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Infrared catastrophe averted by Hertz potential

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A computational scheme for quantum electrodynamics is developed from first principles in which infrared divergences are not encountered. The gist of the method is that the photon propagator is properly $-ig_{\mu\nu}(k^2+i\epsilon)^{-1}[(1/2)k^\sigma\partial/\partial k^\sigma)]_w\ln(-\lambda^{-2}k^2-i\epsilon)$. Here λ is a parameter which plays the role of the photon mass, but which cancels out of cross sections for any finite value ($\lambda\to 0$ is never taken). The subscript w means weak derivative, so a partial integration on k, with neglect of boundary terms, is always understood in Feynman integrals. This makes integrals converge which otherwise would be logarithmically divergent. For sufficiently convergent integrals the partial integration may be undone and the conventional value results because $[(1/2)k^\sigma\partial/\partial k^\sigma)]\ln(-\lambda^{-2}k^2-i\epsilon)=1$. It is shown that the above propagator is not merely an artificial device, but results if the vector potential A^μ is derived from a Hertz potential $\Pi^{\lambda\mu}=-\Pi^{\mu\lambda}$, $A^\mu=\partial_\mu\Pi^{\lambda\mu}$, which is a local field. The real-photon infrared divergences do not occur when the integral over real photons is obtained from the discontinuity in the k^0 plane of the above propagator. A contribution to the real-photon integral at strictly zero frequency, resembling a third polarization state of the photon, is isolated and evaluated explicitly. The S matrix and physical subspaces are defined.

I. INTRODUCTION

Although there is no doubt that coherent states[1-10] provide a basic understanding of the infrared problem in quantum electrodynamics, it is remarkable that the preferred method of calculation does not embody this wisdom. Instead, according to a method[11] brought to a high degree of perfection by Yennie and co-workers[12] a mass λ is attributed to the photon, so the states are not coherent but are Fock states, and λ is finally set equal to zero in cross sections. The greater simplicity of the photon-mass method suggests that it embodies some truth, not yet manifest in the coherent-state method, but implicit somehow in the straightforward prescription of replacing

$$-ig_{\mu\nu}(k^2+i\epsilon)^{-1} \tag{1.1a}$$

by

$$-ig_{\mu\nu}(k^2-\lambda^2+i\epsilon)^{-1}$$

for the free photon propagator.

In the present work, we describe a method in which, hopefully, this truth is distilled, without destroying the coherence of the states. Note that

$$(k^2-\lambda^2+i\epsilon)^{-1} = (k^2+i\epsilon)^{-1}\tfrac{1}{2}(k^\mu\partial/\partial k^\mu)\ln(-\lambda^{-2}k^2+1-i\epsilon),$$

so, with neglect of terms which vanish with λ^2, the photon propogator with mass becomes

$$-ig_{\mu\nu}(k^2+i\epsilon)^{-1}(\tfrac{1}{2}k^\sigma\partial/\partial k^\sigma)_w\ln(-\lambda^{-2}k^2-i\epsilon). \tag{1.1b}$$

As a function, this expression agrees with the usual photon propagator, because $(\frac{1}{2}k^\sigma\partial/\partial k^\sigma)\ln(-\lambda^{-2}k^2-i\epsilon)=1$. However, *as a distribution* it is different, and the subscript w is a reminder that the weak derivative is taken in the distribution-theoretic sense. In other words a partial integration on k with neglect of boundary terms is always understood, before the Feynman integration over k is performed.

The gist of the method developed in the present work is that the photon propagator is truly given by (1.1b). It is understood that λ is a constant with dimensions of mass which plays the role of the photon mass, but *it is a finite parameter which is not set equal to zero.* It cancels out of cross sections for every fixed finite value (although not out of the electron self-mass). In this respect it resembles a renormalization mass. For a properly convergent integral the partial integration implicit in (1.1b) may be undone, and the photon propagator becomes $-ig_{\mu\nu}(k^2+i\epsilon)$, as usual. However, for a logarithmically divergent integral the Euler differential operator $-\frac{1}{2}(\partial/\partial k^\mu)k^\mu\cdots=-\frac{1}{2}(k^\mu\partial/\partial k^\mu+4)\cdots$, which results from the implicit partial integration, annihilates terms that are homogeneous in k of degree -4, which are precisely the cause of the divergence. For example, the typical infrared-divergent integrand is rendered convergent by

$$-\tfrac{1}{2}(k^\lambda\partial/\partial k^\lambda+4)\left(\frac{1}{k^2}\,\frac{p\cdot p'}{(k^2+2p\cdot k)(k^2+2p'\cdot k)}\right) =\frac{p\cdot p'}{(k^2+2p\cdot k)^2(k^2+2p'\cdot k)}+(p\leftrightarrow p'),$$

where a power of k has been added at $k=0$, and similarly for the ultraviolet-divergent integrand,

$$-\tfrac{1}{2}(k^\lambda \partial/\partial k^\lambda + 4)\left(\frac{1}{k^2}\frac{k^\sigma k^\tau}{[(k+p)^2-m^2][(k+p')^2+m^2]}\right)$$

$$=\frac{-1}{k^2}\frac{k^\sigma k^\tau(p\cdot k+p^2-m^2)}{[(k+p)^2-m^2]^2[(k+p')^2-m^2]}+(p\leftrightarrow p'),$$

where a power of k has been removed at $k=\infty$.

The possibilities which the present approach may offer in dealing with ultraviolet divergences have not been explored, although it certainly renders δm and $Z_1=Z_2$ finite in lowest order. However, it does appear that the propagator (1.1b) averts all threatened virtual infrared divergences of quantum electrodynamics.

To be consistent with the change in the photon propagator from (1.1a) to (1.1b), the sum over real-photon states must be changed from

$$\int\frac{d^3k}{2\omega}\cdots=\int d^4k\,\mathrm{disc}(k^2+i\epsilon)^{-1}\cdots \qquad (1.2\mathrm{a})$$

to

$$\int d^4k\,\mathrm{disc}\{(k^2+i\epsilon)^{-1}\tfrac{1}{2}(k^\mu\partial/\partial k^\mu)_w[\ln(-\lambda^2k^2-i\epsilon)]\}\cdots, \qquad (1.2\mathrm{b})$$

where the discontinuity is taken on the positive k^0 axis. Its evaluation in Sec. V is the heart of the present article. Use of the resulting photon phase space averts all real infrared divergences.

Observe that changing the photon propagator and phase space from (1.1a) and (1.2a) to (1.1b) and (1.2b) changes the very notion of photon and photon state. A clear physical interpretation of what these states may be is provided by the reconstruction principle. If $A_\mu(x)$ is the free, in or out vector potential, the generic one-photon state is given by $A(j)\Omega=\int A_\mu(x)j^\mu(x)d^4x\Omega$, where Ω is the vacuum state and $j^\mu(x)$ is a test function, interpreted as a classical current, and so is real and conserved. The basic idea of the reconstruction principle is that states are determined by their inner products with other states. Consider the bilinear form on the pair of currents defined by the inner product,

$$\langle j_1,j_2\rangle\equiv\langle A(j_1)\Omega,A(j_2)\Omega\rangle, \qquad (1.3)$$

calculated in momentum space using Eq. (1.2a) or (1.2b). According to the reconstruction principle, j_2 and j_2' correspond to the same state if and only if $\langle j_1,j_2\rangle=\langle j_1,j_2'\rangle$ for all j_1. This condition defines an equivalence relation between currents, and states are identified with the equivalence classes. Let us see how this works out for the usual photon phase space (1.2a). One finds

$$\langle j_1,j_2\rangle=\int d^3k(2\omega)^{-1}\phi_1^{*\mu}(\vec{k})(-g_{\mu\nu})\phi_2^\nu(\vec{k}), \qquad (1.4)$$

where $\omega=|\vec{k}|$ and

$$\phi^\mu(\vec{k})=(2\pi)^{-3/2}\int e^{i(\omega x^0-\vec{k}\cdot\vec{x})}j^\mu(x)d^4x. \qquad (1.5)$$

In this case test function are equivalent if they have the same Fourier transform on the future light cone, which is, of course, none other than the familiar photon wave function depending on only 3 variables $(\vec{k})$, instead of 4 (x^μ). Because of current conservation, $\partial_\mu j^\mu=0$, the wave functions are transverse, $k_\mu\phi^\mu(k)=\omega\phi^0(\vec{k})-\vec{k}\cdot\phi(k)=0$, which makes $\phi^\mu(k)$ equivalent to $\phi^\mu(k)+k^\mu\alpha(k)$ for any $\alpha(k)$. In this case the reconstruction principle reduces the three independent components of transverse wave function $\phi^\mu(k)$ to the familiar two independent photon polarization states, as it should.

Before the reconstruction principle can be applied to the new photon phase space (1.2b), the properties of the classical currents j^μ must be specified more precisely. If a current carries a net electric charge

$$q=\int j^0(t,\vec{x})d^3x\neq 0,$$

then $j^0(x)$ cannot be of fast decrease in the time direction, for then q would go to zero as $t\to\pm\infty$, but instead it is a constant. To replace the hypothesis of fast decrease, we make what may be called the scattering hypothesis and suppose that at asymptotic distances in space and time, the charge is carried by particles (or a fluid) in uniform motion at subluminal velocities. More precisely we specify

$$\lim_{R\to\infty}R^3j^\mu(Rx)=u^\mu[\rho_+(u)\theta(x^0)+\rho_-(u)\theta(-x^0)]\theta(x^2)(x^2)^{-3/2}, \qquad (1.6)$$

where $u^\mu=x^\mu(x^2)^{-1/2}\mathrm{sgn}(x^0)$ is the unit timelike four-vector representing the asymptotic four-velocity of the charge flow. The functions $\rho_+(u)$ and $\rho_-(u)$ represent the asymptotic charge densities per invariant volume element in velocity space, with total charge q given by

$$q=\int d^3u(u^0)^{-1}\rho_+(u)=\int d^3u(u^0)^{-1}\rho_-(u),$$

where $u^0=(\vec{u}^2+1)^{1/2}$. The motivation for this hypothesis is that the matrix elements of the quantum current operator $J^\mu(x)$ have precisely this property,[13] so divergences will be averted if $A(j)$ is well defined for such test functions. The difference

$$\rho(u) \equiv \rho_+(u) - \rho_-(u) \qquad (1.7)$$

measures the net scattering of electric charge in the flow $j^\mu(x)$. It is an important quantity which we call the "charge-scattering function."

