PHYSICAL REVIEW D VOLUME 19, NUMBER 12 15 JUNE 1979

Classification of SU(2) gauge fields: Lorentz-invariant versus gauge-invariant schemes

M. Carmeli* and M. Fischler
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11794
(Received 26 May 1978)

Two different and complementary schemes for classifying SU(2) gauge fields have recently been suggested. These are Wang and Yang's classification using the rank of a matrix and Carmeli's classification using the eigenspinor-eigenvalue equation. In this paper we interrelate the two classification schemes.

I. INTRODUCTION

Recently two new different methods have been proposed to classify SU(2) gauge fields. The two methods have led to two classification schemes of SU(2) gauge fields. One of these schemes, by Wang and Yang,[1] uses the rank of a Lorentz-invariant matrix in order to obtain the classification. The other scheme, by Carmeli,[2] uses the gauge-invariant eigenspinor-eigenvalue equation for the SU(2) gauge fields. In this paper we make a comparison between the two methods and interrelate the different types of fields of the two schemes of classifications.

In Sec. II we define certain tensors and spinors from the Yang-Mills field strengths. In Sec. III we construct the invariants of the SU(2) gauge fields. In Sec. IV the eigenspinor-eigenvalue equation for the SU(2) gauge fields is given and a detailed comparison is made with the method using the rank of the matrix. In the tables we summarize some of the results obtained.

II. GEOMETRY OF GAUGE FIELDS

Let $F_{k\mu\nu}$ be the gauge field strengths. Here $\mu, \nu = 0, 1, 2, 3$ are the spacetime indices, and k is the isospin index taking the values 1, 2, 3. From the field strengths we may define the four-index tensor

$$R_{\mu\nu\rho\sigma} = -F_{k\mu\nu} F_{k\rho\sigma} . \tag{2.1}$$

The tensor $R_{\mu\nu\rho\sigma}$, which is an SU(2) invariant, satisfies the symmetry properties

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = +R_{\rho\sigma\mu\nu} .$$

Hence the tensor $R_{\mu\nu\rho\sigma}$ is skew symmetric in each of the pair of indices $\mu\nu$ and $\rho\sigma$, and is symmetric under the exchange of these two pairs of indices with each other. These symmetry properties are the same as those of the Riemann curvature tensor, or the Weyl conformal tensor, known from the geometry of curved spacetimes, except that $R_{\mu\nu\rho\sigma}$ does *not* satisfy the cyclic identity, $R_{\mu[\nu\rho\sigma]} = 0$, of the Riemann tensor.

It will be also useful to define another tensor $R^*_{\mu\nu\rho\sigma}$, which is also an SU(2) gauge invariant, by

$$R^*_{\mu\nu\rho\sigma} = -F_{k\mu\nu} {}^*F_{k\rho\sigma} . \tag{2.2}$$

Here ${}^*F_{k\rho\sigma}$ is the dual to the tensor $F_{k\rho\sigma}$,

$${}^*F_{k\rho\sigma} = \tfrac{1}{2}(-g)^{1/2} \epsilon_{\rho\sigma\mu\nu} F_k^{\mu\nu} . \tag{2.3}$$

The tensor $R^*_{\mu\nu\rho\sigma}$ has the same symmetry properties as those of $R_{\mu\nu\rho\sigma}$. From the two tensors $R_{\mu\nu\rho\sigma}$ and $R^*_{\mu\nu\rho\sigma}$ we may then define the complex tensor

$$\tilde{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + iR^*_{\mu\nu\rho\sigma} = -F_{k\mu\nu} F^+_{k\rho\sigma} , \tag{2.4}$$

where

$$F^+_{k\rho\sigma} = F_{k\rho\sigma} + i{}^*F_{k\rho\sigma} . \tag{2.5}$$

The new tensor $\tilde{R}_{\mu\nu\rho\sigma}$ also satisfies the same symmetry properties of $R_{\mu\nu\rho\sigma}$ and $R^*_{\mu\nu\rho\sigma}$.

From the tensor $\tilde{R}_{\mu\nu\rho\sigma}$ we may define the Ricci tensor $\tilde{R}_{\alpha\beta} = \tilde{R}^{\sigma}{}_{\alpha\sigma\beta}$ and the Ricci scalar curvature $\tilde{R} = \tilde{R}^{\alpha}{}_{\alpha}$.

Since the tensor $\tilde{R}_{\alpha\beta\gamma\delta}$ has the same symmetry properties (except for the cyclic identity) as those of the Riemann curvature tensor, we may decompose it as follows[3]:

$$\tilde{R}_{\rho\sigma\mu\nu} = \tilde{C}_{\rho\sigma\mu\nu} + \tfrac{1}{2}(g_{\rho\mu}\tilde{R}_{\sigma\nu} - g_{\rho\nu}\tilde{R}_{\sigma\mu} - g_{\sigma\mu}\tilde{R}_{\rho\nu} + g_{\sigma\nu}\tilde{R}_{\rho\mu}) + \tfrac{1}{6}(g_{\rho\nu}g_{\sigma\mu} - g_{\rho\mu}g_{\sigma\nu})\tilde{R} , \tag{2.6}$$

or in the alternative, but equivalent, form

$$\tilde{R}_{\rho\sigma\mu\nu} = \tilde{C}_{\rho\sigma\mu\nu} + \tfrac{1}{2}(g_{\rho\mu}\tilde{S}_{\sigma\nu} - g_{\rho\nu}\tilde{S}_{\sigma\mu} - g_{\sigma\mu}\tilde{S}_{\rho\nu} + g_{\sigma\nu}\tilde{S}_{\rho\mu}) + \tfrac{1}{12}(g_{\rho\nu}g_{\sigma\mu} - g_{\rho\mu}g_{\sigma\nu})\tilde{R} . \tag{2.7}$$

