Self-dual propagating wave solutions in Yang-Mills gauge theory

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Self-dual propagating wave solutions to the sourceless field equations for Yang-Mills theory are presented. The superposition properties of these solutions are investigated.

I. INTRODUCTION

The importance of self-dual solutions of the Yang-Mills equation of motion in Euclidean space[1] was recognized some time ago. The properties of the instanton solution of Belavin, Polyakov, Schwartz, and Tyupkin have been the focus of much recent investigation, and its physical significance as the signal of quantum tunneling between topologically inequivalent vacuums is well established. Because of the success of the concept of self-duality in Euclidean space, it would be interesting to examine how these ideas carry over into Minkowski space. Indeed, some self-dual Minkowski solutions have already been found,[2] and it has been demonstrated by Rebbi[3] that there exists a complex self-dual solution whose real part still satisfies the Yang-Mills equation of motion and is gauge equivalent to the solution found by de Alfaro, Fubini, and Furlan[4] (dAFF).

Also, in Minkowski space we expect, by analogy with electromagnetism, the existence of some form of propagating solutions which could be regarded as non-Abelian plane waves.[5] In general, however, the nonlinearity of the equations of motion for non-Abelian Yang-Mills fields means that it would not be possible to superpose such solutions as in the electromagnetic case. But by requiring that the fields satisfy the simpler self-duality condition, some linearization of the equations of motion may occur. For example, Cervero[6] has shown that there exists a family of complex solutions for which a superposition principle may be defined, and in particular, that it is possible to add two complex solutions to obtain the dAFF solution.

The above considerations motivated us to search for self-dual solutions of the Minkowski field equations characterized by a four-vector k_μ, defining the direction of propagation of the solution. In Sec. II we define our notations, give general expressions for the field strengths, Lagrangian, and energy-momentum densities, and exhibit the self-duality properties of our solutions. In Sec. III we exhibit the solutions explicitly and discuss their properties. For completeness, we also include the Euclidean version of the solutions.

II. NOTATION AND GENERAL FORM OF SOLUTIONS

The SU(2) gauge potentials,[7] defined as usual by

$$A^\mu = \frac{A_a^\mu \sigma^a}{2i}, \tag{1}$$

where σ^a are the Pauli matrices, satisfy the field equations

$$\begin{aligned} F^{\mu\nu} &= \frac{F_a^{\mu\nu}\sigma^a}{2i} = \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu], \\ \partial_\mu F^{\mu\nu} &+ [A_\mu, F^{\mu\nu}] = 0. \end{aligned} \tag{2}$$

In order to search for propagating solutions in Minkowski space, we begin with the ansatz

$$A^\mu = i\sigma^{\mu\nu} k_\nu f(k\cdot x), \tag{3}$$

where $k_\nu = (k^0, \vec{k})$ is an arbitrary Minkowski four-vector and the $\sigma^{\mu\nu}$ are antisymmetric matrices defined by

$$\begin{aligned} \sigma^{ij} &= \frac{1}{4i}[\sigma^i, \sigma^j] = \tfrac{1}{2}\epsilon^{ijk}\sigma^k, \\ \sigma^{i0} &= \frac{i\sigma^i}{2}. \end{aligned} \tag{4}$$

They satisfy the O(3, 1) commutation relations

$$[\sigma^{\mu\nu}, \sigma^{\zeta\delta}] = -i(g^{\mu\zeta}\sigma^{\nu\delta} - g^{\nu\zeta}\sigma^{\mu\delta} + g^{\mu\delta}\sigma^{\zeta\nu} - g^{\nu\delta}\sigma^{\zeta\mu}). \tag{5}$$

This anzatz is assumed for two reasons. Firstly, $\sigma_{\mu\nu}$ as defined by (4) is the Minkowski analog of the self-dual antisymmetric tensor appearing in the instanton solution and arises as a natural choice for SU(2) gauge groups. The function $f(k\cdot x)$, on the other hand, is motivated by the form of general plane-wave solutions in electromagnetism.

On substitution of (3) we find that (2) reduces to

$$\sigma^{\mu\nu} k_\nu k^2 (f'' - 2f^3) = 0, \tag{6}$$

where the prime denotes differentiation with respect to $u = k\cdot x$.