The one-photon inner product $\langle A(j_1)\Omega, A(j_2)\Omega\rangle$ is evaluated in Sec. V using the new phase space (1.2b) for test functions satisfying the charge-scattering hypothesis (1.6). The result is

$$\langle j_1, j_2\rangle = \tfrac{1}{2}\int d\hat{k}\int_0^\infty d\omega \ln a\omega(-\partial/\partial\omega) \times[\omega^2\phi_1^{*\mu}(\vec{k})(-g_{\mu\nu})\phi_2^\nu(\vec{k})] + \int d^3u_1(u_1^0)^{-1}\int d^3u_2(u_2^0)^{-1} \times \rho_1(u_1)K(u_1,u_2)\rho_2(u_2), \qquad (1.8)$$

where $a = 2\lambda^{-1}$; $\rho_i(u)$, for $i = 1, 2$, is the charge-scattering function, just defined, associated with the flow $j_i^\mu(x)$; $K(u_1, u_2)$ is a kernel given in Eq. (5.35); and the wave function $\phi_i(\vec{k})$, defined in Eq. (1.5) has the infrared limit

$$\lim_{\omega\to 0} \omega\phi_i^\mu(\vec{k}) = \frac{i}{(2\pi)^{3/2}}\int \frac{d^3u}{u_0}\frac{u^\mu\rho_i(u)}{u^0 - \vec{u}\cdot\hat{k}} \qquad (1.9)$$

Unlike (1.4), the inner product (1.8) is finite for such wave functions. If, now, the reconstruction principle is applied to the form (1.8), one finds that besides the two transverse degrees of polarization described by the familiar wave functions $\phi^\mu(\vec{k})$, there is also a scalar wave function $\rho(u)$ needed to describe the extended one-photon state. It is presumably a remnant of the longitudinal state of massive vector mesons, for, as was discovered early in the history of quantum electrodynamics, the wrong result is obtained if the longitudinal degree of polarization is dropped before the photon mass is set equal to zero.[14] We call the first type of photon, described by the wave function $\phi^\mu(\vec{k})$, "radiation photon" since it corresponds to familiar radiation, and carries energy and momentum. We call the second type, described by the wave function $\rho(u)$, "zero-frequency photon" because $\rho(u)$ lies at asymptotic infinity and is translationally invariant, so the zero-frequency photon carries no energy or momentum, although it does have finite angular momentum.

The one-photon inner product (1.8) is indefinite, even for conserved currents. This situation is not unfamiliar in relativistic quantum field theory: The one-particle Klein-Gordon equation also suffers from an indefinite metric, the cure being found in the second-quantized or many-particle theory by expelling the negative-frequency particles. In our case, where the spectrum extends down to zero frequency, the cure is a gain found in the second-quantized or many-photon theory, by also expelling states which are not coherent at zero frequency. As shown in Sec. VII, the states which are coherent at zero frequency, which we call "infrared coherent," provide a space of positive metric which is identified as a physical space.

The organization of the paper is as follows. Those interested in applications may bypass Secs. II and III, take (1.1b) as an ansatz, and pick up the logical development near the end of Sec. IV. Sections II and III are devoted to showing that the photon propagator (1.1b) is not an artificial device, but arises naturally in a theory where the vector potential A^μ is derived from the Hertz[15] potential $A^\mu = \partial_\lambda \Pi^{\lambda\mu}$, $\Pi^{\lambda\mu} = -\Pi^{\mu\lambda}$ which is itself a local field.[16] The basic reason is dimensional: if $A^\cdot(x)A^\cdot(0) \sim x^{-2}$, then $\Pi^\cdot(x)\Pi^\cdot(0) \sim \ln\mu^2x^2$ and a logarithmic function and scale-breaking parameter have entered the theory. Because $\Pi^{\mu\nu}$ is an antisymmetric field, it is convenient, but not essential, to derive it from another vector potential $\Pi_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu$. Regarding A^μ as the father of the field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the Hertz potential $\Pi_{\mu\nu}$ as the father of A and hence the grandfather of the field, F, we may call Π the "grandfather potential" and U the "great grandfather potential." In Sec. III, the free Wightman function $\langle A_\mu(x)U_\nu(0)\rangle \sim g_{\mu\nu}\ln(-\mu^2x^2 + i\epsilon x^0)$ is found. The appearance of the logarithm and the parameter μ with dimensions of mass signify spontaneous breakdown of scale invariance. In Sec. IV the photon propagator (1.1b) is obtained by Fourier transform. In Sec. V the reconstruction principle is employed to answer the question "What is a photon?" The photon inner product (1.8) is found, and its kernel $K(u_1, u_2)$ is evaluated in the Appendix, where the Lorentz invariance of (1.8) is also demonstrated. In Sec. VI, the S matrix is defined and the contribution of the zero-frequency photons is found exactly. Section VII is devoted to the physical subspace and Sec. VIII contains some concluding remarks. For an application of the method we refer the reader to an accompanying article where the spectral composition of coherent bremsstrahlung radiation is calculated nonperturbatively.

II. GRANDFATHER AND GREAT GRANDFATHER POTENTIALS

The Maxwell-Dirac Lagrangian density in first-order form[17]

$$\mathcal{L}_{\text{MD}} = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar\psi(i\not\partial + e\not A - m)\psi \qquad (2.1)$$

does not provide an equation of motion for A^0. A conventional remedy is to add a term which breaks the classical gauge symmetry, $A_\mu \to A_\mu + \partial_\mu \Lambda$, $F_{\mu\nu} \to F_{\mu\nu}$, $\psi \to \psi \exp(ie\Lambda)$, where $\Lambda = \Lambda(x)$ is an arbitrary function of space-time. Making a virtue of necessity, we may tailor the symmetry-breaking term to ensure that A is derived from the Hertz, or grandfather potential $\Pi^{\mu\nu} = -\Pi^{\nu\mu}$,

$$A^\nu = \partial_\mu \Pi^{\mu\nu}. \qquad (2.2)$$

On dimensional grounds we expect that the free propagator $\Pi^{\cdot}\Pi^{\cdot}$ will have the logarithmic form $\sim\ln\mu^2 x^2$ and thereby depend on the mass parameter μ. If we add

$$\mathcal{L}_1 = -I_\nu(A^\nu - \partial_\mu \Pi^{\mu\nu}) \qquad (2.3)$$

to $\mathcal{L}_{\text{MD}}$, where I_ν and $\Pi^{\mu\nu}$ are new fields that are varied independently, we obtain Eq. (2.2) as an equation of motion, but Π^{ij} is left undetermined. A convenient remedy[18] is to add the further term

$$\mathcal{L}_2 = -\tfrac{1}{2}H^{\mu\nu}[(\partial_\mu U_\nu - \partial_\nu U_\mu) - \Pi_{\mu\nu}] + C\partial_\mu U^\mu, \qquad (2.4)$$

where U_ν, $H_{\mu\nu} = -H_{\nu\mu}$, and C are again new fields that are varied independently. Although it seems that a plethora of new fields are introduced, it turns out that $\Pi_{\mu\nu}$, A_ν, and $F_{\mu\nu}$ are all derived from the one-vector field U_ν which is the great grandfather potential

$$\Pi_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu, \qquad (2.5a)$$

$$A^\nu = \partial_\mu \Pi^{\mu\nu} = \partial^2 U^\nu, \qquad (2.5b)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial^2 \Pi_{\mu\nu} = \partial^2(\partial_\mu U_\nu - \partial_\nu U_\mu), \qquad (2.5c)$$

and that I_μ, $H_{\mu\nu}$, and C are free fields which vanish on the physical subspace. As further simplifying elements, the vector fields turn out to be transverse,[19]

$$\partial \cdot U = \partial \cdot A = \partial \cdot I = 0 \qquad (2.6a)$$

and the tensor fields are the curls of the vector fields

$$\Pi_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$
$$H_{\mu\nu} = \partial_\mu I_\nu - \partial_\nu I_\mu. \qquad (2.6b)$$

In fact, on varying the Lagrangian density

$$\mathcal{L} = \mathcal{L}_{\text{MD}} + \mathcal{L}_1 + \mathcal{L}_2 \qquad (2.7)$$

with respect to H, C, I, F, A, Π, U, and $\bar\psi$ in turn we obtain

$$\partial_\mu U_\nu - \partial_\nu U_\mu = \Pi_{\mu\nu}, \qquad (2.8)$$

$$\partial \cdot U = 0, \qquad (2.9)$$

$$\partial_\mu \Pi^{\mu\nu} = A^\nu, \qquad (2.10)$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}, \qquad (2.11)$$

$$\partial_\mu F^{\mu\nu} = I^\nu - e\bar\psi\gamma^\nu\psi, \qquad (2.12)$$

$$\partial_\mu I_\nu - \partial_\nu I_\mu = H_{\mu\nu}, \qquad (2.13)$$

$$\partial_\mu H^{\mu\nu} = \partial^\nu C, \qquad (2.14)$$

$$(i\not\partial + e\not A - m)\psi = 0. \qquad (2.15)$$

Observe that Eq. (2.8) gives $\dot U_i$ and fixes Π_{ij} by constraints, Eq. (2.9) gives $\dot U_0$, Eq. (2.10) gives $\dot\Pi^{0i}$ and fixes A^0 by constraint, Eq. (2.11) gives $\dot A_i$ and fixes F_{ij} by constraint, Eq. (2.12) gives $\dot F^{0i}$ and fixes I^0 by constraint, Eq. (2.13) gives $\dot I_i$ and fixes H_{ij} by constraint, Eq. (2.14) gives $\dot H^{0i}$ and $\dot C$, and Eq. (2.15) gives $\dot\psi$ which accounts for all fields. Apart from ψ and $\bar\psi$ there are thus ten pair of canonical variables A^i and F^{0i}, I^i and Π^{0i}, U^i and H^{0i}, U^0 and C, whereas A^0, F^{ij}, I^0, Π^{ij}, and H^{ij} are fixed by constraints.