Here $\tilde{S}_{\mu\nu}$ is the trace-free Ricci tensor,

$$\tilde{S}_{\mu\nu} = \tilde{R}_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}\tilde{R} , \tag{2.8}$$

which satisfies $\tilde{S}_{\mu}{}^{\mu} = 0$.

Contracting now either of Eq. (2.7) or (2.8) with respect to the indices ρ and μ, we find that the trace of the tensor $\tilde{C}_{\rho\sigma\mu\nu}$ vanishes, $\tilde{C}^{\rho}{}_{\alpha\rho\beta} = 0$. Hence Eqs. (2.7) or (2.8) express the fact that the tensor $\tilde{R}_{\alpha\beta\gamma\delta}$ decomposes into its irreducible components.

The above results may easily be put into the spinor language. The spinor equivalent to the

Yang-Mills field strength $F_{kAB'CD'}$ may then be decomposed,[4]

$$F_{kAB'CD'}=\epsilon_{AC}\bar{\chi}_{kB'D'}+\chi_{kAC}\epsilon_{B'D'}\,, \tag{2.9}$$

where $\chi_{kAC}=\frac{1}{2}\epsilon^{B'D'}F_{kAB'CD'}$. In the above formulas the upper-case letters denote ordinary SL(2, C) spinor indices taking the values 0 and 1. Primed indices transform with the complex-conjugate elements of SL(2, C). The spinors ϵ_{AC} and $\epsilon_{B'D'}$ are the skew-symmetric Levi-Civita metric tensors defined by $\epsilon_{01}=1$. SL(2, C) indices are raised and lowered by means of the above skew-symmetric spinors, using the convention $\alpha^A=\epsilon^{AB}\alpha_B$ and $\alpha^B\epsilon_{BA}=\alpha_A$, thus $\alpha^0=\alpha_1$ and $\alpha^1=-\alpha_0$, for an arbitrary spinor α^A.

The Yang-Mills spinor χ_{kAB} is symmetrical in its two SL(2, C) indices A and B: $\chi_{kAB}=\chi_{kBA}$. Hence it has nine complex independent components. These are χ_{k00}, $\chi_{k01}=\chi_{k10}$, and χ_{k11}. These nine complex components are equivalent to the original eighteen real components of the gauge field strengths $F_{k\mu\nu}$.

We may also find the spinor equivalent to the tensor ${}^*F_{k\mu\nu}$, the dual to $F_{k\mu\nu}$. We then find

$${}^*F_{kAB'CD'}=i(\epsilon_{AC}\bar{\chi}_{kB'D'}-\chi_{kAC}\epsilon_{B'D'}). \tag{2.10}$$

Subsequently, the spinor equivalent to the tensors $R_{\mu\nu\rho\sigma}$ and $R^*_{\mu\nu\rho\sigma}$ may be found. So may the spinor equivalent to $F^*_{k\rho\sigma}$. As a result, the spinor equivalent to the tensor $R_{\mu\nu\rho\sigma}$ is given by

$$R_{AB'CD'EF'GH'}=-F_{kAB'CD'}F_{kEF'GH'}\,.$$

Using now Eq. (2.9) we then obtain

$$\begin{aligned}R_{AB'CD'EF'GH'}=-(&\xi_{ACEG}\epsilon_{B'D'}\epsilon_{F'H'}+\epsilon_{EG}\zeta_{ACF'H'}\epsilon_{B'D'}\\&+\epsilon_{AC}\bar{\zeta}_{B'D'EG}\epsilon_{F'H'}+\epsilon_{AC}\epsilon_{EG}\bar{\xi}_{B'D'F'H'})\,.\end{aligned} \tag{2.11}$$

In Eq. (2.11) the two spinors ξ_{ABCD} and $\zeta_{ABC'D'}$ are defined by[5]

$$\xi_{ABCD}=\chi_{kAB}\chi_{kCD} \tag{2.12}$$

and

$$\zeta_{ABC'D'}=\chi_{kAB}\bar{\chi}_{kC'D'}\,, \tag{2.13}$$

respectively.

