Statistical model of inclusive distributions: Formulation of the model and mean-field approximation

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We formulate a statistical model of inclusive distributions. The source function of the observed particle is regarded as a random variable on phase space; its distribution is determined by a Lorentz-invariant probability density. We suggest that the Lorentz invariance of the correlation functions is broken spontaneously; this may account for the observed anisotropy of the inclusive distributions.

I. INTRODUCTION

Statistical models of particle reactions are useful in that they account for some gross features of the distributions in a relatively simple manner. Although explicit constructions differ in many technical details, the basic physical assumption underlying all statistical models is the same. Roughly speaking, one assumes that dynamical details of the process in question are so complicated that they become practically irrelevant. The measurable particle distributions *on the average* exhibit simple features which can be described "statistically," i.e., in terms of some random variables.

In constructing a statistical model of particle reactions, two basic issues have to be settled.

(i) One has to determine an appropriate (possibly multidimensional) random variable on the phase space, in terms of which particle distributions are described.

(ii) One has to determine a suitable (i.e., physically "reasonable") measure on the space of random variables.

Once this is done, particle distributions, correlation functions, etc. can be calculated (in principle) as appropriate moments of this measure.

One of the latest developments in this direction has been the model of inclusive reactions proposed by Scalapino and Sugar[1] (see also Ref. 2). In effect, these authors start from the well-known fact that at sufficiently high energies the inclusive cross section for a particle c produced in the reaction $a+b\to c+X$ can be formally written as a source correlation function, viz.,

$$E\frac{d\sigma}{d^3p}\propto\int d^4x d^4y e^{i(px-py)}\langle j(x)j(y)\rangle \equiv\langle\phi(p)\phi^*(p)\rangle\,, \quad (1.1)$$

where the expectation value is taken with respect to the initial state $(a+b)$. The operator $j(x)$ is the source operator of particle c, (assumed to be Hermitian), while $\phi(p)$ is its on-mass-shell Fourier transform. Scalapino and Sugar[1] propose to replace the quantum-mechanical expectation value in (1.1) by a statistical one. The quantity $\phi(p)$ is to be regarded as a random variable, and the expectation value of $\phi(p)\phi^*(p)$ is to be calculated as the "equal-p limit" of an "ensemble average,"

$$\langle\phi(p)\phi^*(p)\rangle=\lim_{\substack{p-p'\to 0\\(p^2=p'^2=m^2)}}\int D\phi D\phi^*\rho(\phi,\phi^*)\phi(p)\phi^*(p')\,. \quad (1.2)$$

This assumption effectively settles issue (i): the random variable chosen to describe inclusive distributions is the on-shell Fourier transform of the source function of the observed particle.

Issue (ii) is settled in Ref. 1 for the special case of spinless observed particles considered on a one-dimensional projection of the phase space (the rapidity axis) only. (The restriction to spinless particles is physically reasonable: Pions make up some 80% to 90% of all the particles produced.) A one-dimensional Laudau-Ginzburg weight is chosen in (1.2), viz.,

$$\rho(\phi,\phi^*)\approx\exp\left[-\int dy\left(\frac{1}{2}\frac{d\phi^*}{dy}\frac{d\phi}{dy}+\frac{a}{2}\phi\phi^*+\frac{b}{4}(\phi\phi^*)^2\right)\right], \quad (1.3)$$

where y is the conventional rapidity variable.

The authors of Ref. 1 show that such an assumption for ρ describes the main qualitative features of inclusive distributions on the longitudinal phase space.

In this series of papers we propose to extend the main idea expressed in Ref. 1 to inclusive distributions on the full phase space. Such an extension is physically well-motivated, since there exists a large amount of data describing inclusive distributions as a function of both rapidities and transverse momenta. Despite its innocent appearance, however, such an extension is highly nontrivial from the theoretical point of view. As it turns out, the problem in question is equivalent to the problem of solving a Euclidean quantum field theory on a three-dimensional curved space. Considerable theoretical efforts notwithstanding, this is still largely an open problem. Accordingly, we have to be guided by physical intuition and by results available on analogous theories in flat space. The basic purpose of this paper is to give a proper formulation of the problem and to investigate some rough qualitative features of the solutions.