These 10 pair (20 variables) reduce to the 2 pair (4 variables) which describe physical photons, as follows. The equation for ψ implies conservation of the electric current $J_\mu = -e\bar\psi\gamma_\mu\psi$,

$$\partial_\mu J^\mu = 0, \qquad (2.16)$$

so the divergence of Eq. (2.12) reads

$$\partial_\mu I^\mu = 0, \qquad (2.17)$$

and the divergence of Eq. (2.14) reads

$$\partial^2 C = 0. \qquad (2.18)$$

Thus C is a free massless field. Substitution of Eqs. (2.12) and (2.17) into Eq. (2.14) gives

$$\partial^2 I_\nu = \partial_\nu C. \qquad (2.19)$$

The last three equations describe a free massless transverse vector field. (If, in the last three equations, I_ν were replaced by A_ν they would be the equations satisfied by the free vector potential in the Landau gauge). Suppose first that the fields are classical and that at $t=0$ the following eight classical dynamical variables vanish, $I^i = H^{0i} = C = \partial_i F^{i0} - J^0 = 0$, the last condition expressing Gauss's law. Then, by the constraints

$I_0 = H_{ij} = 0$ at $t = 0$ and, by the equations of motion, $\dot{C} = \dot{I}_i = 0$ at $t = 0$ and by Eq. (2.17), $\dot{I}_0 = 0$ at $t = 0$. It follows from Eq. (2.18) that $C = 0$ at all times and thus from Eq. (2.19), $I_\nu = 0$ at all times, and hence also $H_{\mu\nu} = 0$ for all t. Thus the eight classical variables vanish at all times if they vanish at $t = 0$. Let their vanishing be imposed as a subsidiary condition. It guarantees the validity of Maxwell's equations and reduces the degrees of freedom from 20 to 12. The quantum-mechanical analog of this condition is the requirement that physical states be annihilated by the negative-frequency part of I_ν,

$$I_\nu^{(-)}(x)\Phi = 0 , \tag{2.20}$$

which implies that I_ν, $H_{\mu\nu}$, and C vanish between physical states. This condition is covariant since, by Eq. (2.18) and (2.19), $\partial^2\partial^2 I_\nu = 0$, so I_ν has support only on the light cone in momentum space. The vertex of the light cone will be examined in detail in Sec. V, and positivity in Sec. VII.

In the following sections we will consider only states which, in their dependence on photon variables, are obtained by applying functions of $A(j) = \int A_\mu(x) j^\mu(x) d^4x$ to the vacuum, where $j^\mu(x)$ is a transverse vector test function, $\partial_\mu j^\mu(x) = 0$. The subsidiary condition (2.20) will be satisfied because the commutator $[I_\nu^-(x), A_\mu(j)]$ is proportional to $\partial \cdot j = 0$, in virtue of Eq. (2.30b), below.

A further reduction in degrees of freedom comes from the restricted gauge invariance of the second kind, of the Lagrangian $\mathcal{L}$. Observe that $\mathcal{L}_2$ is invariant under

$$U_\mu \to U_\mu + M_\mu , \tag{2.21a}$$

$$\Pi_{\mu\nu} \to \Pi_{\mu\nu} + \partial_\mu M_\nu - \partial_\nu M_\rho , \tag{2.21b}$$

$$H_{\mu\nu} \to H_{\mu\nu} , \tag{2.21c}$$

$$C \to C , \tag{2.21d}$$

provided the classical vector function M_μ satisfies $\partial_\mu M^\mu = 0$, and that $\mathcal{L}_1$ is invariant under substitution (2.21b) and

$$A_\mu \to A_\mu + \partial_\mu \Lambda , \tag{2.21e}$$

$$I_\mu \to I_\mu , \tag{2.21f}$$

provided also $\partial^2 M_\mu = \partial_\mu \Lambda$, and $\mathcal{L}_{\rm MD}$ is invariant under this and

$$F_{\mu\nu} \to F_{\mu\nu} , \tag{2.21g}$$

$$\psi \to \psi \exp(ie\Lambda) . \tag{2.21h}$$

The total Lagrangian density $\mathcal{L}$ is thus invariant under the restricted gauge transformations of the second kind defined by Eqs. (2.21), provided the gauge functions M_μ and Λ satisfy the restrictions

$$\partial_\mu M^\mu = 0 , \tag{2.22a}$$

$$\partial^2 M_\mu = \partial_\mu \Lambda , \tag{2.22b}$$

which imply

$$\partial^2 \Lambda = 0 . \tag{2.22c}$$

These last three equations again describe a free massless transverse vector field, like I_ν. It has 8 independent variables (which may be taken to be M_i, $\dot{M}_i$, M_0, and Λ). Thus, of the 12 degrees of freedom which remain after the eight subsidiary conditions are imposed, another eight describe a gauge freedom and only four, two pair, are physically meaningful, as required. Observables, by definition, must be invariant under the gauge transformations (2.21).

The ten pair of canonical variables satisfy ten nonvanishing equal-time canonical commutation relations,

$$[F^{0i}(\vec{x}), A^j(\vec{y})] = [\Pi^{0i}(\vec{x}), I^j(\vec{y})] = [H^{0i}(\vec{x}), U^j(\vec{y})] = -i\delta_{ij}\delta^3(\vec{x} - \vec{y}) , \tag{2.23a}$$

$$[C(\vec{x}), U^0(\vec{y})] = -i\delta^3(\vec{x} - \vec{y}) , \tag{2.23b}$$

the other equal-time commutators of the canonical variables being zero. Because C, I_μ, and $H_{\mu\nu}$ are free their commutators with all fields for arbitrary space-time intervals may be found using the field equations. From $\dot{C} = \partial_i H^{i0}$ and $\partial^2 C = 0$ we find

$$[C(x), U_\mu(y)] = -i\partial_\mu D(x - y) \tag{2.24}$$

and that C commutes with all other fields.

Here $D(x)$ is the Pauli-Jordan function satisfying

$$\partial^2 D(x) = 0, \quad D(0, \vec{x}) = 0, \quad \dot{D}(0, \vec{x}) = \delta^3(\vec{x}), \tag{2.25a}$$

$$D(x) = (2\pi)^{-1}\delta(x^2)\,\mathrm{sgn}(x^0) . \tag{2.25b}$$

From $\partial^2 I_\mu(x) = \partial_\mu C(x)$ we conclude that the commutator of I_μ with Π, A, F, I, H, C, and ψ satisfies the wave equation as does $[I_\mu(x), U_\nu(y)] + i\partial_\mu\partial_\nu E(x - y)$. Here $E(x)$ is the invariant function defined by

$$\partial^2 E(x) = D(x), \quad E(0, \vec{x}) = \dot{E}(0, \vec{x}) = 0 , \tag{2.26a}$$

with explicit form

$$E(x) = (8\pi)^{-1}\theta(x^2)\,\mathrm{sgn}(x^0) , \tag{2.26b}$$

and the properties

$$\partial^2\partial^2 E(x) = \ddot{E}(0, \vec{x}) = 0, \quad \dddot{E}(0, \vec{x}) = \delta^3(\vec{x}) . \tag{2.26c}$$

From the constraint $I^0 = \partial_i F^{i0} - J^0$ and the equal-time commutator of the charge density

$$[J^0(\vec{x}), \psi(\vec{y})] = e\delta^3(\vec{x} - \vec{y})\psi(\vec{y}) , \tag{2.27}$$

we have

$$[I^0(\vec{x}), \psi(\vec{y})] = -e\delta(\vec{x}-\vec{y})\psi(\vec{y}), \tag{2.28}$$

and, using the canonical commutators and $\partial \cdot I = 0$,

$$[I_\mu(x), \psi(y)] = -e\partial_\mu D(x-y)\psi(y), \tag{2.29}$$

$$[I_\mu(x), U_\nu(y)] = i[g_{\mu\nu}D(x-y) - \partial_\mu\partial_\nu E(x-y)]. \tag{2.30a}$$

Finally, from $\partial^2 U_\nu = A_\nu$, we have

$$[I_\mu(x), A_\nu(y)] = -i\partial_\mu\partial_\nu D(x-y). \tag{2.30b}$$

Similarly,

$$[I_\mu(x), I_\nu(y)] = 0 \tag{2.30c}$$

follows from

$$\partial^2 A_\nu = I_\nu + J_\nu. \tag{2.31}$$

Renormalization must be effected in such a way that these commutators of the free fields I_ν, $H_{\mu\nu} = \partial_\mu I_\nu - \partial_\nu I_\mu$ and C are maintained for the renormalized Heisenberg fields. For the remaining fields a perturbative procedure is necessary.

III. SPONTANEOUS SCALE BREAKING

To initiate a perturbative procedure, we set the electric charge to zero, $e=0$. Equation (2.31) becomes

$$\partial^2 A_\nu = I_\nu \tag{3.1}$$

and it follows from the commutator (2.30a) that

$$[A_\mu(x), U_\nu(y)] - i[g_{\mu\nu}E(x-y) - \partial_\mu\partial_\nu F(x-y)]$$

satisfies the wave equation. Here $F(x)$ is the invariant function defined by

$$\partial^2 F(x) = E(x), \quad F(0,\vec{x}) = \dot{F}(0,\vec{x}) = 0, \tag{3.2a}$$

with solution

$$F(x) = (64\pi)^{-1}x^2\theta(x^2)\,\text{sgn}x^0. \tag{3.2b}$$

From $\partial^2 U_\nu(y) = A_\nu(y)$ we obtain

$$[A_\mu(x), A_\nu(y)] = i[g_{\mu\nu}D(x-y) - \partial_\mu\partial_\nu E(x-y)], \tag{3.3}$$

which, together with the equal-time commutation relations, gives

$$[A_\mu(x), U_\nu(y)] = i[g_{\mu\nu}E(x-y) - \partial_\mu\partial_\nu F(x-y)], \tag{3.4}$$

and similarly

$$[U_\mu(x), U_\nu(y)] = i[g_{\mu\nu}F(x-y) - \partial_\mu\partial_\nu G(x-y)]. \tag{3.5}$$

Here $G(x)$ is the invariant function defined by

$$\partial^2 G(x) = F(x), \quad G(0,\vec{x}) = \dot{G}(0,\vec{x}) = 0, \tag{3.6a}$$

$$G(x) = (1536\pi)^{-1}(x^2)^2\theta(x^2)\,\text{sgn}(x^0), \tag{3.6b}$$

with the property

$$(\partial^2)^4 G = 0. \tag{3.6c}$$

We shall now construct Wightman functions satisfying these commutation relations. They will be characterized by an invariant decomposition of $U(x)$ [whose support in momentum space lies on the light cone, $(\partial^2)^4 U(x) = 0$] into positive- and negative-frequency parts

$$U(x) = U^{(-)}(x) + U^{(+)}(x), \tag{3.7a}$$

$$U^{(+)}(x) = U^{(-)}(x)^\dagger, \tag{3.7b}$$

and an invariant vacuum state Ω, unique to within a phase, which is annihilated by the negative-frequency part of $U(x)$,

$$U^{(-)}(x)\Omega = 0. \tag{3.8}$$

As a first step we introduce a decomposition of the invariant functions into positive- and negative-frequency parts, starting with $G(x)$:

$$G(x) = G^{(-)}(x) + G^{(+)}(x), \tag{3.9a}$$

$$G^{(+)}(x) = G^{(-)}(x)^*, \tag{3.9b}$$

$$G^{(-)}(x) = (3072\pi^2 i)^{-1}(x^2)^2\ln(-\mu_2^2 x^2 + i\epsilon x^0). \tag{3.9c}$$