From the definition of the spinor ξ_{ABCD} we see that it satisfies the following symmetry properties:

$$\xi_{ABCD}=\xi_{BACD}=\xi_{ABDC}=\xi_{CDAB}\,.$$

Hence it can be decomposed into the sum of a totally symmetric spinor η_{ABCD} and a scalar P,

$$\xi_{ABCD}=\eta_{ABCD}+\tfrac{1}{6}P(\epsilon_{AC}\epsilon_{BD}+\epsilon_{AD}\epsilon_{BC})\,. \tag{2.14}$$

Here the scalar P is the trace of the spinor ξ_{ABCD},

$$P=\xi_{AB}{}^{AB}=\tfrac{1}{4}F_{k\mu\nu}(F_k^{\mu\nu}+i{}^*F_k^{\mu\nu})\,. \tag{2.15}$$

A simple calculation, moreover, shows that

$$\xi_{AC}{}^{C}{}_{B}=\tfrac{1}{2}P\epsilon_{AB}\,.$$

The spinor ξ_{ABCD} resembles in its properties the gravitational field spinor χ_{ABCD} which combines the Weyl conformal spinor and the Ricci scalar curvature. The difference between the two spinors being only in their trace structure, the trace of the gravitational field spinor is $\chi_{AB}{}^{AB}=-R/4$, where R is the Ricci scalar curvature, which is a real quantity. Here, however, the invariant P is a *complex* function. The role of P in gauge fields, nevertheless, seems to be similar to that of the cosmological constant of general relativity theory, even though it is now a complex function.

The spinor η_{ABCD} in Eq. (2.14), on the other hand, is a totally symmetrical spinor in all of its four indices and is given by

$$\eta_{ABCD}=\tfrac{1}{3}(\xi_{ABCD}+\xi_{ACBD}+\xi_{ADBC})\,. \tag{2.16}$$

It is therefore completely analogous to the Weyl conformal spinor, and has only five independent complex components: $\eta_0=\eta_{0000}$, $\eta_1=\eta_{0001}$, $\eta_2=\eta_{0011}$, $\eta_3=\eta_{0111}$, and $\eta_4=\eta_{1111}$.

The other spinor $\zeta_{ABC'D'}$ appearing in Eq. (2.11), defined by Eq. (2.13), satisfies the same symmetries that the trace-free Ricci spinor $\phi_{ABC'D'}$ satisfies, namely

$$\zeta_{ABC'D'}=\zeta_{BAC'D'}=\zeta_{ABD'C'}=\bar{\zeta}_{C'D'AB}\,.$$

It, therefore, has nine real independent components. The spinor $\zeta_{ABC'D'}$ is, moreover, irreducible. Its physical meaning lies in the fact that it is proportional to the energy-momentum tensor of the Yang-Mills field (see details in Ref. 4).

From the spinor $R_{AB'CD'EF'GH'}$ given by Eq. (2.11) we may define the Ricci spinor $R_{CD'GH'}=R^{EF'}{}_{CD'EF'GH'}$. We then find that

$$R_{CD'GH'}=2\zeta_{CGD'H'}-\tfrac{1}{2}(P+\bar{P})\epsilon_{CG}\epsilon_{D'H'}\,.$$

We also find for the Ricci scalar curvature

$$R=R^{GH'}{}_{GH'}=-2(P+\bar{P})\,.$$

We now find the spinor equivalent to the tensor $R^*_{\alpha\beta\gamma\delta}$ defined by Eq. (2.2). It is given by

$$R^*_{AB'CD'EF'GH'}=-F_{kAB'CD'}{}^*F_{kEF'GH'}\,.$$

Using Eqs. (2.9) and (2.10) we obtain

$$\begin{aligned}R^*_{AB'CD'EF'GH'}=i(&\xi_{ACEG}\epsilon_{B'D'}\epsilon_{F'H'}-\epsilon_{EG}\zeta_{ACF'H'}\epsilon_{B'D'}\\&+\epsilon_{AC}\bar{\zeta}_{B'D'EG}\epsilon_{F'H'}-\epsilon_{AC}\epsilon_{EG}\bar{\xi}_{B'D'F'H'})\,.\end{aligned} \tag{2.17}$$

The Ricci spinor and Ricci scalar curvature are subsequently given by

$$R^{*}_{CD'GH'}=\tfrac{1}{2}(P-\bar{P})\epsilon_{CG}\epsilon_{D'H'},$$
$$R^{*}=2i(P-\bar{P}),$$

respectively.

Finally, the spinor equivalent to the complex tensor $\tilde{R}_{\alpha\beta\gamma\delta}$, defined by Eq. (2.4), is given by

$$\tilde{R}_{AB'CD'EF'GH'}=R_{AB'CD'EF'GH'}+iR^{*}_{AB'CD'EF'GH'}.$$

We then find that

$$\tilde{R}_{AB'CD'EF'GH'}=-2(\xi_{ACEG}\epsilon_{B'D'}+\epsilon_{AC}\bar{\xi}_{B'D'EG})\epsilon_{F'H'}. \quad (2.18)$$

The Ricci spinor and Ricci scalar curvature are then given by

$$\tilde{R}_{CD'GH'}=2\zeta_{CGD'H'}-P\epsilon_{CG}\epsilon_{D'H'},$$
$$\tilde{R}=-4P,$$

respectively.