The Lagrangian and energy-momentum density for the potentials (3) are given by

$$\mathcal{L}=\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$$
$$=-\tfrac{3}{2}k^4(f'^2+f^4) \quad (7)$$

and

$$\theta_{\mu\nu}=2\ \mathrm{Tr}F_{\mu\alpha}F_\nu{}^\alpha-g_{\mu\nu}\mathcal{L}$$
$$=(2k^2k_\mu k_\nu-\tfrac{1}{2}g_{\mu\nu}k^4)(f^4-f'^2)\ . \quad (8)$$

The components of the electric and magnetic fields can be expressed as

$$E^{ia}=F^{0ia}=i\ \mathrm{Tr}(F^{0i}\sigma^a)$$
$$=(ik^ik^a-i\delta^{ia}k_0{}^2-\epsilon^{ija}k_jk^0)(f'+f^2)+i\delta^{ia}k^2f^2 \quad (9)$$

and

$$B^{ia}=\tfrac{1}{2}\epsilon^{ijk}F^{jka}=\tfrac{1}{2}\epsilon^{ijk}\ \mathrm{Tr}(F^{jk}\sigma^a)$$
$$=(i\epsilon^{ija}k_0k_j+k^ik^a+\delta^{ia}k^jk_j)(f'+f^2)-\delta^{ia}k^2f^2\ . \quad (10)$$

The self-duality condition[3] may be written as

$$*F^{\mu\nu}=\pm F^{\mu\nu} \quad (11)$$

with

$$*F^{\mu\nu}=\tfrac{1}{2}i\epsilon^{\mu\nu}{}_{\alpha\beta}F^{\alpha\beta}\ . \quad (12)$$

From (3) we find this corresponds to[8]

$$f'=\mp f^2 \quad (13)$$

or

$$E^{ia}=\mp iB^{ia}\ . \quad (14)$$

In addition, we note that upon choosing the minus sign in (11), $k^\alpha k_\alpha=k^2=0$ is also a solution.

It is clear from (9) and (10) that the electric and magnetic fields are complex. In fact, as is easily seen from (13), the energy-momentum tensor vanishes as a consequence of the self-duality condition. This is only possible for nonzero $F_{\mu\nu}$ if the fields can take on complex values. Consequently, since our solution has nonzero values for the gauge-covariant quantities $F_{\mu\nu}$, it cannot be made real by a complex gauge transformation. We can, however, obtain an equivalent formulation in terms of real gauge fields by extending our gauge group SU(2) to the larger noncompact group SL (2, C).[9]

III. DISCUSSION OF SOLUTIONS

We are now in a position to investigate the solutions of (6). We note that the solutions of the equation of motion fall into two classes. Class 1 is

$$\text{(i)}\ \ f''-2f^3=0\ . \quad (15)$$

This equation may be integrated once to give

$$f^4-f'^2=M\ , \quad (16)$$

where M is an arbitrary positive constant.

We see from (13) that, unless $M=0$, the field configuration obtained will not be self-dual, and, furthermore, since the components of the energy-momentum tensor of such a solution would be constant, the energy density integrated over all space would diverge. Equation (16) may be integrated to obtain $f(u)$ in terms of Jacobi elliptic functions. For completeness we note the solution is given by

$$f(u)=M^{1/4}\mathrm{nc}(\sqrt{2}\,M^{1/4}(u+C)\,|\,\tfrac{1}{2}), \quad (17)$$

where C is a constant of integration.

This solution becomes singular whenever

$$u=k\cdot x=\frac{(2n+1)}{\sqrt{2}M^{1/4}}K(\tfrac{1}{2})-C,\quad n=0,1,2,\ldots$$

where $K(\frac{1}{2})$ denotes the complete elliptic integral of the first kind with parameter $\frac{1}{2}$.