We have chosen an arbitrary scale of length μ_2^{-1} (reserving μ_1 for use shortly) in order to make the logarithm well defined. A change in the value of μ would add to $G^{(-)}(x)$ a term proportional to $(x^2)^2$ and subtract it from $G^{(+)}(x)$. Such a term, as indeed any polynomial in (x^2) is left indeterminate by the separation into positive- and negative-frequency parts, since it has support at the origin in momentum space. The decomposition of $G(x)$ induces a corresponding decomposition of $F(x) = \partial^2 G$, by

$$F^{(\pm)}(x) = \partial^2 G^{(\pm)}(x), \tag{3.10}$$

$$F^{(-)}(x) = (128\pi^2 i)^{-1}x^2\ln(-\mu_1^2 x^2 + i\epsilon x^0), \tag{3.11a}$$

$$F^{(+)}(x) = F^{(-)}(x)^*, \tag{3.11b}$$

$$\mu_2^2 = \mu_1^2 e^{-5/6}. \tag{3.12}$$

The commutators of $U^{(+)}(x)$ and $U^{(-)}(x)$ are determined to within a polynomial by Eq. (3.5). A natural choice for these commutators is

$$[U^{(-)}(x), U^{(-)}(y)] = [U^{(+)}(x), U^{(+)}(y)] = 0, \tag{3.13a}$$

$$[U_\mu^{(-)}(x), U_\nu^{(+)}(y)] = i[g_{\mu\nu}F^{(-)}(x-y) - \partial_\mu\partial_\nu G^{(-)}(x-y)]. \tag{3.13b}$$

This completely determines all the Wightman functions of the field U, for the generic Wightman function or vacuum expectation value $\langle\Omega, U_{\mu_1}(x_1)\cdots U_{\mu_n}(x_n)\Omega\rangle$ is found by commuting all $U^{(-)}(x)$ to the right and all $U_\mu^{(+)}(x)$ to the left. The result is the Wightman functions of the generalized free field with two-point function,

$$W_{\mu\nu}{}^{UU}(x-y) \equiv \langle \Omega, U_\mu(x) U_\nu(y)\Omega\rangle , \tag{3.14}$$

$$W_{\mu\nu}{}^{UU}(x) = i[g_{\mu\nu}F^{(-)}(x) - \partial_\mu\partial_\nu G^{(-)}(x)] , \tag{3.15a}$$

$$W_{\mu\nu}{}^{UU}(x) = (128\pi^2)^{-1}[g_{\mu\nu}x^2\ln(-\mu_1{}^2x^2+i\epsilon x^0) - \partial_\mu\partial_\nu(24)^{-1}(x^2)^2 \times\ln(-\mu_2{}^2x^2+i\epsilon x^0)] . \tag{3.15b}$$

The Wightman functions of the other fields are obtained by differentiation, in particular

$$W_{\mu\nu}{}^{AU}(x-y) \equiv \langle\Omega, A_\mu(x)U_\nu(y)\Omega\rangle \tag{3.16a}$$

$$\equiv \partial_x{}^2 W_{\mu\nu}{}^{UU}(x-y) , \tag{3.16b}$$

$$W_{\mu\nu}{}^{AU}(x) = (16\pi^2)^{-1}[g_{\mu\nu}\ln(-\mu^2x^2+i\epsilon x^0) - \partial_\mu\partial_\nu 8^{-1}x^2\ln(-\mu_1{}^2x^2+i\epsilon x^0)] . \tag{3.17}$$

Here we have introduced the mass μ, related to μ_1 by

$$\mu_1{}^2 = \mu^2 e^{-3/2} . \tag{3.18}$$

A representation of the free fields may be obtained from these Wightman functions by the reconstruction theorem[20] modified, because of lack of positivity, as effected previously in a model quantum electrodynamics.[21] In Sec. V this will be done for the physically relevant subspace obtained by applying functions of $A(j) = \int A_\mu j^\mu d^4x$, $\partial\cdot j = 0$, to the vacuum. Although the field U_μ does not appear explicitly in $A(j)$, it is implicit, because A_μ is the weak derivative $A_\mu = \partial^2 U_\mu$.

The action $S = \int \mathcal{L} d^4x$ corresponding to the Lagrangian (2.7) (with $e=0$) is invariant under the scale transformation

$$U_\mu(x) \to s^{-1}U_\mu(sx) , \tag{3.19a}$$

$$\Pi_{\mu\nu}(x) \to \Pi_{\mu\nu}(sx) , \tag{3.19b}$$

$$A_\mu(x) \to sA_\mu(sx) , \tag{3.19c}$$

$$F_{\mu\nu}(x) \to s^2 F_{\mu\nu}(sx) , \tag{3.19d}$$

$$I_\mu(x) \to s^3 I_\mu(sx) , \tag{3.19e}$$

$$H_{\mu\nu}(x) \to s^4 H_{\mu\nu}(sx) , \tag{3.19f}$$

$$C(x) \to s^4 C(sx) , \tag{3.19g}$$

and thus, so are the field equations. However, the Wightman functions clearly are not, due to the $\ln(-\mu^2x^2)$ dependence. Thus the occurrence of the parameter μ with the dimensions of mass reveals a spontaneous breakdown of scale invariance. It is clear on dimensional grounds that this will happen in any theory in which A is the derivative of a higher-order potential.

IV. PHOTON PROPAGATOR AND ELIMINATION OF VIRTUAL INFRARED DIVERGENCES

The time-ordered product follows directly from the Wightman function (3.17),

$$T_{\mu\nu}{}^{AU}(x-y) = \langle\Omega, T[A_\mu(x)U_\nu(y)]\Omega\rangle , \tag{4.1}$$

$$T_{\mu\nu}{}^{AU}(x) = (16\pi^2)^{-1}[g_{\mu\nu}\ln(-\mu^2x^2+i\epsilon) - \partial_\mu\partial_\nu 8^{-1}x^2\ln(-\mu_1{}^2x^2+i\epsilon)] , \tag{4.2}$$

where $\mu_1{}^2 = e^{-3/2}\mu^2$.

As a preliminary step to finding the momentum-space propagator, it is convenient to calculate the Fourier transform

$$L(k) \equiv \int d^4x e^{ik\cdot x}\ln(-\mu^2x^2+i\epsilon) = \mu^2\partial_k{}^2\int d^4x e^{ik\cdot x}\frac{\ln(-\mu^2x^2+i\epsilon)}{(-\mu^2x^2+i\epsilon)} . \tag{4.3}$$

We rewrite this as

$$L(k) = \mu^2\partial_k{}^2 \lim_{\nu\to 0}\frac{\partial}{\partial\nu} I(k,\nu) , \tag{4.4}$$

where ν may approach zero from above or below, and

$$\begin{aligned} I(k,\nu) &\equiv \int d^4x e^{ik\cdot x}(-\mu^2x^2+i\epsilon)^{\nu-1} \\ &= e^{i(\nu-1)\pi/2}\int d^4x e^{ik\cdot x}(\epsilon+i\mu^2x^2)^{\nu-1} \\ &= \frac{e^{i(\nu-1)\pi/2}}{\Gamma(1-\nu)}\int_0^\infty ds\, s^{-\nu}\int d^4x e^{ik\cdot x}e^{-i\mu^2x^2s} \\ &= \frac{e^{i\nu\pi/2}}{\Gamma(1-\nu)}\pi^2\int_0^\infty \frac{ds\, s^{-\nu}}{(\mu^2 s)^2}\exp(ik^2/4\mu^2 s) \\ &= \frac{e^{i\nu\pi/2}}{\Gamma(1-\nu)}\frac{\pi^2}{\mu^2}\int_0^\infty dt\, t^\nu \exp[-(\epsilon - ik^2/4\mu^2)t] , \end{aligned} \tag{4.5}$$

$$I(k,\nu) = \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{(-i)16\pi^2 a_0{}^4}{(-a_0{}^2k^2-i\epsilon)^{1+\nu}} , \tag{4.6}$$

with

$$a_0 \equiv (2\mu)^{-1} . \tag{4.7}$$

This allows $L(k)$ to be written in the alternative forms,

$$L(k) = \lim_{\nu\to 0}\frac{\partial}{\partial\nu}\left(i\nu(\nu+1)\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{16\pi^2 a_0{}^4}{(-a_0{}^2k^2-i\epsilon)^{2+\nu}}\right) \tag{4.8}$$

or

$$L(k)=\partial_k{}^2\left(\frac{4\pi^2 i}{(-k^2-i\epsilon)}\ln(-e^{2\gamma}a_0{}^2k^2-i\epsilon)\right), \tag{4.9a}$$

$$L(k)=\frac{\partial}{\partial k^\mu}\left(\frac{8\pi^2 ik^\mu}{(-k^2-i\epsilon)^2}\ln(-e^{2\gamma-1}a_0{}^2k^2-i\epsilon)\right), \tag{4.9b}$$

where $\gamma=-\int_0^\infty dt\, e^{-t}\ln t$ is Euler's constant. If one effects the operation $\lim_{\nu\to 0}\partial/\partial\nu$ in Eq. (4.8), one obtains $L(k)=16\pi^2 i(-k^2-i\epsilon)^{-2}$ which cannot be the correct Fourier transform of $\ln(-\mu^2x^2-i\epsilon)$, since it is independent of μ. However, it is a very convenient representation if used as in analytic renormalization; namely one integrates first over k and then effects $\lim_{\nu\to 0}\partial/\partial\nu$. Similarly, in formula (4.9) $L(k)$ is understood to be a distribution and the derivative gets applied to the function it is integrated over.

With this result we find for the Fourier transform of $T^{AU}(x)$, Eq. (4.2),

$$\tilde{T}_{\mu\nu}{}^{AU}(k)=\int e^{ik\cdot x}T_{\mu\nu}{}^{AU}(x)d^4x\,, \tag{4.10}$$

$$\tilde{T}_{\mu\nu}{}^{AU}(k)=\left(g_{\mu\nu}+\frac{k_\mu k_\nu}{-k^2-i\epsilon}\right)\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\left(i\nu(\nu+1)\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{a_0{}^4}{(-a_0{}^2k^2-i\epsilon)^{2+\nu}}\right), \tag{4.11}$$

or

$$\tilde{T}_{\mu\nu}{}^{AU}(k)=\left(g_{\mu\nu}+\frac{k_\mu k_\nu}{-k^2-i\epsilon}\right)\left(\frac{\partial}{\partial k^\lambda}\right)_w\left(\tfrac{1}{2}i\frac{k^\lambda}{(-k^2-i\epsilon)^2}\ln(-\tfrac{1}{4}a^2k^2-i\epsilon)\right), \tag{4.12}$$

where

$$a=2a_0e^{\gamma-1/2}=e^{\gamma-1/2}\mu^{-1}\,. \tag{4.13}$$

The subscript w signifies weak derivative.