The contents of this paper is therefore organized as follows. In the next section we list the fundamental assumptions of the model, motivated by the work done in Ref. 1. Section III is mostly pedagogical in its nature: We derive some elementary properties of the phase space (the "playground" of the model). While such properties are generally well-known, they are rederived and listed here for the purpose of future reference. (Perhaps a consistent geometrical treatment of particle kinematics adds some unusual flavor to the discussion.) We conjecture that the remarkable stability of distributions in transverse momentum (i.e., their weak dependence on energy and on the nature of the colliding particles) is due to intrinsic properties of the random distribution on phase space (the "Feynman gas"). In particular, we investigate whether the Feynman gas can develop an anisotropic "condensate" (Sec. IV). We cannot give a complete answer; nevertheless, by investigating the qualitative properties of the random distribution, it becomes plausible that his is a real possibility. The results are discussed in Sec. V.

II. FORMULATION OF THE MODEL

Following Ref. 1, we consider inclusive production of one kind of spinless particles in hadronic (or semihadronic) reactions. All the following considerations can be easily extended to more general cases, viz. particles carrying spin and/or intrinsic quantum numbers. Correspondingly, we assume that the inclusive distributions can be described in terms of a Hermitian random variable $\phi(p)$ (being the on-shell Fourier transform of a source function) defined on the single-particle phase space, $p_0^2-\vec{p}^2=m^2$, where m is the mass of the observed particle. In what follows, units are chosen in such a way that $\hbar=c=m=1$. The Hermiticity of the random variable is equivalent to the relationship

$$\phi(-p)=\phi^*(p)\,, \tag{2.1}$$

where the asterisk stands for complex conjugation. As a consequence, the domain of definition of $\phi(p)$ may be restricted to the hyperboloid, H,

$$p_0^2-\vec{p}^2=1\,, \quad p_0>0\,. \tag{2.2}$$

Obviously, (2.2) defines the single-particle phase space for the emitted particles. In the spirit of statistical models, we assume that the space defined by (2.2) is the support of all distributions considered. Eventually, we impose restrictions in the center-of-mass system (c.m.s.) of the colliding primaries of the form

$$Q\geqslant p_0>0\,, \tag{2.3}$$

where Q stands for the "c.m.s. energy available for particle production" and, hence, it is of the order of magnitude $\approx\sqrt{s}$, for sufficiently high incident energies. However, in this paper, we shall be dealing mostly with the "infinite-energy limit," i.e., with the space defined by (2.2) alone.

Clearly, (2.2) and (2.3) do not take energy-momentum conservation into account properly. However, kinematic restrictions are expected to influence the shape of the distributions substantially only near the boundary of the phase space. At sufficiently high energies and "inside" the phase space, kinematic effects are expected to be small.

We now assume that the generating functional of inclusive distributions is given by the functional integral:

$$Z[j]=\int D\phi D\phi^*\exp(-\beta W)\exp\left[i\int dV(\phi^*j+\phi j^*)\right], \tag{2.4}$$

where W is an integral of a local "entropy density," S: $W=\int dVS_p$ and β is a constant. Following Ref. 1, we conjecture

$$S_p=\tfrac{1}{2}g^{ab}\nabla_a\phi^*\nabla_b\phi+U_p(\phi^*\phi)\,, \tag{2.5}$$

where ∇_a $(1>a>3)$ stands for a derivative with respect to independent coordinates on the phase space. In these formulas, g^{ab} is the metric tensor and dV is the invariant volume element on phase space. Most of the time we shall also assume that U is a quadratic polynomial in $\phi^*\phi$, as it is in a standard Landau-Ginzburg model, although some of the discussion applies to more general functions U as well. Obviously, the existence of Z requires $U\to+\infty$ as $\phi^*\phi\to\infty$.

