int \theta(x)\delta(x) = \frac{1}{2}$. Using the above and the differential equation (6) we may then obtain the fundamental relation

$$i\frac{d}{dx}(\psi_1\chi_2' - \psi_2\chi_1') = \tfrac{1}{2}(\xi' - \xi)(\psi_1\chi_2' + \psi_2\chi_1')\,, \tag{A4}$$

where $\psi \equiv \psi(x,\xi)$ and $\chi' \equiv \chi(x,\xi')$. It is also useful to note that ψ and χ commute,

$$[\psi_i(x,\xi), \chi_j(x,\xi')] = 0\,. \tag{A5}$$

From ψ and χ we may obtain two more normal-ordered solutions to the equation (6) given by

$$\begin{aligned}
\tilde{\psi} &= \begin{pmatrix}\psi_2^*\\ \psi_1^*\end{pmatrix},\\
\tilde{\chi} &= \begin{pmatrix}\chi_2^*\\ \chi_1^*\end{pmatrix}.
\end{aligned} \tag{A6}$$

All the results (A3), (A4), and (A5) are still valid with ψ replaced by $\tilde{\psi}$ and/or χ replaced by $\tilde{\chi}$.

With these definitions and preliminaries we may now prove the following theorem:

Theorem 1. Let Λ denote any of the functions $a(\xi'), a^*(\xi'), b(\xi'), b^*(\xi')$. Then the commutator of $a(\xi)$ with Λ may be expressed in the form

$$[a(\xi), \Lambda] = i\sqrt{c}\int\left(dx\ \psi_2(x,\xi)\frac{\delta\Lambda}{\delta\phi^*(x)}\chi_2(x,\xi) - \psi_1(x,\xi)\frac{\delta\Lambda}{\delta\phi(x)}\chi_1(x,\xi)\right)\,. \tag{A7}$$

Proof. We first note that the series expansion (11a) for $a(\xi)$ may be brought to the form

$$a(\xi)=\sum_{N=0}^{\infty}c^N\prod_{i=1}^{N}\left(\int dx_i e^{i\xi x_i}dy_i e^{-i\xi y_i}\right)\theta(x_1<y_1<x_2<y_2<\cdots<x_N<y_N)$$
$$\times\{\phi^*(x_1)[\phi^*(x_2)\phi(y_1)][\phi^*(x_3)\phi(y_2)]\cdots[\phi^*(x_N)\phi(y_{N-1})]\phi(y_N)\}\,. \tag{A8}$$

We see that in this expression the fields are "almost" ordered in the same way as their arguments—the only exceptions being the pairs in square brackets which are reversed. Considering for example the $N=2$ term in this expansion we find

$$[a^{(2)}(\xi),\Lambda]\equiv c^2\int dx_i e^{i\xi x_i}dy_i e^{-i\xi y_i}\theta(x_1<y_1<x_2<y_2)[\phi^*(x_1)\phi^*(x_2)\phi(y_1)\phi(y_2),\Lambda]$$
$$=c^2\int dx_i e^{i\xi x_i}dy_i e^{-i\xi y_i}\theta(x_1<y_1<x_2<y_2)\{[\phi^*(x_1),\Lambda]\phi^*(x_2)\phi(y_1)\phi(y_2)+\phi^*(x_1)\phi(y_1)[\phi^*(x_2),\Lambda]\phi(y_2)$$
$$+\phi^*(x_1)[\phi(y_1),\Lambda]\phi^*(x_2)\phi(y_2)+\phi^*(x_1)\phi^*(x_2)\phi(y_1)[\phi(y_2),\Lambda]\}\,, \tag{A9}$$

where the ordering has been obtained from the naively expected one by noting that the quantity $[[\phi^*(x_2),\Lambda],\phi(y_1)]+[\phi^*(x_2),[\phi(y_1),\Lambda]]$ vanishes by virtue of the Jacobi identity. Relabeling the integration variables and using $[\phi^*(x),\Lambda]=-\delta\Lambda/\delta\phi(x)$ and $[\phi(x),\Lambda]=\delta\Lambda/\delta\phi^*(x)$ we see that (A9) is just the term of order c^2 in the expansion of the right-hand side of the theorem. By treating each term in the expansion of $a(\xi)$ in a similar manner, the theorem is proved.