Only the photon propagator $D_{\mu\nu}(k)\equiv\tilde{T}_{\mu\nu}{}^{AA}(k)$ is required. From $A_\mu(x)=\partial^2U_\mu(x)$, one has $D_{\mu\nu}(k)=-k^2\tilde{T}_{\mu\nu}{}^{AU}(k)$, namely

$$D_{\mu\nu}(k)=(-g_{\mu\nu}k^2+k_\mu k_\nu)\lim_{\nu\to}\frac{\partial}{\partial\nu}\left(i\nu(\nu+1)\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{a_0{}^4}{(-a_0{}^2k^2-i\epsilon)^{2+\nu}}\right), \tag{4.14}$$

or

$$D_{\mu\nu}(k)=(-g_{\mu\nu}k^2+k_\mu k_\nu)\left(\frac{\partial}{\partial k^\lambda}\right)_w\left(\frac{1}{2}\frac{ik^\lambda}{(-k^2-i\epsilon)^2}\ln(-\tfrac{1}{4}a^2k^2-i\epsilon)\right). \tag{4.15}$$

If the operations $\lim_{\nu\to 0}\partial/\partial\nu$ or $\partial/\partial k^\lambda$ were effected before integration one would obtain

$$D_{\mu\nu}(k)=\frac{i}{(k^2+i\epsilon)^2}(-g_{\mu\nu}k^2+k_\mu k_\nu)\,,$$

which is the standard expression for the photon propagator in the Landau gauge that, however, leads to virtual infrared divergences.

For a simple example of the elimination of these divergences by the propagator (4.15), consider the on-shell electron vertex function

$$e\bar{u}(p')\{F_e(\psi)\gamma^\mu+F_m(\psi)(4m)^{-1}[\not{k},\gamma^\mu]\}u(p)\,, \tag{4.16}$$

where the form factors $F_e(\psi)$ and $F_m(\psi)$ are functions of the hyperbolic angle $\psi\geq 0$ defined by

$$(p'-p)^2=-2m^2(\cosh\psi-1)\,. \tag{4.17}$$

In zeroth order, $F_e=1$ and $F_m=0$. From the triangle graph, using the propagator (4.15), one finds

$$F_e(\psi)=1-\frac{\alpha}{\pi}\left[\left(\frac{\psi}{\tanh\psi}-1\right)[\ln(\tfrac{1}{2}am)-\tfrac{1}{2}]+\frac{R(\psi)-2R(\psi/2)}{\tanh\psi}-\frac{1}{2}\left(\frac{\psi/2}{\tanh(\psi/2)}-1\right)\right], \tag{4.18a}$$

$$F_m(\psi)=\frac{\alpha}{2\pi}\frac{\psi}{\sinh\psi}\,, \tag{4.18b}$$

where

$$R(\psi)\equiv\int_0^\psi dx\frac{x}{\tanh x}\,. \tag{4.19}$$

This result holds[22] for any *finite* value of the parameter a. The photon-mass method gives a charge vertex function of the form

$$F_e(\lambda,\psi)=A(\psi)+B(\psi)\ln\lambda+O(\lambda)\,, \tag{4.20}$$

where the last term vanishes as λ goes to zero. Only after the last term is dropped do the two expressions agree[23] provided

$$\lambda^{-1}=a/2\ . \tag{4.21}$$

This equality is somewhat misleading because a is a finite parameter whereas the agreement holds only after terms which vanish with λ are dropped. A more precise statement is that the $\lambda\to 0$ limit of massive vector-meson theory contains a finite parameter with dimensions of mass. Its role here is dimensional transmutation.[24] For the logarithms which appear in each order of perturbation theory sum to an anomalous power.

V. WHAT IS A PHOTON?

In this section it will be seen how the correct one-photon states eliminate real infrared divergences. Our main tool will be the reconstruction principle which resurrects the states buried in the Wightman functions found in Sec. III. It is not our purpose to construct a representation of all the nonphysical degrees of freedom counted in Sec. II, but to proceed as rapidly as possible to the physical states.

In this section, as in the last two, $A_\mu(x)$ designates a free field, in the sense that its commutator is a c number. Here it should be thought of as the in or out field A^{in} or A^{out} and the states we find will populate the in or out asymptotic state spaces.

Because $A_\mu(x)$ is a free field, it is sufficient to consider the one-quantum states. The many-quantum states may be obtained as symmetrized products of one-quantum states. The one-quantum states of interest are spanned, using complex coefficients, by vectors of the form

$$A(j)\Omega\ , \tag{5.1}$$

where Ω is the vacuum state, unique to within a phase,

$$A(j)=\int A_\mu(x)j^\mu(x)d^4x\ , \tag{5.2}$$

and $j^\mu(x)$ is a real C^∞ test function, which is interpreted as an electric current, and is conserved

$$\partial_\mu j^\mu(x)=0\ . \tag{5.3}$$

Because the corresponding charge

$$q(t)=\int j^0(t,\vec{x})d^3x \tag{5.4}$$

is a constant, $q(t)=q(\pm\infty)=q$, $j^\mu(x)$ cannot be of fast decrease in all directions of space-time for nonzero charge. This conclusion holds also if $q=0$ but $j^\mu(x)$ describes separation of charge, as in ionization of a neutral atom, with positive and negative charges ultimately traveling off in different directions. To allow for these physical situations, the assumption of fast decrease is replaced by what may be called the charge-scattering hypothesis. Namely, we suppose instead that in asymptotically distant regions of space-time, the current $j^\mu(x)$ describes a fluid in uniform rectilinear motion with material velocity $\vec{v}$, less than that of light, and impact parameter negligible compared to $\vec{v}t$, so the asymptotic motion is given by $\vec{x}=\vec{v}t$. These conditions are summarized in the requirement that the limit

$$\lim_{\lambda\to\infty}\lambda^3 j^\mu(\lambda x)=0\ ,\quad x^2\leq 0 \tag{5.5a}$$

$$=\frac{u^\mu\rho_\pm(u)}{(x^2)^{3/2}}\ ,\quad x^2\geq 0\ ,\quad \pm x^0>0 \tag{5.5b}$$

exist, where $\rho_\pm(u)$ may, and usually will, be a distribution, and $u^\mu\equiv(\text{sgn}x^0)x^\mu(x^2)^{-1/2}$, defined for $x^2>0$, is a four-velocity, namely a unit future four-vector. The upper and lower signs refer to the asymptotically distant future and past. The motivation for this hypothesis is that the matrix elements of the quantum-mechanical current operator have this property.[13]

Upon change of variable, $\vec{x}=\vec{v}t$, Eq. (5.4) gives

$$q(t)=\int |t|^3 j^0(t,t\vec{v})d^3v$$

and so, with Eq. (5.5),

$$q=q(\pm\infty)=\int_{v^2<1}\frac{\rho_\pm(u)}{(1-v^2)^2}d^3v\ ,$$

where $u^\mu=(u^0,\vec{u})=(1-v^2)^{-1/2}(1,\vec{v})$, or

$$q=\int\rho_\pm(u)\frac{d^3u}{u^0}\ , \tag{5.6}$$

with $u^0=(\vec{u}^2+1)^{1/2}$. Because $(u^0)^{-1}d^3u$ is the invariant volume element on the unit hyperboloid, $\rho_\pm(u)$ represents the asymptotic charge density per invariant volume in velocity space. For example, if, at asymptotic times, the charge is carried by an extended particle with asymptotic four-velocity u_f or u_i then

$$\rho_{+,-}(u)=qu^0\delta^3(\vec{u}-\vec{u}_{f,i})\ . \tag{5.7}$$

The behavior at large x^μ described by Eq. (5.5) implies a singularity of a precise nature at the origin in momentum space. Let

$$\tilde{j}^\mu(k)=\int e^{ik\cdot x}j^\mu(x)d^4x \tag{5.8}$$

be the Fourier transform of $j^\mu(x)$, satisfying $k\cdot\tilde{j}(k)=0$, and consider

$$\lim_{\eta\to 0}\eta\tilde{j}^\mu(\eta k)=\lim_{\eta\to 0}\int\eta j^\mu(x)e^{i\eta k\cdot x}d^4x\ .$$

On making the change of variable $x'=\eta x$, one has

$$\lim_{\eta\to 0}\eta\tilde{j}^{\mu}(\eta k)=\int\lim_{\eta\to 0}\left(\frac{1}{\eta}\right)^{3}j^{\mu}\left(\frac{x}{\eta}\right)e^{ik\cdot x}d^{4}x$$
$$=\int d^{4}x\theta(x^{2})(x^{2})^{-3/2}e^{ik\cdot x}$$
$$\times u^{\mu}[\theta(x^{0})\rho_{+}(u)+\theta(-x^{0})\rho_{-}(u)],$$

where $u^{\mu}=\pm x^{\mu}(x^{2})^{-1/2}$. Changing variables according to $x^{\mu}=\pm Ru^{\mu}$, one has

$$\lim_{\eta\to 0}\eta\tilde{j}^{\mu}(\eta k)$$
$$=\int\frac{d^{3}u}{u^{0}}u^{\mu}\int_{0}^{\infty}dR[e^{iRk\cdot u}\rho_{+}(u)+e^{-iRk\cdot u}\rho_{-}(u)], \tag{5.9}$$

$$\lim_{\eta\to 0}\eta\tilde{j}^{\mu}(\eta k)=\int\frac{d^{3}u}{u^{0}}u^{\mu}\left[\frac{\rho_{+}(u)}{\epsilon-ik\cdot u}+\frac{\rho_{-}(u)}{\epsilon+ik\cdot u}\right],$$

which gives the form of the singularity at $k=0$. If k lies inside or on the light cone, the denominators do not vanish, and one has simply

$$\lim_{\eta\to 0}\eta\tilde{j}^{\mu}(\eta k)=i\int\frac{u^{\mu}}{u\cdot k}\rho(u)\frac{d^{3}u}{u^{0}}, \tag{5.10}$$

where

$$\rho(u)\equiv\rho_{+}(u)-\rho_{-}(u), \tag{5.11a}$$

$$\rho(u)=\lim_{\lambda\to\infty}\lambda^{3}[u\cdot j(\lambda u)-u\cdot j(-\lambda u)]. \tag{5.11b}$$