III. GEOMETRY OF THE PHASE SPACE

The hyperboloid (2.2) is isomorphic to the symmetric space, SO(3,1)/SO(3). Conveniently, the standard SO(3) subgroup is chosen to stabilize the point P_0 ($p_0=1$, $\vec{p}=0$). Since the action of SO(3,1) is transitive, each coordinatization of (2.2) corresponds to a particular "boost convention" and *vice versa.* A metric on the hyperboloid can be chosen as the natural metric induced by the Cartan-Killing form on SO(3,1). Equivalently, if ξ^i are a set of three independent boost parameters (i.e., coordinates of the homogeneous space), such that $p^\mu = p^\mu(\xi)$ satisfies (2.2), the metric can be read off from the elementary formula for the invariant distance:

$$ds^2 = \eta_{\mu\nu}\frac{\partial p^\mu}{\partial \xi^i}\frac{\partial p^\nu}{\partial \xi^k}d\xi^i d\xi^k \equiv g_{ik}d\xi^i d\xi^k , \tag{3.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$. From the point of view of applications to this model, a boost convention is convenient if it satisfies two criteria. First, it contains a "preferred direction," since eventually we shall be looking for anisotropic distributions with (at most) an axial symmetry. (Thus, for instance a "Wigner boost," i.e., one along a geodesic connecting P_0 with the desired point P on the hyperboloid, does not give a convenient parametrization.) Second, we should attempt to obtain as simple a parametrization of the metric as possible. In particular, we want to exhibit the maximal possible number of cyclic coordinates. [There are two cyclic coordinates on SO(3,1)/SO(3).]

These criteria leave us essentially with two boost conventions [modulo elements of SO(3), the stabilizer of P_0]. Either we first boost in the plane transverse to the preferred direction and afterwards along that direction ("Feynman boost"), or we boost in the reverse order (as is done in usual "null-plane parametrizations"[3]). We exhibit both types of boosts below, together with the expressions of the metric tensor. We use the representation of momenta in terms of 2×2 matrices, so that for any momentum [not necessarily on the hyperboloid (2.2)] we have

$$P = \sigma_0 p^0 + \sigma_k p^k = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}. \tag{3.2}$$

In the last equation σ_0 and σ_k stand for a standard set of Pauli matrices σ_0 being the unit matrix. The Lorentz transform of P is given by

$$P' = APA^\dagger , \quad \det A = 1 .$$

A. Feynman boost

Choose

$$A = \exp(\tfrac{1}{2}y\sigma_3)\exp(\tfrac{1}{2}i\varphi\sigma_3)\exp(\tfrac{1}{2}\sinh^{-1}t^{1/2}\sigma_1). \tag{3.3}$$

This results in the following parametrization of the momentum components:

$$p^0 = \cosh y\,(1+t)^{1/2}, \quad p^3 = \sinh y\,(1+t)^{1/2}, \\ p^1 = \sqrt{t}\cos\varphi , \quad p^2 = \sqrt{t}\sin\varphi . \tag{3.4}$$

On calculating ds^2 from (3.1) we find

$$ds^2 = (1+t)dy^2 + \frac{1}{4}\frac{dt^2}{t(1+t)} + t\,d\varphi^2 , \tag{3.5}$$

so that the nonvanishing components of the inverse metric tensor and the invariant volume element become

$$g^{yy} = \frac{1}{1+t} , \quad g^{tt} = 4t(1+t), \quad g^{\varphi\varphi} = \frac{1}{t} , \\ dV = \tfrac{1}{2}dy\,dt\,d\varphi . \tag{3.6}$$

Notice, in particular, that the invariant Laplacian has the simple expression

$$\nabla^2\psi = 4\frac{\partial}{\partial t}\left(t(1+t)\frac{\partial\psi}{\partial t}\right) + \frac{1}{1+t}\frac{\partial^2\psi}{\partial y^2} + \frac{1}{t}\frac{\partial^2\psi}{\partial\varphi^2} . \tag{3.7}$$

It is easily verified that ∇^2 is an elliptic operator everywhere on the phase space. The rapidity y and the azimuthal angle φ are cyclic coordinates.

B. "Lightlike boost"

We choose

$$A = (\sigma_0 + \tfrac{1}{2}\sigma_A v_A + \tfrac{1}{2}i\epsilon_{AB}\sigma_A v_B) \\ \times\left(l^{-1/2}\frac{\sigma_0+\sigma_3}{2} + l^{1/2}\frac{\sigma_0-\sigma_3}{2}\right), \tag{3.8}$$

where $1 \leq A, B, \ldots \leq 2$. The transverse boost [the first factor in (3.8)] is an element of a Galilean subgroup of SO(3,1): It leaves a null ray, $p_0 + p_3 = 0$, invariant.