Choosing $M=0$, (15) may be integrated directly to give

$$f=\frac{1}{C\pm k\cdot x}\ , \quad (18)$$

where the plus sign corresponds to the self-dual case and the minus sign to the anti-self-dual case. The electric and magnetic fields are thus given explicitly by

$$E^{ia}=-iB^{ia}=i\delta^{ia}k^2f^2\ \ \text{for}\ f=\frac{1}{C+k\cdot x} \quad (19)$$

and

$$E^{ia}=+iB^{ia}$$
$$=[2ik^ik^a-i\delta^{ia}(k_0{}^2-k_jk^j)-2\epsilon^{ija}k_jk^0]f^2\ \text{for}\ f=\frac{1}{C-k\cdot x}\ . \quad (20)$$

We note that f is singular on the surface $C\pm k\cdot x=0$ where the electric and magnetic fields diverge quadratically.

These solutions may be interpreted as a propagating disturbance characterized by a four-vector k_μ. The quantity $k_0/|\vec{k}|$ determines the phase speed of the solution. If k_μ is spacelike, that is, if $|\vec{k}|>k^0$, then the solution propagates with a phase speed less than c. If k_μ is timelike, then the phase speed is greater than c. The interpretation of a wave with a phase velocity greater than c would seem to be a difficult one. A conservative attitude is to neglect such a solution completely as unphysical. Solutions with $k^2\neq 0$ cannot be superposed.

The second class is

$$\text{(ii)}\quad k^\alpha k_\alpha = (k^0)^2 - (k^i)^2 = 0. \tag{21}$$

In this case, f remains completely arbitrary and, in particular, may be a plane wave, $\exp(ik\cdot x)$. These solutions have several interesting properties which are similar to electromagnetic waves. They are anti-self-dual and their electric and magnetic fields are given by

$$E^{ia} = iB^{ia} = (ik^i k^a - i\delta^{ia}k_0{}^2 - \epsilon^{ija}k_j k^0)(f' + f^2)\,. \tag{22}$$

Since $k^0 = |\vec{k}|$, these solutions propagate at the speed of light. Nevertheless, k_μ is not an energy-momentum four-vector as in the electromagnetic case because the components of $\theta^{\mu\nu}$ all vanish.

Solutions with the same k_μ may also be linearly superposed. That is, if A_1^μ and A_2^μ are solutions such that

$$A_1^\mu = i\sigma^{\mu\nu}k_\nu f_1(k\cdot x)\,,$$

$$A_2^\mu = i\sigma^{\mu\nu}k_\nu f_2(k\cdot x)\,,$$

then the sum $A_1^\mu + A_2^\mu$ is also a solution of the equation of motion. We stress that these are genuine non-Abelian waves which nevertheless have a superposition property. It is clear from (22) that since the electric and magnetic fields contain a term which depends on f^2, these fields do not add up linearly when the potentials are superposed. This is to be contrasted with the analogous situation for electromagnetism described by the Abelian Maxwell equations where such nonlinear terms do not arise.

In the expressions given for A_1^μ and A_2^μ above, both potentials have the same k_μ. In fact, it is always possible to superpose waves going in the same direction. That is, if we have

$$A_1^\mu = i\sigma^{\mu\nu}(k_1)_\nu f_1(k_1\cdot x)\,,$$

$$A_2^\mu = i\sigma^{\mu\nu}(k_2)_\nu f_2(k_2\cdot x)\,,$$

then $A_1^\mu + A_2^\mu$ is also a solution provided that $k_1{}^2 = k_2{}^2 = 0$ and

$$(k_1)_\mu = \alpha(k_2)_\mu\,, \tag{23}$$

where α is an arbitrary (real) constant.

In the electromagnetic case, the sourceless Maxwell equations also allow for a solution of the form $A_\mu = \epsilon_\mu f(k\cdot x)$ where f is an arbitrary function, provided that $k^\mu k_\mu = \epsilon^\mu k_\mu = 0$. Our $k^2 = 0$ solutions are the generalizations of this in the non-Abelian case.

We also note that an analogous set of self-dual or anti-self-dual solutions may be constructed from the ansatz

$$\overline{A}^\mu = i\overline{\sigma}^{\mu\nu}k_\nu f(k\cdot x)\,, \tag{24}$$

where

$$\overline{\sigma}^{ij} = \sigma^{ij}\,,$$

$$\overline{\sigma}^{i0} = -\sigma^{i0}\,.$$

These solutions have the properties that

$$*\overline{F}^{\mu\nu} = \overline{F}^{\mu\nu}$$

if

$$f' = f^2 \text{ or } k^2 = 0$$

and

$$*\overline{F}^{\mu\nu} = \overline{F}^{\mu\nu}$$

if

$$f' = -f^2.$$

The Lagrangian density for these solutions is also given by (7) and all components of the energy-momentum tensor vanish.