The quantity $\rho(u)$ will be very important in the following. It represents the net scattering of electric charge, per invariant volume element in velocity space, in the flow $j^{\mu}(x)$ and will be called the "charge-scattering function." It satisfies

$$\int\rho(u)\frac{d^{3}u}{u^{0}}=0, \tag{5.12}$$

which represents equality of initial and final total charge. Corresponding to the example (5.7), one has

$$\rho(u)=qu^{0}[\delta^{3}(\vec{u}-\vec{u}_{f})-\delta^{3}(\vec{u}-\vec{u}_{i})], \tag{5.13a}$$

$$\lim_{\eta\to 0}\eta\tilde{j}^{\mu}(\eta k)=iq\left(\frac{u_{f}}{u_{f}\cdot k}-\frac{u_{i}}{u_{i}\cdot k}\right). \tag{5.13b}$$

The inner product $\langle A(j_{1})\Omega, A(j_{2})\Omega\rangle$ defines a Hermitian symmetric form on the currents $\langle j_{1},j_{2}\rangle=\langle j_{2},j_{1}\rangle^{*}$ whose kernel is the two-point Wightman function

$$\langle j_{1},j_{2}\rangle=\langle A(j_{1})\Omega,A(j_{2})\Omega\rangle, \tag{5.14a}$$

$$\langle j_{1},j_{2}\rangle=\int d^{4}x\,d^{4}y\,j_{1}^{\mu}(x)W_{\mu\nu}(x-y)j_{2}^{\nu}(y), \tag{5.14b}$$

$$W_{\mu\nu}(x-y)=\langle\Omega,A_{\mu}(x)A_{\nu}(y)\Omega\rangle. \tag{5.14c}$$

In momentum space this reads

$$\langle j_{1},j_{2}\rangle=(2\pi)^{-4}\int\tilde{j}_{1}^{\mu *}(k)\tilde{W}_{\mu\nu}(k)\tilde{j}_{2}^{\nu}(k)d^{4}k, \tag{5.15}$$

where

$$W_{\mu\nu}(x)=(2\pi)^{-4}\int\tilde{W}_{\mu\nu}(k)e^{-ik\cdot x}d^{4}k. \tag{5.16}$$

For $\tilde{W}_{\mu\nu}(k)$ we may use our previous result for the propagator $D_{\mu\nu}(k)$, Eq. (4.14), with

$$\tilde{W}_{\mu\nu}(k)=D_{\mu\nu}(k)_{+}-D_{\mu\nu}(k)_{-}, \tag{5.17}$$

where the discontinuity is evaluated along the right-hand axis of the k^{0} plane. This gives

$$\tilde{W}_{\mu\nu}(k)=(-g_{\mu\nu}k^{2}+k_{\mu}k_{\nu})$$
$$\times\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\left(-2\sin(\pi\nu)\nu(\nu+1)\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{a_{0}^{4}\theta(k^{0}-|\vec{k}|)}{(a_{0}^{2}k^{2})^{2+\nu}}\right), \tag{5.18}$$

and thus, using current conservation,

$$\langle j_{1},j_{2}\rangle=\lim_{\nu\to 0}(2\pi)^{-3}\frac{\partial}{\partial\nu}$$
$$\times\left(\frac{-\sin\pi\nu}{\pi}\nu(\nu+1)\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}J(\nu)\right), \tag{5.19a}$$

where

$$J(\nu)\equiv a_{0}^{2}\int d^{4}k\frac{\theta(k^{0}-|\vec{k}|)}{(a_{0}^{2}k^{2})^{1+\nu}}\tilde{j}_{1}^{\mu *}(k)(-g_{\mu\nu})\tilde{j}_{2}^{\nu}(k). \tag{5.19b}$$

Our immediate goal is to rewrite this so it is clear that only values of k on the future light cone contribute. To this end we introduce as variables of integration

$$k^{\mu}=(\omega,\omega v\hat{k}),\quad \omega\geq 0,\quad 0\leq v\leq 1,$$
$$J(\nu)=a_{0}^{2}\int d\hat{k}\int_{0}^{\infty}d\omega\,\omega^{3}\int_{0}^{1}dv\,v^{2}[a_{0}^{2}\omega^{2}(1-v^{2})]^{-1-\nu}$$
$$\times\tilde{j}_{1}^{\mu *}(k)(-g_{\mu\nu})\tilde{j}_{2}^{\nu}(k). \tag{5.20}$$

Assuming $\nu<0$ we may integrate by parts on ω^{2} and v^{2},

$$J(\nu)=\frac{-1}{4\nu^{2}}\int d\hat{k}\int_{0}^{\infty}d\omega^{2}\int_{0}^{1}dv^{2}[a_{0}^{2}\omega^{2}(1-v^{2})]^{-\nu}$$
$$\times\frac{\partial}{\partial\omega^{2}}\frac{\partial}{\partial v^{2}}G(k), \tag{5.21}$$

where

$$G(k)=\omega^{2}v\tilde{j}_{1}^{\mu *}(k)(-g_{\mu\nu})\tilde{j}_{2}^{\nu}(k). \tag{5.22}$$

For the limit $\nu\to 0$ in Eq. (5.19a) it will be sufficient to put

$$[a_{0}^{2}\omega^{2}(1-v^{2})]^{-\nu}=1-\nu\ln[\tfrac{1}{4}(1-v^{2})]-\nu\ln 4a_{0}^{2}\omega^{2}, \tag{5.23}$$

and so

$$J(\nu)=\frac{-1}{4\nu^2}\int d\hat{k}\left\{\nu\int_0^1 dv\ln[\tfrac{1}{4}(1-v^2)]\frac{\partial}{\partial v}G(k)\right|_{\omega=0}+\int_0^\infty d\omega[1-2\nu\ln 2a_0\omega]\frac{\partial}{\partial\omega}G(k)\left|_{v=1}\right\}. \tag{5.24}$$

To the same order in ν we have

$$-\nu(\nu+1)\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{\sin\pi\nu}{\pi}=-\nu^2(1+\nu-2\gamma\nu),$$

where γ is Euler's constant

$$\gamma=-\Gamma'(1)=-\int_0^\infty e^{-s}\ln s\,ds. \tag{5.25}$$

We thus find from Eq. (5.19)

$$\langle j_1,j_2\rangle=\tfrac{1}{4}(2\pi)^{-3}\int d\hat{k}\left\{\int_0^1 dv\ln[\tfrac{1}{4}(1-v^2)]\frac{\partial}{\partial v}G(k)\right|_{\omega=0}-2\int_0^\infty d\omega\ln a\omega\frac{\partial}{\partial\omega}G(k)\left|_{v=1}\right\}, \tag{5.26}$$

where

$$a=2a_0e^{\gamma-1/2}=e^{\gamma-1/2}\mu^{-1}. \tag{5.27}$$

Recall that $k^\mu=(\omega,\omega v\hat{k})$, so the first term has its suppport at the vertex of the cone and the second is on the mantle of the cone with ω the conventional photon frequency.

The function

$$\phi^\mu(\vec{k})\equiv(2\pi)^{-3/2}\tilde{j}^\mu(k)|_{k^0=|\vec{k}|=\omega}=(2\pi)^{-3/2}\int e^{ik\cdot x}j^\mu(x)d^4x|_{k^0=|\vec{k}|=\omega}, \tag{5.28}$$

defined on the mantle of the future cone, is the conventional photon wave function. The second term in Eq. (5.26) is a sesquilinear form on the photon wave functions given by

$$\langle\phi_1,\phi_2\rangle\equiv-\tfrac{1}{2}\int d\hat{k}\int_0^\infty d\omega\ln a\omega\frac{\partial}{\partial\omega}\times[\omega^2\phi_1^{\mu}{}^*(\vec{k})(-g_{\mu\nu})\phi_2^\nu(\vec{k})]. \tag{5.29}$$

If the wave functions were regular at $\omega=0$ an integration by parts would give

$$\tfrac{1}{2}\int d\hat{k}\int_0^\infty\frac{d\omega}{\omega}\omega^2\phi_1^{\mu}{}^*(\vec{k})(-g_{\mu\nu})\phi_2^\nu(\vec{k}),$$

which is the conventional inner product. They are not regular at $\omega=0$, for we have, rather, from Eq. (5.10),

$$\lim_{\omega\to 0}[\omega\phi^\mu(\vec{k})]=\frac{i}{(2\pi)^{3/2}}\int\frac{u^\mu}{u^0-\vec{u}\cdot\hat{k}}\rho(u)\frac{d^3u}{u^0}, \tag{5.30}$$

and the conventional inner product is divergent. However, the inner product (5.29) is finite.

Because $k^\mu=(\omega,\omega v k)$, the first term in Eq. (5.26) depends only on the zero energy and momentum limit of the currents, and thus, by the asymptotic limit (5.10), it is bilinear form on the charge defect functions

$$\langle\rho_1,\rho_2\rangle\equiv\tfrac{1}{4}(2\pi)^{-3}\int d\hat{k}\int_0^1 dv\ln[\tfrac{1}{4}(1-v^2)]\frac{\partial}{\partial v}\times[v\omega^2\tilde{j}_1^{\mu}{}^*(k)\times(-g_{\mu\nu})\tilde{j}_2^\nu(k)]|_{\omega=0}, \tag{5.31}$$

$$\langle\rho_1,\rho_2\rangle=\int\frac{d^3u_1}{u_1^0}\frac{d^3u_2}{u_2^0}\rho(u_1)K(u_1,u_2)\rho(u_2), \tag{5.32}$$

with kernel

$$K(u_1,u_2)=\frac{1}{4}\frac{1}{(2\pi)^3}\int d\hat{k}\int_0^1 dv\ln[\tfrac{1}{4}(1-v^2)]\frac{\partial}{\partial v}\left(\frac{v(-u_1\cdot u_2)}{(u_1^0-\vec{u}_1\cdot v\hat{k})(u_2^0-\vec{u}_2\cdot v\hat{k})}\right). \tag{5.33}$$

By rotational invariance, $K(u_1,u_2)$ is a function of three hyperbolic angles

$$K(u_1,u_2)=K(\psi,\psi_1,\psi_2) \tag{5.34a}$$

defined by

$$u_1^0=\cosh\psi_1,\quad u_2^0=\cosh\psi_2,\quad u_1\cdot u_2=\cosh\psi, \tag{5.34b}$$

which may be chosen to be non-negative, and which form the three sides of a hyperbolic triangle. The integral is evaluated in the Appendix, with the result

$$K(\psi,\psi_1,\psi_2)=\frac{1}{(2\pi)^2}\frac{1}{\tanh\psi}\times\{R[\tfrac{1}{2}(\psi+\psi_1+\psi_2)]-R[\tfrac{1}{2}(\psi_1+\psi_2-\psi)]+R[\tfrac{1}{2}(\psi_1+\psi-\psi_2)]+R[\tfrac{1}{2}(\psi_2+\psi-\psi_1)]-R(\psi)\}, \tag{5.35}$$

where

$$R(\psi)\equiv\int_0^\psi dx\frac{x}{\tanh x}. \tag{5.36}$$

This explicit expression for K gives a large part of the radiative corrections. The latter part of the Appendix is devoted to the Lorentz transformation properties of $\langle\rho_1,\rho_2\rangle$ and $\langle\phi_1,\phi_2\rangle$.