The boost (3.8) gives rise to the following parametrization of the points on (2.2):

$$p^0 = \frac{1}{2}\left(l + \frac{1+v_A v_A}{l}\right), \quad p^3 = \frac{1}{2}\left(-l + \frac{1-v_A v_A}{l}\right), \quad p^A = \frac{v^A}{l} . \tag{3.9}$$

We find the following expression for the invariant distance:

$$ds^2 = (1/l^2)(dl^2 + dv_A dv_A) ; \tag{3.10}$$

hence, this coordinatization brings the metric into a conformally flat form. The transverse Galilean velocities are cyclic coordinates. The components of the inverse metric tensor and the invariant volume element become

$$g^{ll} = l^2 , \quad g^{AB} = l^2\delta^{AB} , \quad dV = \frac{dl\,dv^1 dv^2}{l^3} . \tag{3.11}$$

In this coordinatization, the invariant Laplacian has the following expression:

$$\nabla^2\psi = l^2\left[l\frac{\partial}{\partial l}\left(\frac{1}{l}\frac{\partial\psi}{\partial l}\right) + \partial_A\partial_A\psi\right] . \tag{3.12}$$

The coordinatization (3.9) of the phase space may not be the most convenient one from the point of view of a direct comparison of a theory with experimental data. However, owing to the simplicity of the expression of the metric, this coordinate system is suitable for the formulation of a statistical model of the production of spinning particles. We briefly outline the construction of such a model.

The random variable describing the inclusive production of a particle of spin j is the $(2j+1)$-component source function $\vec{\phi}(p)$. Obviously, $\vec{\phi}$ is Hermitian, and it transforms according to the representation (j) of the stability group of the point P of the hyperboloid (2.2). It is now straightforward to generalize expressions (2.4) and (2.5) for this case. In particular, since the "potential term" U in (2.5) is local, it simply generalizes to a term of the form $U(\vec{\phi}^*(p)\cdot\vec{\phi}(p))$. However, the correlation term has to be modified. We have

$$S_p=\tfrac{1}{2}g^{ab}(\nabla_a\vec{\phi})^*\cdot(\nabla_b\vec{\phi})+U(\vec{\phi}^*\cdot\vec{\phi})\,, \tag{3.13}$$

where $\nabla_a\vec{\phi}$ is now the *covariant derivative* of $\vec{\phi}$,

$$\nabla_a\vec{\phi}=(\partial_a+i\Gamma^c_{ba}J^b_c)\vec{\phi}\,, \tag{3.14}$$

where $a,b,\ldots$ run through the values of $1,2,l$ [cf. (3.9)]. The matrices J^b_c are the generators of local rotations ("Wigner rotations"), and in a local orthonormal frame they have the standard expression in terms of angular momentum matrices J^α_λ. The connection between the J^α_λ and the J^a_b is easily established by using (3.10). In fact, we can read off the *triad coefficients* by inspection. Define the one-forms ω^α $(\alpha=1,2,l)$ such that

$$ds^2=\delta_{\alpha\beta}\omega^\alpha\omega^\beta\,.$$

Clearly, $\omega^\alpha=e^\alpha_a dx^a$, where the coordinate differentials (dl,dv^A) have been collectively denoted by dx^a. Obviously,

$$e^\alpha_a=\delta^\alpha_a\frac{1}{l}\,,\quad e^a_\alpha=l\delta^a_\alpha\,,$$

so that (3.15)

$$J^a_b=e^a_\chi e^\lambda_b J^\chi_\lambda=\delta^a_\chi\delta^\lambda_b J^\chi_\lambda\,.$$

The connection coefficients can be easily read off either directly or by making use of the transformation law of a Riemann connection under conformal transformations, see, e.g., Hawking and Ellis.[4] We find

$$\Gamma^a_{bc}=(-1/l)(\delta^a_b\delta^l_c+\delta^Q_c\delta^l_b-\delta_{bc}\delta^a_l)\,. \tag{3.16}$$