For completeness we also exhibit the Euclidean-space version of our solutions. In this case we begin with the ansatz

$$A^\mu = i\sigma_E^{\mu\nu}k^\nu f(k\cdot x)\,, \tag{25}$$

where

$$\sigma_E^{ij} = \frac{1}{4i}[\sigma^i, \sigma^j]\,,$$

$$\sigma_E^{i0} = \tfrac{1}{2}\sigma^i\,,$$

and

$$k\cdot x = k^1x^1 + k^2x^2 + k^3x^3 + k^4x^4\,.$$

The $\sigma_E^{\mu\nu}$ satisfy the usual O(4) commutation relations.

The Euclidean equation of motion thus reduces to

$$\sigma_E^{\mu\nu}k^\nu k^\alpha k^\alpha(f'' - 2f^3) = 0\,, \tag{26}$$

where the prime denotes differentiation with respect to $v = k\cdot x$.

As in the Minkowski case, we find that f must satisfy the differential equation

$$f'' - 2f^3 = 0. \tag{27}$$

Note, however, that the analog of the $k^2 = 0$ solutions in Euclidean space corresponds to the trivial vacuum solution since all components of k^μ must be zero for k^2 to vanish in the Euclidean metric.

Expressions may be obtained for the Lagrangian density, energy-momentum tensor, and the topological quantum number. These are given by

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{Tr}\,F^{\mu\nu}F^{\mu\nu}$$

$$= \tfrac{3}{2}k^4(f'^2 + f^4)\,, \tag{28}$$

$$\theta^{\mu\nu} = 2\ \mathrm{Tr}(F^{\mu\lambda}F^{\nu\lambda}) - \tfrac{1}{2}\delta^{\mu\nu}\ \mathrm{Tr}(F^{\xi\lambda}F^{\xi\lambda})$$
$$= (2k^2k^\mu k^\nu - \tfrac{1}{2}\delta^{\mu\nu}k^4)(f^4 - f'^2)\,, \qquad (29)$$

$$q = -\frac{1}{16\pi^2}\int d^4x\ \mathrm{Tr}({}^*F^{\mu\nu}F^{\mu\nu})$$
$$= \frac{6k^4}{16\pi^2}\int d^4x\, f'f^2\,. \qquad (30)$$

The only self-dual or anti-self-dual solutions of (27) correspond to $f' = \pm f^2$ or

$$f = \frac{1}{C \mp k\cdot x}\,.$$

This solution is singular on the surface $C \pm k\cdot x = 0$ and hence the Lagrangian density, energy-momentum tensor, and topological charge density are not well defined there.

In conclusion, we remark that our search for propagating wave like solutions of the Yang-Mills equation, motivated by a simple analogy with electromagnetism, leads us to consider potentials of the form (3) or (24). Because of the richer structure of non-Abelian theories, we find there are solutions with a propagation four-vector such that $k^2 = 0$, which, since our choice of $f(k\cdot x)$ in the gauge potential is arbitrary, can be chosen to be perfectly well behaved. These solutions exhibit some superposition properties and may be regarded as the non-Abelian analogs of plane electromagnetic waves.

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[7]Our metric is $g_{\mu\nu} = \mathrm{diag}\,(+---)$.

[8]In order to simplify the calculation, we may write

$$\sigma^{\mu\nu} = \tfrac{1}{2}\eta^{i\mu\nu}\sigma^i,\quad i = 1, 2, 3$$

where $\eta^{i\mu\nu} = \epsilon^{i\mu\nu 0} - ig^{i\mu}g^{\nu 0} + ig^{i\nu}g^{\mu 0}$. A useful identity is

$$\epsilon^{\mu\nu\alpha}{}_{\beta}\,\eta^{i\xi\beta} = i(\eta^{i\alpha\nu}g^{\xi\mu} + \eta^{i\mu\alpha}g^{\xi\nu} + \eta^{i\nu\mu}g^{\xi\alpha})\,.$$

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