The principal result of this section may now be stated, namely $\langle j_1,j_2\rangle \equiv \langle A(j_1)\Omega, A(j_2)\Omega\rangle$ is given by

$$\langle A(j_1)\Omega, A(j_2)\Omega\rangle = \langle \rho_1,\rho_2\rangle + \langle\phi_1,\phi_2\rangle , \tag{5.37}$$

where the two inner products on the right-hand side are defined in Eqs. (5.32) and (5.29). According to the reconstruction principle, the state $A(j)\Omega$ is the equivalence class of test functions, modulo test functions orthogonal to all test functions.[25] Equation (5.37) shows that the equivalence classes, and thus the states, may be characterized by the pair of functions $\rho(u)$ and $\phi^\mu(k)$, which are thus the wave functions of the state $A(j)$, and we may write

$$A(j)\Omega = \begin{pmatrix}\rho(u)\\ \phi^\mu(k)\end{pmatrix} . \tag{5.38}$$

The second wave function,

$$\phi^\mu(\vec{k}) = (2\pi)^{-3/2}\int \exp[i(|\vec{k}|x^0 - \vec{k}\cdot\vec{x})]j^\mu(x)d^4x ,$$

is the familiar photon wave function, although the inner product $\langle\phi_1,\phi_2\rangle$, Eq. (5.29), is a regularized one. Because of transversality, the wave functions $\phi_\mu(k)$ and $\phi_\mu(k)+k_\mu f(k)$ are equivalent, so $\phi_\mu(k)$ represents two degrees of polarization, as usual. The wave function $\rho(u)$, given by Eq. (5.11b) is the charge-scattering function of the flow $j^\mu(x)$. It is the wave function of a quantum whose energy-momentum four-vector is strictly zero, but which carries four-dimensional angular momentum. It is a remnant of the third degree of polarization of massive vector mesons. To distinguish the two types of photons, we may call the photons described by $\phi^\mu(k)$ "radiation photons," since they are the familiar quanta of radiation, and the quanta described by the charge-scattering function "zero-frequency photons" since they carry no energy or momentum. The generic one-photon state may thus be said to be a direct sum of a radiation-photon and a zero-frequency-photon state.

We have been using the term "state" to designate any vector in the representation space of the fields, which is an indefinite-metric space. To obtain physical states which are elements of a positive-metric space, we turn to the many-particle space, and seek positive-metric subspaces. However, we wish to emphasize that the indefiniteness of the inner product (5.37) does not arise as in the usual Gupta-Bleuler[26] formalism from the indefiniteness of the Minkowski metric. [This has been overcome by the transversality condition $k_\mu \tilde{j}^\mu(k) = 0$, which implies $k_\mu\phi^\mu(k)=0$.] Instead it comes from the regularization of infrared divergent integrals by what is, in structure, a linear substraction. On the coherent states the regularization is multiplicative, and hence maintains positivity.

VI. THE S MATRIX

The many-particle states are obtained from the one-particle states by the method of second quantization. Let Ω be the vacuum state, unique to within a phase, and let $a_\gamma^\dagger(\phi)$ and $a_z^\dagger(\rho)$ be creation operators for radiation photons and zero-frequency photons, where ϕ and ρ are radiation-photon and zero-frequency-photon wave functions. They satisfy

$$a_\gamma(\phi)\Omega = a_z(\rho)\Omega = 0 , \tag{6.1}$$

$$[a_\gamma(\phi_1), a_\gamma^\dagger(\phi_2)] = \langle\phi_1,\phi_2\rangle , \tag{6.2}$$

$$[a_z(\rho_1), a_c^\dagger(\rho_2)] = \langle\rho_1,\rho_2\rangle , \tag{6.3}$$

where $\langle\phi_1,\phi_2\rangle$ and $\langle\rho_1,\rho_2\rangle$ are the inner products defined previously, Eqs (5.29) and (5.32), and the remaining commutators vanish.

The free (in or out) vector potential,

$$A(j) = \int A_\mu(x) j^\mu(x) d^4x , \tag{6.4}$$

is represented by

$$A(j) = a_\gamma^\dagger(\phi) + a_z^\dagger(\rho) + a_\gamma(\phi) + a_z(\rho) , \tag{6.5}$$

where the wave functions are expressed in terms of j by Eqs. (5.11b) and (5.28). This provides an invariant decomposition of $A(j)$ into creation and annihilation parts

$$A(j) = A^+(j) + A^-(j) , \tag{6.6}$$

$$A^-(j) = a_\gamma(\phi) + a_z(\rho) , \tag{6.7}$$

$A^+(j) = [A^-(j)]^\dagger$, each of which is invariantly decomposed into radiation-photon and zero-frequency-photon parts.

Scattering of electromagnetic radiation by a given external classical current $j^\mu(x)$ is described by the scattering operator

$$S(j) = T\exp\left[-i\int A_\mu(x)j^\mu(x)d^4x\right] , \tag{6.8}$$

which by the usual Wick ordering may be expressed as

$$S(j) = C(j)\exp[-iA^+(j)]\exp[-iA^-(j)] , \tag{6.9}$$

$$C(j) = \exp\left[-\tfrac{1}{2}\int j^\mu(x) D_{\mu\nu}(x-y) j^\nu(y) d^4x d^4y\right] . \tag{6.10}$$

To separate out zero-frequency-photon coordinates we decompose $A^-(j)$ according to $A^-(j) = a_\gamma(\rho) + a_z(\rho)$, when ϕ and ρ are expressed in terms of j by Eqs. (5.11b) and (5.28), and similarly for $A^+(j)$, and write

$$\exp[-ia_z^\dagger(\rho)]\exp[-ia_z(\rho)]=\exp[\tfrac{1}{2}\langle\rho,\rho\rangle]U_z(\rho)\,, \tag{6.11}$$

where

$$U_z(\rho)=\exp\{-i[a_z^\dagger(\rho)+a_z(\rho)]\} \tag{6.12}$$

is a pseudounitary operator depending only on the zero-frequency-photon coordinates. This provides a factorization of the scattering operator into a radiation factor and a zero-frequency-photon factor, each of which is separately pseudounitary,

$$S(j)=S_\gamma(j)\,U_z(\rho)\,. \tag{6.13}$$

Here

$$S_\gamma(j)=C_\gamma(j)\exp[-ia_\gamma^\dagger(\phi)]\exp[-ia_\gamma(\phi)], \tag{6.14}$$

with

$$C_\gamma(j)=\exp(\tfrac{1}{2}\langle\rho,\rho\rangle)\times\exp\left[-\tfrac{1}{2}\int j^\mu(x)D_{\mu\nu}(x-y)j^\nu(y)d^4x\,d^4y\right], \tag{6.15}$$

is a pseudounitary operator depending only on photon coordinates. The state of the zero-frequency photons is a matter of supreme indifference because they carry no momentum or energy, so $U_z(\rho)$ may be ignored and we may identify $S_\gamma(j)$ as the physical scattering operator. In the photon-mass formalism, the contribution of $U_z(\rho)$ would be eliminated in the sum over final undetectable photons of zero frequency.

Consider now the quantum-electrodynamical scattering operator

$$S=T\exp\left(-i\int J_\mu A^\mu d^4x\right), \tag{6.16}$$

where the current J_μ is expressed in terms of the free Dirac spinor field ψ,

$$J_\mu=-e\bar\psi\gamma_\mu\psi\,. \tag{6.17}$$

(The generalization to more than one type of charged particle is clear.) Wick's theorem for the A field gives

$$S=T_\psi\left\{\exp\left[-\tfrac{1}{2}\int J^\mu(x)D_{\mu\nu}(x-y)J^\nu(y)d^4x\,d^4y\right]\times\exp\left(-i\int J^\mu A_\mu^+d^4x\right)\times\exp\left(-i\int J^\mu A_\mu^-d^4x\right)\right\}, \tag{6.18}$$

where the time-ordering symbol T_ψ acts only on the ψ and $\bar\psi$ fields. Here the exponentials are expanded in power series and subsequently time ordered in J. It is convenient to take matrix elements of S between initial and final charged particle states, each such matrix element being an operator on the states of the electromagnetic field,

$$\langle p_f,e_f|S|p_ie_i\rangle=\left\langle p_f,e_f\right|T_\psi\left\{\exp\left[-\tfrac{1}{2}\int J^\mu(x)D_{\mu\nu}(x-y)J^\nu(y)d^4x\,d^4y\right]\times\exp\left(-i\int J^\mu A_\mu^+d^4x\right)\exp\left(-i\int J^\mu A_\mu^-d^4x\right)\right\}\left|p_i,e_i\right\rangle, \tag{6.19}$$

where p_i,e_i (p_f,e_f) represent the momenta and charges of the initial (final) charged particles.

For these matrix elements, as for the classical current, we may factor out a pseudounitary operator depending only on zero-frequency-photon coordinates. These appear only in the factor

$$\exp\left(-i\int J_\mu A^{\mu-}d^4x\right)=\exp[-iA_\gamma^-(J)]\exp[-iA_z^-(J)] \tag{6.20}$$

and its Hermitian conjugate. Here the notation means that, after the J's are time ordered and the matrix element taken so $A_\gamma^-(J)$ and $A_z^-(J)$ have become contractions with c-number functions, then the replacements $A_\gamma^-(J)\to a_\gamma(\phi)$ and $A_z^-(J)\to a_z(\rho)$ are effected.