Finally, upon introducing the standard Cartesian components of the angular momentum operators by the relations

$$J_A=\epsilon_{AB}J^B_l\,,\quad J_3=\tfrac{1}{2}\epsilon_{AB}J^{AB}\,,$$

we find

$$\nabla_l\vec{\phi}=\frac{\partial\vec{\phi}}{\partial l}\,,$$

$$\nabla_A\vec{\phi}=\frac{\partial\vec{\phi}}{\partial v^A}+\frac{i}{l}\epsilon_{AB}J_B\vec{\phi}\,,$$

so that the explicit expression of the entropy density for a spinning particle reads as follows:

$$S_p=\tfrac{1}{2}l^2\left[\frac{\partial\vec{\phi}^*}{\partial l}\frac{\partial\vec{\phi}}{\partial l}+\delta^{AB}\left(\frac{\partial\vec{\phi}}{\partial v^A}+\frac{i}{l}\epsilon_{AR}J_R\vec{\phi}\right)^*\cdot\left(\frac{\partial\vec{\phi}}{\partial v^B}+\frac{i}{l}\epsilon_{BS}J_S\vec{\phi}\right)\right]+U(\vec{\phi}^*\vec{\phi})\,. \tag{3.17}$$

IV. SPONTANEOUS ANISOTROPY IN THE FEYNMAN GAS

The entropy densities as written down in Sec. III, are invariant under the group SO(3,1). As long as we approximate the available phase-space volume by the full hyperboloid (2.2), the generating functional Z is Lorentz invariant. This is clearly unphysical. In fact, it immediately follows that the correlation functions (1.2) are *constant.* [Outline of proof: $\langle\phi(p)\Phi^*(p)\rangle$ is a Lorentz invariant for spinless particles. Since the hyperboloid is a symmetric space, we can apply a Lorentz transformation to obtain $\langle\phi(p)\phi^*(p)\rangle=\langle\phi(0)\phi^*(0)\rangle$, Q.E.D.]

As mentioned in the Introduction, we conjecture that the observed anisotropy of inclusive distributions develops largely as a consequence of a spontaneous breakdown of the Lorentz invariance. (Using the language of statistical mechanics, the Feynman gas develops an anisotropic condensate.) The existence of a condensate is characterized by a (locally) stable minimum of the functional W introduced in Sec. II, viz., $\delta W=0$, $\delta^2W>0$. We immediately remark that the condensate need not correspond to an absolute minimum of W. In fact, the colliding primaries define a preferred direction in the c.m.s. hence the *source* of the Feynman gas is definitely anisotropic. However, we conjecture that the Feynman gas develops under its own rules of dynamics, with the source acting as a small perturbation only. In such a case, the main role of a small anisotropic perturbation is just to stabilize a certain local minimum of W. For future reference we record that *if* there exists a stable condensate Φ_0, then the expression of the normalized correlation function reads in the leading ("mean-field") approximation

$$\frac{1}{\sigma_T} E\frac{d\sigma}{d^3p} \propto \frac{1}{Z}\langle\phi(p)\phi^*(p)\rangle \simeq |\phi_0(p)|^2. \tag{4.1}$$

We use Feynman coordinatization of the phase space, and look for cylindrically symmetric functions. The condition $\delta W=0$ is then equivalent to the Euler-Lagrange equation

$$\left[\frac{\partial}{\partial t}\left(t(1+t)\frac{\partial\phi}{\partial t}\right)+\frac{1}{4(t+1)}\frac{\partial^2\phi}{\partial y^2}\right]-\frac{1}{2}\frac{\partial U}{\partial\phi^*}=0, \tag{4.2}$$

whereas the second variation of W reads

$$\delta^2 W=\frac{1}{2}\int dV\psi^*\left[-\frac{\partial}{\partial t}\left(t(1+t)\frac{\partial\psi}{\partial t}\right)-\frac{1}{4(t+1)}\frac{\partial^2\psi}{\partial y^2} -\frac{1}{4t}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{\partial^2 U}{\partial\phi^*\partial\phi}\bigg|_{\Phi=\Phi_0}\psi\right] +(\text{surface terms}). \tag{4.3}$$