A fourth spinor that can be constructed out of the Yang-Mills spinor is given by

$$\chi_{ABCDEF}=\epsilon_{ijk}\chi_{iAB}\chi_{jCD}\chi_{kEF}. \quad (2.19)$$

It satisfies the following symmetry:

$$\chi_{ABCDEF}=\chi_{BACDEF}=\chi_{ABDCEF}=\chi_{ABCDFE}.$$

In addition, the spinor χ_{ABCDEF} keeps or changes its sign, depending upon whether the pairs of indices AB, CD, EF are an even or an odd permutation of the apirs of numbers $00, 01(=10), 11$, and zero otherwise. Hence it can be decomposed as follows:

$$\chi_{ABCDEF}=\tfrac{1}{24}Q(\epsilon_{AG}\epsilon_{BE}\epsilon_{DF}+\epsilon_{AF}\epsilon_{BC}\epsilon_{DE}+\epsilon_{AC}\epsilon_{BF}\epsilon_{DE}+\epsilon_{AE}\epsilon_{BC}\epsilon_{DF}+\epsilon_{AD}\epsilon_{BF}\epsilon_{CE}+\epsilon_{AD}\epsilon_{BE}\epsilon_{CF}+\epsilon_{AF}\epsilon_{BD}\epsilon_{CE}+\epsilon_{AE}\epsilon_{BD}\epsilon_{CE}), \quad (2.20)$$

where Q is a complex quantity, the trace of the spinor χ_{ABCDEF}:

$$Q=\chi_{A}{}^{C}{}_{C}{}^{E}{}_{E}{}^{A}=\epsilon^{CB}\epsilon^{ED}\epsilon^{AF}\chi_{ABCDEF}. \quad (2.21)$$

Finally, two more mixed indices spinors, with unprimed and primed indices, can be defined as follows:

$$\phi_{ABCDE'F'}=\epsilon_{ijk}\chi_{iAB}\chi_{jCD}\bar{\chi}_{kE'F'}, \quad (2.22)$$

$$\phi_{ABC'D'E'F'}=\epsilon_{ijk}\chi_{iAB}\bar{\chi}_{jC'D'}\bar{\chi}_{kE'F'}. \quad (2.23)$$

The relationship between them can easily be found.

III. INVARIANTS OF THE YANG-MILLS FIELD

The invariants of the SU(2) gauge fields may now be constructed from the spinors defined in the last section. Other invariants that occur in the coupled Yang-Mills and the gravitational or the electromagnetic fields will also be discussed.

We already have two complex invariants P and Q defined in the last section by Eqs. (2.15) and (2.21), respectively. More invariants may be constructed as follows[5-7]:

$$R=\zeta_{ABC'D'}\zeta^{ABC'D'}, \quad (3.1)$$

$$S=\xi_{ABCD}\xi^{ABCD}=G+\tfrac{1}{3}P^{2}, \quad (3.2)$$

$$T=\phi_{ABCDE'F'}\phi^{ABCDE'F'}, \quad (3.3)$$

where the invariant G is given by

$$G=\eta_{ABCD}\eta^{ABCD}. \quad (3.4)$$

We may define two more invariants F and H by means of

$$F=\xi_{AB}{}^{CD}\xi_{CD}{}^{EF}\xi_{EF}{}^{AB}=H+PG+\tfrac{1}{9}P^{3}, \quad (3.5)$$

and

$$H=\eta_{AB}{}^{CD}\eta_{CD}{}^{EF}\eta_{EF}{}^{AB}. \quad (3.6)$$

It will be noted that the seven invariants P, Q, S, T, F, G, and H are complex functions, whereas R (not to be confused with the Ricci scalar curvature R) is real. The reality of the invariant R can easily be seen if we calculate its complex conjugate:

$$\bar{R}=\bar{\zeta}_{A'B'CD}\bar{\zeta}^{A'B'CD}$$
$$=\zeta_{CDA'B'}\zeta^{CDA'B'}=\zeta_{ABC'D'}\zeta^{ABC'D'}=R.$$

Finally, two more real invariants R' and R'' may be defined as follows:

$$R'=\zeta_{AB}{}^{G'H'}\zeta^{AB}{}_{C'D'}\zeta_{EF}{}^{C'D'}\zeta^{EF}{}_{G'H'}, \quad (3.7)$$

$$R''=\xi^{AB}{}_{CD}\zeta_{AB}{}^{G'H'}\bar{\xi}^{E'F'}{}_{G'H'}\bar{\zeta}_{E'F'}{}^{CD}. \quad (3.8)$$

The reality of the invariants R' and R'' may easily be verified.

The above invariants may also be defined in a somewhat different way by means of the gauge-invariant, but Lorentz-dependent, 3×3 symmetrical matrix

$$\xi_{ij}=\xi_{ABCD}e_{i}^{AB}e_{j}^{CD}, \quad (3.9)$$

where e_{i}^{AB} is some basis in the spinor space. One then finds, for instance, that

$$P=\mathrm{Tr}\,\xi,\quad S=\mathrm{Tr}\,\xi^{2},$$
$$F=\mathrm{Tr}\,\xi^{3},\quad T=\epsilon_{ijk}\epsilon_{mnp}\xi_{im}\xi_{jn}\bar{\xi}_{kp}. \quad (3.10)$$

Other invariants can be written in terms of those of Eq. (3.10). The invariants G and H, for instance, may be written in terms of S, P, and F by means of Eqs. (3.2) and (3.5).