We may derive a similar theorem for the operator $b(\xi)$:

Theorem 2. Let Λ denote either $a^*(\xi')$ or $b(\xi')$. Then the commutator of $b(\xi)$ with Λ is given by

$$[b(\xi),\Lambda]=\int dx\left[\psi_2(x,\xi)\frac{\delta\Lambda}{\delta\phi^*(x)}\tilde{\chi}_2(x,\xi)-\psi_1(x,\xi)\frac{\delta\Lambda}{\delta\phi(x)}\tilde{\chi}_1(x,\xi)\right]. \tag{A10}$$

Proof. The proof follows closely that of theorem 1 except that the restriction on the operator Λ requires comment. The expansion of $b(\xi)$ analogous to (A9) is given by

$$b(\xi)=\sum_{N=0}^{\infty}c^N\left(\prod_{i=1}^{N+1}\int dx_i e^{i\xi x_i}\right)\left(\prod_{i=1}^{N}\int dy_i e^{-i\xi y_i}\right)\theta(x_1<y_1<x_2<\cdots<x_N<y_N<x_{N+1})$$
$$\times\{\phi^*(x_1)[\phi^*(x_2)\phi(y_1)][\phi^*(x_3)\phi(y_2)]\cdots[\phi^*(x_{N+1})\phi(y_N)]\}\,. \tag{A11}$$

The essential new point is that the pairs which are out of order now include the two fields whose arguments are furthest to the right, and a careful analysis shows that if the field in Λ whose argument is furthest to the right is ϕ (rather than ϕ^*) then the change of integration variables required to prove the theorem is invalid. This condition eliminates a and b^* from the allowed choices for Λ.

Similar theorems may be proved for a^* and b^*, but in fact this is not necessary since commutators involving a^* or b^* may be obtained from those involving a or b by Hermitian conjugation; theorems 1 and 2 are sufficient to calculate all commutators among a, a^*, b, b^* except $[b(\xi),b^*(\xi')]$. To illustrate the method we will evaluate $[a(\xi),b(\xi')]$. Applying theorem 1 and computing the variational derivatives $\delta b/\delta\phi$ and $\delta b/\delta\phi^*$ we obtain

$$[a(\xi),b(\xi')]=i\sqrt{c}\int dx(\psi_1\psi_2'\tilde{\chi}_2'\chi_1-\psi_2\psi_1'\tilde{\chi}_1'\chi_2)$$
$$=i\sqrt{c}\int dx(\psi_1\tilde{\chi}_2'\psi_2'\chi_1-\psi_2\tilde{\chi}_1'\psi_1'\chi_2)\,, \tag{A12}$$

where $\psi=\psi(x,\xi)$ and $\psi'=\psi(x,\xi')$, etc., and the second line follows from the first by (A5). Using (A4) we see that the integrand is a total derivative

$$\psi_1\tilde{\chi}_2'\psi_2'\chi_1-\psi_2\tilde{\chi}_1'\psi_1'\chi_2=\frac{i}{\xi-\xi'}\frac{d}{dx}[(\psi_1\tilde{\chi}_2'-\psi_2\tilde{\chi}_1')(\psi_1'\chi_2-\psi_2'\chi_1)]\,. \tag{A13}$$

Recalling that $a(\xi)$ is analytic in the lower half plane we may make the replacement $\xi\to\xi-i\epsilon$ and evaluate the contribution at infinity to obtain

$$[a(\xi),b(\xi')]=-\frac{ic}{\xi-\xi'-i\epsilon}b(\xi')a(\xi)\,. \tag{A14}$$

The other commutators, with the exception of $[b(\xi), b^*(\xi')]$, may be evaluated in a similar manner, and yield the results (24) to (27). Note that the restriction on the allowed operators Λ in theorem 2 is very important, since if we had erroneously used theorem 2 with $\Lambda = a$ to evaluate $[a(\xi), b(\xi')]$ we would have obtained the terms on the right-hand side of (A14) in the reverse order.

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[2]After completion of this work, we received an interesting report by J. Honerkamp, P. Weber, and A. Wiesler [University of Freiberg Report No. THEP 79/1 (unpublished)]. These authors have reached a similar conclusion regarding the role of the scattering data operator $b(\xi)$ [Eq. (11b)] as the creation operator for the Bethe eigenstates. They also consider the analogous procedure for the massive Thirring model. We have also been informed by L. Faddeev (private communication) that similar results appear in L. Faddeev and E. Sklyanin, Dokl. Acad. Sci. USSR 243, 1430 (1978) and E. Sklyanin, *ibid.* 244, 1337 (1979).

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