In Eq. (4.3) ψ stands for the deviation from the "condensate" Φ_0, which is the solution of (4.2). Equation (4.2) is to be solved subject to the boundary conditions $\Phi_0=0$ on the boundary of phase space, and, hence, ψ also has to vanish there. As a consequence, the surface terms in Eq. (4.3) can be shown to vanish. For any reasonable potential (except for one which is linear in $\Phi^*\Phi$), Eq. (4.2) is a nonlinear elliptic partial differential equation; hence, its solution poses a highly nontrivial problem. In what follows, we restrict ourselves to a qualitative discussion of the solutions, keeping in mind mainly polynomial functions U, typically of the Landau-Ginzburg form,

$$U=A\phi^*\phi+\tfrac{1}{2}B(\phi^*\phi)^2, \tag{4.4}$$

where A can be of either sign; however, necessarily $B>0$, cf. Sec. II.

One expects that if Φ_0 has anything to do with the observed distributions, it is a relatively slowly varying function of y far from the boundary of phase space, whereas it should be a rapidly decreasing function of t. This observation allows us to approximately reduce our problem to a one-dimensional one. In fact, if we are sufficiently far away from the boundary of phase space, the dependence on y and t is approximately uncorrelated, i.e., $\Phi_0\approx f(t)g(y)$, where both f and g may be chosen to be real. Since g is supposed to be slowly varying, we approximately replace (4.2) and (4.3) by their averages taken over a sufficiently large rapidity interval, viz.,

$$\frac{d}{dt}\left(t(1+t)\frac{df}{dt}\right)+\frac{f}{4(t+1)}\left\langle\frac{1}{g}\frac{d^2g}{dy^2}\right\rangle-\frac{1}{2}\left\langle\frac{1}{g}\frac{\partial U}{\partial\phi^*}\right\rangle=0, \tag{4.5}$$

and similarly, $\partial^2U/\partial\Phi^*\partial\Phi$ in (4.3) is replaced by its average over rapidity.

With this approximate reduction, the linear operator standing in square brackets in (4.3) can also be reduced to a one-dimensional one. In fact, we can now separate variables by putting

$$\psi=\sum_{m,n}\chi_{mn}(t)\exp[i(m\varphi+ny)],$$

so that,

$$\delta^2W=\frac{\pi}{2}Y\sum_{m,n}\int_0^\infty dt\chi^*_{m,n}\left[-\frac{d}{dt}\left(t(1+t)\frac{d\chi}{dt}\right)+\left(\frac{m^2}{4t}+\frac{n^2}{4(1+t)}\right)\chi_{mn}+\left\langle\frac{\partial^2U}{\partial\phi^*\partial\phi}\bigg|_{\Phi=\Phi_0}\right\rangle\chi_{mn}\right].$$

On substituting further

$$t=\tfrac{1}{2}(\cosh e^Z-1),\quad \chi_{mn}=e^{Z/2}(\sinh e^Z)^{-1/2}\rho_{mn},$$

the expression of the second variation becomes

$$\delta^2W=\frac{\pi Y}{2}\sum_{m,n}\int_0^\infty dz\rho^*_{mn}\left(-\frac{d^2\rho_{mn}}{dZ^2}+F\rho_{mn}\right), \tag{4.6}$$

with

$$F=\left[\left\langle\frac{\partial^2U}{\partial\phi^*\partial\phi}\bigg|_{\Phi=\Phi_0}\right\rangle+\frac{1}{2}\left(\frac{m^2}{\cosh e^Z-1}+\frac{n^2}{\cosh e^Z+1}+\frac{e^{-2Z}}{2}-\frac{1}{2}(\coth e^Z)^2+1\right)\right]e^{2Z}.$$

In these equations, Y stands for the length of the rapidity interval available. This is of the order of $\log Q$, and it is roughly independent of the transverse momentum. [One may remember that the boundary of the phase space is given by $(1+t)^{1/2}\cosh y = Q \approx s^{1/2}$.] The positivity of $\delta^2 W$ is obviously equivalent to the condition that all the eigenvalues of the operator $(-d^2/dz^2+F)$ are positive. Hence, in order to decide the stability of any particular solution of (4.5), one has only to determine the sign of the lowest eigenvalue of this operator. [Formally, this is a one-dimensional Schrödinger equation with boundary conditions $\rho(z=\pm\infty)=0$.]