We recall that the number of invariants of the SU(2) guage fields in terms of real functions is nine. Hence we obviously have interdependence relations between the above-defined invariants. A selection should be made here that is based on physical grounds just as in the gravitational and the electromagnetic cases. It will also be noted that the two invariants G and H are constructed from the totally symmetric spinor η_{ABCD} in precisely the same way as the gravitational field invariants I and J are constructed from the totally symmetric Weyl conformal spinor ψ_{ABCD}. The following five sets of invariants,

$$P,\ Q,\ R,\ S,\ T; \tag{3.11a}$$

$$P,\ T,\ R,\ G,\ H; \tag{3.11b}$$

$$P,\ F,\ R,\ S,\ T; \tag{3.11c}$$

$$P,\ S,\ F,\ R,\ R',\ R''; \tag{3.11d}$$

$$P,\ G,\ H,\ R,\ R',\ R''; \tag{3.11e}$$

may be taken, for instance, as the invariants of the SU(2) gauge fields. Each of the sets of invariants given by Eqs. (3.11) does not form a complete system of invariants by itself in the sense of the theory of invariants. Therefore, not every algebraic invariant which is constructed from the spinor χ_{kAB} can be expressed as a polynomial in terms of each of the above sets of invariants. The invariant Q defined by Eq. (2.21), for instance, is not a rational function of the set of invariants P, S, F, R, R', and R'' given by Eq. (3.17d). This fact can easily be seen since such a function should be of even order in the Yang-Mills spinor χ_{kAB}, whereas the invariant Q is of odd order in χ_{kAB}. The invariant Q is, nevertheless, algebraically dependent on the set of invariants P, S, F, R, R', and R''. In fact, the square of Q may be written in the form

$$Q^2 = F - \tfrac{3}{2}PS + \tfrac{1}{2}P^3 .$$

We face a situation here similar to that of gravitation and electrodynamics.

IV. LORENTZ-INVARIANT vs GAUGE-INVARIANT SCHEMES

We now write the eigenspinor-eigenvalue equation

$$\xi^{AB}{}_{CD}\phi^{CD} = \lambda\phi^{AB} \tag{4.1}$$

for the SU(2) gauge fields. Here ξ_{ABCD} is the gauge-invariant spinor defined by Eq. (2.12) and ϕ^{AB} is a symmetrical spinor, the eigenspinor. Using Eq. (2.14) expressing the spinor ξ_{ABCD} in terms of the totally symmetric spinor η_{ABCD}, the eigenspinor equation (4.1) may then be written in the form

$$\eta^{AB}{}_{CD}\phi^{CD} = \lambda'\phi^{AB} , \tag{4.2}$$

where the new eigenvalues λ' are related to λ by $\lambda' = \lambda - P/3$, and P is the field invariant given by Eq. (2.15).

The classification of the spinor ξ_{ABCD} is accordingly reduced to the classification of the completely symmetric spinor η_{ABCD}. The eigenvalue equation obtained from Eq. (4.2) can easily be shown to be given by

$$f(\lambda') \equiv \lambda'^3 - \tfrac{1}{2}G\lambda' - \tfrac{1}{3}H = 0 , \tag{4.3}$$

where G and H are the two field invariants given by Eqs. (3.4) and (3.6), respectively. We then have

$$\begin{aligned} &\lambda_1' + \lambda_2' + \lambda_3' = \eta_{AB}{}^{AB} = 0 , \\ &\lambda_1'^2 + \lambda_2'^2 + \lambda_3'^2 = \eta_{ABCD}\eta^{ABCD} = G , \\ &\lambda_1'^3 + \lambda_2'^3 + \lambda_3'^3 = \eta_{AB}{}^{CD}\eta_{CD}{}^{EF}\eta_{EF}{}^{AB} = H , \end{aligned} \tag{4.4}$$

where λ_1', λ_2', and λ_3' are the eigenvalues which may or may not be distinct.

The spinor η_{ABCD}, and therefore the spinor ξ_{ABCD}, can now be classified according to the possible numbers of distinct eigenvalues and eigenspinors. The maximum number of eigenvalues is three. Corresponding to each eigenvalue there is at least one eigenspinor. Hence when we have three distinct eigenvalues, we have three eigenspinors. This is the general type I field. Since we have two cases for which $P \neq 0$ and $P = 0$, we obtain the fields of types Ip and Io, respectively. The symmetrical spinor η_{ABCD} will then have the general form

$$\eta_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)} , \tag{4.5}$$

where α_A, β_B, γ_C, and δ_D are four arbitrary one-index spinors and parentheses indicate symmetrization, thus giving 24 terms in Eq. (4.5). Detailed analysis of Eqs. (4.4) show that in this case we have $G^3 \neq 6H^2$, in complete analogy to gravitation where $I^3 \neq 6J^2$ for Petrov type I and the Weyl conformal spinor having an identical expression to that given by Eq. (4.5).