For the sake of definiteness, from now on we concentrate mainly on functions U of the form (4.4). We have then with $\Phi_0 \approx f(t)g(y)$,

$$\left\langle \frac{1}{g}\frac{\partial U}{\partial \phi^*}\right\rangle = Af + B\langle g^2\rangle f^3 \equiv Af + bf^3 , \qquad (4.7)$$
$$\left\langle \frac{\partial^2 U}{\partial\phi^*\partial\phi}\right\rangle = B\langle g^2\rangle f^2 = bf^2 , \quad b>0 .$$

On introducing the notation $\langle g^{-1}d^2g/dy^2\rangle = 4c$, Eq. (4.5) becomes

$$\frac{d}{dt}\left(t(1+t)\frac{df}{dt}\right)+\frac{C}{t+1}f-\frac{A}{2}f-\frac{b}{2}f^3=0 . \qquad (4.8)$$

We want to solve this equation for $0\le t<\infty$ with $f(0)<\infty$, $f(\infty)=0$; in particular, we are looking for nonoscillating solutions, since these are expected to minimize W by giving the lowest possible value to the correlation term, $g^{ab}\nabla_a\Phi^*\nabla_b\Phi$. By inspection we discover that the singular points of (4.7) are at $t=-1,0,\infty$, hence, we are dealing with a singular boundary-value problem. Singular boundary-value problems for a large class of differential equations have been discussed in a recent paper by Hartman.[5] We refrain from reproducing Hartman's results; rather, we give an intuitive discussion of the properties of the solutions. (Needless to say, the properties we discover, at least for $c=0$, are proved in Ref. 5.) First, we notice that if we want $f\to 0$ $(t\to\infty)$, then asymptotically we have the linear equation

$$\frac{d}{dt}\left(t^2\frac{df}{dt}\right)-\frac{A}{2}f=0 . \qquad (4.9)$$

This is solved by $f=t^\alpha$, with

$$\alpha=\tfrac{1}{2}[-1\pm(1+2A)^{1/2}] . \qquad (4.10)$$

Hence, for any $A>-\frac{1}{2}$, there exists a solution decreasing as a power of t.

Experimentally, the observed inclusive distributions are indeed roughly proportional to t^{-N} with $2\lesssim N\lesssim 4$. Discarding the positive root in (4.10), we thus need $0\lesssim A\lesssim 4$.

In order to discuss the solutions further, we notice that (4.5) is formally equivalent to the equation of motion of a nonrelativistic particle in one dimension with a Lagrangian,

$$L=\tfrac{1}{2}t(1+t)\left(\frac{df}{dt}\right)^2 - V . \qquad (4.11)$$

Here,

$$V=\frac{1}{2}\frac{c}{1+t}f^2-\frac{1}{2}U(f) ,$$

$U(f)$ being the rapidity-averaged potential.

The mass of the fictitious particle is "time"-dependent, $m=t(1+t)$. Correspondingly, the energy changes in time:

$$H=\frac{1}{2}\frac{1}{t(1+t)}p^2+V , \qquad (4.12)$$
$$\frac{dH}{dt}=\frac{\partial H}{\partial t}=-(t+\tfrac{1}{2})\left(\frac{df}{dt}\right)^2-\frac{1}{2}\frac{C}{(1+t)^2}f^2 .$$

On choosing $A>0$ as suggested by the discussion following Eq. (4.10), we see that at $t\to\infty$ the particle has to climb to a maximum of V. Apart perhaps from a short initial time, energy is dissipated monotonically for either sign of c. This implies that at $t=0$ the particle has to start with some initial coordinates, $f(0)>0$, and with initial velocity determined by the equation of motion itself, viz.,

$$\left.\frac{df}{dt}\right|_{t=0}+\left.\frac{dV}{df}\right|_{t=0}=0 ,$$

such that the initial energy is positive. (Since $t=0$ is a singular point, we are not free to prescribe both the coordinate and the velocity there.) If $C<0$, this is clearly impossible with the simple form (4.4) for U, since with $A>0$, $C<0$, the potential V is monotonically decreasing from $V=0$. One can contemplate other more complicated forms of U, for instance, by taking a polynomial which is cubic in $\Phi^*\Phi$. In such cases one can always choose the initial coordinate in such a way that the particle arrives at the "top of the hill" at $f=0$ with exactly zero energy; hence, the solution $f(t)$ does not oscillate. However, all such solutions are unstable: The operator in (4.6) has at least one nonpositive eigenvalue. This can be easily verified: The function F dips below zero, and one can verify either by numerical calculation or by means of a WKB estimate that a negative eigenvalue is present.