When two of the eigenvalues, let us say λ_1' and λ_2', are equal we have two and three distinct eigenspinors. The classes of fields are now of types II and D, respectively. Again we have two cases: $P \neq 0$ and $P = 0$. The spinor η_{ABCD} will then have the form where two of the four one-index spinors are identical,

$$\eta_{ABCD} = \alpha_{(A}\alpha_B\gamma_C\delta_{D)} , \tag{4.6}$$

for types IIp and IIo, and where the four spinors

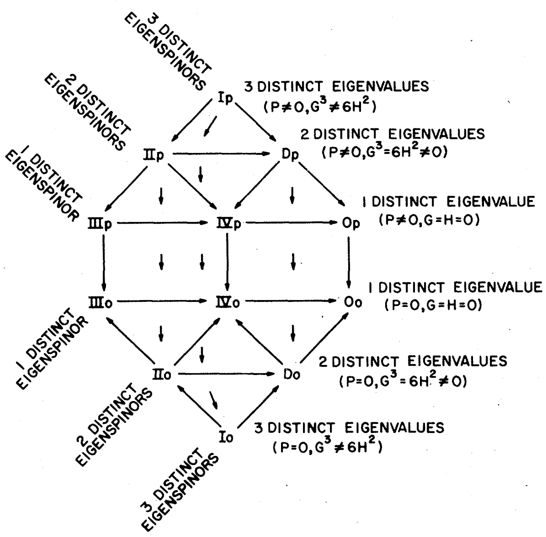

FIG. 1. Diagram of classification of SU(2) gauge fields (Carmeli, Ref. 2). The completely symmetrical spinor η_{ABCD} has identical decomposition to the Weyl conformal spinor ψ_{ABCD} for each one of the twelve classes of fields in the diagram. For types Ip and Io we have $\eta_{ABCD}=\alpha_{(A}\beta_B\gamma_C\delta_{D)}$; for types II$p$ and IIo: $\eta_{ABCD}=\alpha_{(A}\alpha_B\gamma_C\delta_{D)}$; for types Dp and Do: $\eta_{ABCD}=\alpha_{(A}\alpha_B\delta_C\delta_{D)}$; for types III$p$ and IIIo: $\eta_{ABCD}=\alpha_{(A}\alpha_B\alpha_C\delta_{D)}$; for types IV$p$ and IVo: $\eta_{ABCD}=\alpha_A\alpha_B\alpha_C\alpha_D$; and for types Op and Oo: $\eta_{ABCD}=0$. The invariants G and H are given by $G=\eta_{ABCD}\eta^{ABCD}$ and $H=\eta_{AB}{}^{CD}\eta_{CD}{}^{EF}\eta_{EF}{}^{AB}$ in complete analogy to the two gravitational invariants in terms of the Weyl conformal spinor ψ_{ABCD}.

are identical in pairs,

$$\eta_{ABCD}=\alpha_{(A}\alpha_B\delta_C\delta_{D)}\,, \tag{4.7}$$

for types Dp and Do. Equations (4.4) then show that $G^3=6H^2\neq 0$ for these four cases. Again the analogy with gravitation is remarkable.

Finally, if $\lambda_1'=\lambda_2'=\lambda_3'$ then we may have one, two or three eigenspinors. The fields obtained are of types III, IV, or 0, respectively. The spinors η_{ABCD} will then specialize where three of the four spinors are identical,

$$\eta_{ABCD}=\alpha_{(A}\alpha_B\alpha_C\delta_{D)}\,, \tag{4.8}$$

for types IIIp and IIIo fields, where all the four spinors are identical,

$$\eta_{ABCD}=\alpha_A\alpha_B\alpha_C\alpha_D\,, \tag{4.9}$$

for types IVp and IVo fields, and $\eta_{ABCD}=0$ for the fields Op and Oo. Equations (4.4) now show that $G=H=0$ for these six cases, just as in general relativity theory.

The results of the above analysis is summarized in Fig. 1. For each case the spinor ξ_{ABCD} is obtained by adding to η_{ABCD} the expression with the P term according to Eq. (2.14). Notice that Op is not a zero field since $P\neq 0$, whereas Oo includes the zero field since $\xi_{ABCD}=0$ in this case. The analogy with gravitation is most remarkable since the spinor η_{ABCD} and the invariants G and H satisfy the same conditions that the Weyl conformal spinor ψ_{ABCD} and the gravitational invariants I and J satisfy.

Using now the expression $\lambda'=\lambda-P/3$ and Eq. (3.2) in the eigenvalue equation (4.3), we then obtain for the latter

$$\lambda^3+m_1\lambda^2+m_2\lambda+m_3=0\,, \tag{4.10}$$

where

$$m_1=-P\,, \tag{4.11a}$$

$$m_2=\tfrac{1}{2}(P^2-S)\,, \tag{4.11b}$$

$$m_3=-\tfrac{1}{3}(F-\tfrac{3}{2}PS+\tfrac{1}{2}P^3)\,. \tag{4.11c}$$

Here the invariants P, S, and F may be written

TABLE I. Case 1 of Wang-Yang (Ref. 1) classification scheme for which the rank of the matrix Δ is equal to 3, and its corresponding types of fields in Carmeli's classification scheme (Ref. 2).

Subcases of Wang-Yang scheme	Invariants P	G	H	Relations between invariants	Corresponding Carmeli type
1A	0	0	√	$G^3\neq 6H^2$	Io
1B	0	√	√	$G^3\neq 6H^2$	Io
1C	0	√	√	$G^3=6H^2\neq 0$	IIo, Do
1D	√	0	0	$G=H=0$	IIIp, IVp, Op
1E	√	0	√	$G^3\neq 6H^2$, $P^3\neq -9H$	Ip
1F	√	√	0	$G^3\neq 6H^2$, $P^2\neq 9G/2$	Ip
1G	√	√	√	$G^3\neq 6H^2$ and P is not a root of the polynomial (4.15).	Ip
1H	√	√	√	$G^3=6H^2\neq P^6/6^3$	IIp, Dp

TABLE II. Case 2 of Wang-Yang classification scheme (Ref. 1) for which the rank of Δ is 2, and its corresponding types of fields in Carmeli's scheme of classification (Ref. 2).

Subcases of Wang-Yang scheme	Invariants P	G	H	Relations between invariants	Corresponding Carmeli type
2A	0	√	0	$G^3 \neq 6H^2$	Io
2B	0	0	0	$G=H=0$. $\xi = A\tilde{A}$, $\Delta = \tilde{A}A$. Rank $\xi = 1$. Rank $\Delta = 2$. $A = \begin{pmatrix} 1 & 0 & 0 \\ i & \lambda & i\lambda \\ 0 & 1 & i \end{pmatrix}$ (λ is complex).	IVo
2C	√	0	√	$G^3 \neq 6H^2$. [3 distinct roots for the polynomial (4.15)].	Ip
2D	√	√	0	Same as above.	Ip
2E	√	√	√	Same as above.	Ip
2F	√	√	√	$G^3 = 6H^2 \neq 0$. [2 distinct roots for the polynomial (4.15)].	IIp, Dp

as

$$P = \mathrm{Tr}\,\xi = \mathrm{Tr}\,\Delta, \tag{4.12a}$$

$$S = \mathrm{Tr}\,\xi^2 = \mathrm{Tr}\,\Delta^2, \tag{4.12b}$$

$$F = \mathrm{Tr}\,\xi^3 = \mathrm{Tr}\,\Delta^3, \tag{4.12c}$$

where ξ is the gauge-invariant symmetrical matrix obtained from the spinor ξ_{ABCD} when expanded in an appropriate basis in the spin space (see Sec. III), and Δ is a matrix whose elements are given by $\Delta_{ab} = \chi_{aAB}\chi_b^{AB}$.

The symmetrical matrix Δ is Lorentz invariant and has been used by Wang and Yang for classifying the SU(2) gauge fields according to its rank. One easily finds that

$$m_3 = -\det\xi = -\det\Delta = \tfrac{1}{54}(9PG - 18H - 2P^3). \tag{4.13}$$

The two matrices Δ and ξ have a simple presentation. If $A_{ia} = E_{ia} + iH_{ia}$, where E_{ia} and H_{ia} are the "electric" and "magnetic" parts of the Yang-Mills field, then $A_{ia} = e_i^{AB}\chi_{aAB}$, where e_i^{AB} is an appropriate basis in spinor space and $\Delta = \tilde{A}A$ and $\xi = A\tilde{A}$. their transformation rules are then given by $A' = AG$ and $A' = LA$, thus

$$\Delta' = \tilde{G}\Delta G, \quad \xi' = L\xi\tilde{L},$$

where G is a three-dimensional orthogonal real matrix and L is a three-dimensional complex orthogonal matrix, both with determinants unity. We finally notice that the eigenvalues λ_1, λ_2, and λ_3 of the spinor ξ_{ABCD} satisfy

$$\begin{aligned} \sum \lambda_i &= \xi_{AB}{}^{AB} = P, \\ \sum \lambda_i{}^2 &= \xi_{ABCD}\xi^{ABCD} = S, \\ \sum \lambda_i{}^3 &= \xi_{AB}{}^{CD}\xi_{CD}{}^{EF}\xi_{EF}{}^{AB} = F, \end{aligned} \tag{4.14}$$

when expressed in terms of the invariants P, S, and F.

TABLE III. Cases 3 and 4 of Wang-Yang classification scheme (Ref. 1) for which the ranks of Δ are equal to 1 and 0, respectively, and their corresponding types of fields in Carmeli's classification scheme (Ref. 2).

Subcases of Wang-Yang scheme	Invariants P	G	H	Relations between invariants	Corresponding Carmeli type
3A	√	√	√	$G^3 = 6H^2 \neq 0$. [2 distinct roots for the polynomial (4.15)].	IIp, Dp
3B	0	0	0	$G=H=0$	IIIo, IVo, Oo
4	0	0	0	$G=H=0$	IVo, Oo

TABLE IV. Corresponding cases of Wang and Yang using the method of the rank of Δ (cases 1, 2, 3, 4 denote the ranks of $\Delta=3, 2, 1, 0$) vs field types of Carmeli using the eigenspinor equation.