The situation is radically different for $C>0$, however. Even with a simple form (4.4), it can be easily arranged that the solution becomes stable in the sense that the function F in (4.6) never dips below zero, and, hence, (4.6) can never develop a negative eigenvalue. In terms of the fictitious mechanical system (4.11), $C>0$ corresponds to a repulsive force, proportional to $(1+t)^{-1}f$, which then decreases with increasing time. At large values of t, the first term in V becomes ineffective and the particle eventually stops at the top of the hill, $f=0$.

Although we are unable to give any rigorous result at present, experimentation with various simple forms of U suggests that the condition $c>0$ may be necessary in order to obtain a stable, nonoscillating solution.

Physically, the condition $C>0$ has a very simple meaning. In fact, we recall that the "constant" is just the average of the function $g^{-1}g''$ over a sufficiently large rapidity interval, and, hence, it is practically independent of y. The function g is expected to be relatively slowly varying; in particular, it should not have roots: This would give rise to sharp dips in the distribution (4.1). We may therefore assume $g>0$. It is then clear that the main contribution to c comes from regions on the y axis, where g is *small.* We obtain $C>0$ if g is convex wherever it is small. In other words, *g must have at least one sufficiently deep minimum* in the rapidity range over which averages are taken: Otherwise, we cannot obtain a stable condensate in the Feynman gas. However, the existence of one or several dips in rapidity corresponds precisely to the existence of "rapidity clusters," or in other words, to a multifireball picture, which seems to be supported by experimental data.

V. FINAL REMARKS

Although the preceding discussion is not a rigorous one, the emerging physical picture is certainly suggestive. In particular, we find a connection between rapidity clustering and the sharp falloff in transverse momentum, provided the observed distributions are dominated by the condensate. The existence of a condensate, in turn, has already been suggested by the work of Scalapino and collaborators[1,2]: They compare their model to a superconducting wire.

We find it also interesting that a power-behaved inclusive cross section at large transverse momenta is a natural (and almost inevitable) consequence of the model. This does not contradict the usual explanation of this behavior in terms of a quark model: The latter is certainly more fundamental. However, it is amusing to recover this aspect of inclusive distributions from a simple phenomenological picture.

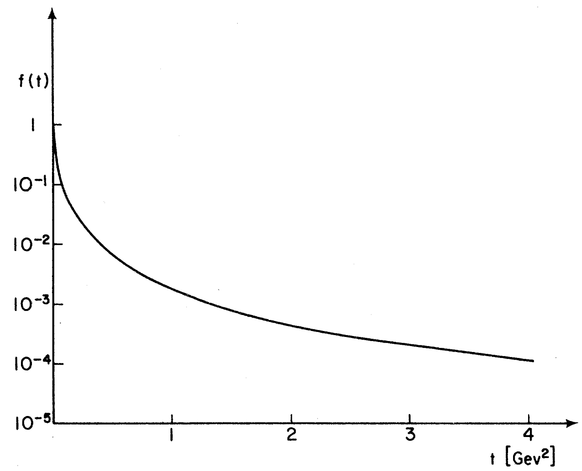

FIG. 1. Normalized expectation value of the random source function in the mean field approximation.

At this point it would be premature to make a detailed comparison between our calculations and the observed distributions. However, it is worth noting that a sample calculation with $A=4$, $B=0.2$, $C=1.3$ gives a qualitatively fair representation of the general shape of inclusive distributions (see Fig. 1). (We have not made a systematic search for parameter values which give the "best fit.") By varying the parameters, one can convince oneself that the rapid initial falloff in t is due to the "repulsion" exerted by the term proportional to c; this confirms one's expectation based on the qualtitative discussion given in Sec. IV. At higher values of t, the solution goes smoothly over to a power behavior, essentially governed by Eq. (4.9).

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