Wang-Yang case scheme	Carmeli corresponding type
1	Ip, IIp, Dp, IIIp, IVp, Op, Io, IIo, Do
2	Ip, IIp, Dp, Io, IVo
3	IIp, Dp, IIIo, IVo, Oo
4	IVo, Oo

In order to compare the classification schemes using the ranks of the matrices Δ and ξ and the spinor method, we now consider in some detail the case for which the ranks of Δ and ξ are equal to three, namely their determinants are different from zero. From Eq. (4.13) we see that the polynomial

$$m_3(P)=-\tfrac{1}{27}P^3+\tfrac{1}{6}GP-\tfrac{1}{3}H \tag{4.15}$$

of third order in the invariant P should not vanish. Changing variables from P to z by $z=-P/3$, we obtain the polynomial

$$f(z)=z^3-\tfrac{1}{2}Gz-\tfrac{1}{3}H\,. \tag{4.16}$$

This is exactly the polynomial one obtains from the eigenspinor equation (4.1), as it should be.

From the above it is clear that there exists a guage in which Δ is realized by a matrix whose elements are constructed out of the invariants P, G, and H. A possible presentation of such a matrix is given by

$$\Delta=\begin{pmatrix}\tfrac{1}{3}P & \alpha_3 & \alpha_2\\ \alpha_3 & \tfrac{1}{3}P & \alpha_1\\ \alpha_2 & \alpha_1 & \tfrac{1}{3}P\end{pmatrix} \tag{4.17}$$

for instance, with $\sum\alpha_i{}^2=G/2$ and $\alpha_1\alpha_2\alpha_3=H/6$. One may indeed easily check that $\mathrm{Tr}\Delta=P$, $\mathrm{Tr}\Delta^2=P^2/3+G$, $\mathrm{Tr}\Delta^3=P^3/9+PG+H$, and $\det\Delta$ satisfies Eq. (4.13).

Coming now back to the polynomial (4.15) and the condition $\det\Delta=(2P^3-9GP+18H)/54\neq 0$. This is case 1 in Wang and Yang's scheme of classification. It yields eight subcases ($1A-1H$) which are listed in Table I. They correspond to nine types of fields in Carmeli's classification scheme. These are Ip, IIp, Dp, IIIp, IVp, Op, Io, IIo, and Do. When $\det\Delta=0$ and $\mathrm{rank}\Delta=2$, which is case 2 in Wang and Yang's classification scheme, we have six subcases ($2A-2F$). These are listed in Table II. They correspond to five types of fields in Carmeli's classification scheme. These are Io, IVo, Ip, IIp, and Dp. The results for cases 3 and 4 (ranks of Δ are 1 and 0) are listed in Table III. Summaries of the correspondence between the two methods are given in Tables IV and V.

TABLE V. Types of fields using the eigenspinor equation and their corresponding cases using the rank of the matrix Δ (cases 1, 2, 3, 4 denote ranks of Δ equal to 3, 2, 1, 0).

Carmeli type	Wang-Yang corresponding case scheme
Ip	1, 2
IIp, Dp	1, 2, 3
IIIp, IVp, Op	1
Io	1, 2
IIo, Do	1
IIIo	3
IVo	2, 3, 4
Oo	3, 4

Useful details about these classification schemes and other related schemes appear in a series of papers by Anandan, Roskies, and Tod,[8-10] and in a report by the authors.[11]

ACKNOWLEDGMENTS

It is a pleasure to acknowledge stimulating conversations and discussions with Max Dresden, Werner Israel, Moshe Kugler, Garry Ludwig, John Moffat, Roger Penrose, and in particular Chen Ning Yang. This work was supported in part by the National Science Foundation under Contract No. Phys.-76-15328.

*On leave from Department of Physics, Ben Gurion University, Beer Sheva 84120, Israel.

[1]L.-L. Wang and C. N. Yang, Phys. Rev. D 17, 2687 (1978).
[2]M. Carmeli, Phys. Lett. 77B, 188 (1978).
[3]L. P. Eisenhart, *Riemannian Geometry* (Princeton Univ. Press, Princeton, N.J., 1926).
[4]M. Carmeli, Ch. Charach, and M. Kaye, Nuovo Cimento 45B, 310 (1978).
[5]M. Carmeli, in *Differential Geometrical Methods in Mathematical Physics*, edited by K. Bleuler, H. R. Petry, and A. Reetz (Springer, Heidelberg, 1978).
[6]R. Roskies, Phys. Rev. D 15, 1722 (1977).
[7]M. Carmeli, Phys. Rev. Lett. 39, 523 (1977).
[8]J. Anandan and K. P. Tod, Phys. Rev. D 18, 1144 (1978).
[9]J. Anandan and R. Roskies, Phys. Rev. D 18, 1152 (1978).
[10]J. Anandan, J. Math. Phys. 20, 260 (1979).
[11]M. Carmeli and M. Fischler, Stony Brook Report No. ITP-SB-78-33 (unpublished).