We first demonstrate that in the matrix element of the T_ψ product

$$\exp[-iA_z^-(J)]=\exp[-ia_z(\rho_{p,e})]\,, \tag{6.21}$$

where

$$\rho_{p,e}(u)=\sum_a e_a\delta^3(\vec{u}-\vec{p}_a/m_a)u^0 \tag{6.22}$$

and a is an index that runs over initial and final particles, with $e_a=-e_i$ when a refers to an initial particle and $e_a = +e_f$ when a refers to a final particle, so

$$\rho_{p,e}(u) = u^0\Big[\sum_f e_f \delta^3(\vec{u} - \vec{p}_f/m_f) - \sum_i e_i \delta^3(\vec{u} - \vec{p}_i/m_i)\Big]. \tag{6.23}$$

Note that $A_z^-(J)$ depends on the current $J^\mu(x)$ only through the asymptotic charge-density operator

$$\rho(u) = \lim_{\lambda\to\infty} \lambda^3[u\cdot J(\lambda u) - u\cdot J(-\lambda u)], \tag{6.24}$$

where u is a unit future timelike vector. Because of the T_Ψ product, as λ approaches infinity, $J(\lambda u)$ gets moved to the left in the matrix element up against $\langle p_f, e_f|$ and $J(-\lambda u)$ gets moved to the right up against $|p_i, e_i\rangle$. Moreover the current assumes an asymptotic form in the limit[6,13]

$$\lim_{\lambda\to\infty} \lambda^3 u\cdot J(\lambda u) = -e\int \frac{d^3p}{2E}\sum_s [b_s^\dagger(p) b_s(p) - d_s^\dagger(p) d_s(p)]\delta^3(\vec{u} - \vec{p}/m)u^0, \tag{6.25}$$

where $b_s(p)$ and $d_s(p)$ are annihilation operators of electrons and positrons. The states $|p_i, e_i\rangle$ are eigenstates of this operator with eigenvalue

$$\sum_i e_i \delta^3(\vec{u} - \vec{p}_i/m)u^0 \tag{6.26}$$

and likewise for $i\to f$, and Eq. (6.21) is thereby established. The desired factorization of zero-frequency-photon variables now follows as for the classical external current and we have

$$\langle p_f, e_f|S|p_i, e_i\rangle = \langle p_f, e_f|S|p_i, e_i\rangle_\gamma U_z(\rho_{p,e}), \tag{6.27}$$

where

$$U_z[\rho_{p,e}] = \exp[-ia^\dagger(\rho_{p,e}) - ia(\rho_{p,e})] \tag{6.28}$$

as before, and the scattering operator on radiation variables is

$$\langle p_f, e_f|S|p_i, e_i\rangle_\gamma = \exp\Big[\tfrac{1}{2}\sum_{a,b} e_a e_b K(u_a, u_b)\Big]\Big\langle p_f, e_f\Big|T_\Psi\Big\{\exp\Big[-\tfrac{1}{2}\int J^\mu(x) D_{\mu\nu}(x-y) J^\nu(y)\, d^4x\, d^4y\Big] \times \exp\Big(-i\int A_\gamma^{\mu+} J_\mu d^4x\Big)\exp\Big(-i\int A_\gamma^{\mu-} J_\mu d^4x\Big)\Big\}\Big|p_i, e_i\Big\rangle. \tag{6.29}$$

Here we have used Eqs. (5.32) and (6.22) for $\langle\rho_{p,e}, \rho_{p,e}\rangle$

$$\langle\rho_{pe}, \rho_{p,e}\rangle = \sum_{a,b} e_a e_b K(u_a, u_b). \tag{6.30}$$

Again $U_z(\rho_{p,e})$ is a pseudounitary operator that may be dropped as a matter of indifference,[27] and the physical scattering operator is identified with $\langle p_f, e_f|S|p_i, e_i\rangle_\gamma$.

VII. PHYSICAL SUBSPACE

The representation space of the electromagnetic field, $\mathcal{I}_{em}$, generated from the vacuum by applying the creation operators of radiation photons $a_\gamma^+(\phi)$ and of zero-frequency photons $a_z^+(\rho)$, has an indefinite metric. However, the old familiar space of (radiation) photons with square integrable wave functions and no zero-frequency photons is contained in it as a subspace $\mathcal{I}_0$, which has positive norm. It may be completed in the norm[10] to give a physical Hilbert space $\mathcal{K}_0$.

A convenient characterization of $\mathcal{I}_0$ is provided by

$$a_z(\rho)\mathcal{I}_0 = 0. \tag{7.1}$$

This states that all zero-frequency-photon wave functions vanish identically. According to Eq. (5.30), the residue of a radiation-photon wave function at zero frequency,

$$\phi_R^\mu(\hat{k}) \equiv \lim_{w\to 0} \omega\phi^\mu(\omega, \hat{k}), \tag{7.2}$$

is expressible in terms of the related zero-frequency-photon wave function

$$\phi_R^\mu(\hat{k}) = \frac{i}{(2\pi)^{3/2}}\int \frac{u^\mu}{u^0 - \vec{u}\cdot\hat{k}}\rho(u)\frac{d^3u}{u^0}. \tag{7.3}$$

If it vanishes, the inner product (5.29) may be integrated by parts[28] with the result, for $\phi_1, \phi_2 \in \mathcal{I}_0$,

$$\langle\phi_1, \phi_2\rangle = \tfrac{1}{2}\int d\hat{k}\int d\omega\, \omega\phi_1^{\mu *}(k)(-g_{\mu\nu})\phi_2^\nu(k). \tag{7.4}$$

This is the old familiar inner product. It is true for each variable in a many-photon wave function in the space defined by Eq. (7.1), which establishes the assertion.

It is convenient to define the annihilation operator of radiation photons, depending on a single four-momentum $a_\gamma^\mu(k)$, $k=\omega(l,\hat{k})$, and of zero-frequency photons, depending on a single four-velocity $a_z(u)$, by

$$a_\gamma^\mu(k)\Omega = a_z(u)\Omega = 0\,, \tag{7.5}$$

$$[a_\gamma^\mu(k), a_\gamma^\dagger(\phi)] = \phi^\mu(k)\,, \tag{7.6a}$$

$$[a_z(u), a_z^\dagger(\rho)] = \rho(u)\,, \tag{7.6b}$$

the other commutators being zero. [The Hermitian conjugate quantities, $a_\gamma^{\mu\dagger}(k)$, $a_z^\dagger(u)$, are not operators, but operator-valued distributions.] The residue of the radiation-photon annihilation operator at zero frequency, defined by

$$a_R^\mu(\hat{k}) \equiv \lim_{\omega\to 0} \omega a^\mu(\omega,\hat{k})\,, \tag{7.7}$$

satisfies[29]

$$a_R^\mu(\hat{k}) = \frac{i}{(2\pi)^{3/2}}\int \frac{u^\mu}{u^0 - \vec{u}\cdot\hat{k}}\, a_z(u)\frac{d^3u}{u^0}\,, \tag{7.8}$$

which is the operator expression of, Eq. (7.3) for every wave function. Thus, Eq. (7.1) implies

$$a_R^\mu(\hat{k})\mathcal{G}_0 = 0\,, \tag{7.9}$$

which is the operator statement that all radiation-photon residues in $\mathcal{G}_0$ vanish.

In case only neutral particles are present it is reasonable to suppose that the electromagnetic field is in a state of $\mathcal{G}_0$. We may learn about the state of the electromagnetic field associated with charged particles by producing them from the scattering of neutral particles.[30] In this case the electromagnetic field is in a state of $S\mathcal{G}_0$. We saw, Eq. (6.27), that for states with well-defined charged particle momenta, the dependence of the S matrix on zero-frequency-photon coordinates is entirely contained in the factor $U_z(\rho_{p,e}) = \exp[-ia_z^\dagger(\rho_{p,e}) - ia_z(\rho_{p,e})]$, so the zero-frequency photons $S\mathcal{G}_0$ are in the state

$$\Phi_{p,e} = U_z(\rho_{p,e})\Omega_z\,,$$

where Ω_z is the zero-frequency-photon vacuum. This is a completely coherent state characterized by

$$a_z(u)\Phi_{p,e} = \rho_{p,e}(u)\Phi_{p,e}$$

$$= -i\Big(\sum_f e_f \delta^3(\vec{u} - \vec{p}_f/m_f)u^0\Big)\Phi_{p,e}\,. \tag{7.10}$$

This condition may be expressed in operator language, for $\Phi \in S\mathcal{G}_0$,

$$a_z^{\text{out}}(u)\Phi = -i\rho_{\text{qu}}^{\text{out}}(u)\Phi\,, \tag{7.11}$$

where

$$\rho_{\text{qu}}^{\text{out}}(u) = -e\int \frac{d^3p}{2E}\sum_s [b_s^\dagger(p)b_s(p) - d_s^\dagger(p)d_s(p)]^{\text{out}}\delta^3(\vec{u} - \vec{p}/m)u^0 \tag{7.12}$$

is the quantum-mechanical operator representing the charge density of outgoing particles (electrons and positrons for simplicity) per invariant volume element in velocity space. Its integral gives the total-charge operator

$$Q = \int \rho_{\text{qu}}^{\text{out}}(u)\frac{d^3u}{u^0} = -e\int\frac{d^3p}{2E}\sum_s [b_s^\dagger(p)b_s(p) - d_s^\dagger(p)d_s(p)]^{\text{out}}\,. \tag{7.13}$$

It would be nice if we could adopt

$$a_z^{\text{out}}(u)\Phi = -i\rho_{\text{qu}}^{\text{out}}(u)\Phi \tag{7.14}$$

as a subsidiary condition defining the physical subspace in the presence of charged particles. In fact the last section shows that

$$a_z^{\text{out}}(u) + i\rho_{\text{qu}}^{\text{out}}(u) = a_z^{\text{in}}(u) + i\rho_{\text{qu}}^{\text{in}}(u)\,, \tag{7.15}$$

so the above condition, Eq. (7.14) which characterizes the states of charged particles produced from neutrals, is stable under subsequent scatterings. (This equality allows us to suppress in or out labels in the remainder of this section.) Unfortunately, starting from neutrals, we can only learn about states of total-charge zero, and this condition violates transversality in the sectors of nonvanishing total charge.[31] For by Eq. (7.8) we have

$$a_R^0(\hat{k}) - \hat{k}\cdot a_R(\hat{k}) = \frac{i}{(2\pi)^{3/2}}\int\frac{d^3u}{u^0}a_z(u)\,. \tag{7.16}$$

The left-hand side vanishes by transversality so

$$\int\frac{d^3u}{u^0}a_z(u) = 0\,, \tag{7.17}$$

and the proposed subsidiary condition (7.14) leads to $Q\Phi = 0$.

One solution of this dilemma[32] is to generalize the infrared-coherence conditions to

$$a_z(u)\Phi = -i[\rho_{qu}(u) - \rho_{cl}(u)]\Phi\,, \tag{7.18}$$

where $\rho_{cl}(u)$ is a fixed classical function, or distribution, whose integral $q = \int \rho_{cl}(u)(u^0)^{-1}d^3u$ is an eigenvalue of the charge operator Q. The quantity $-\rho_{cl}(u)$ may be interpreted as the asymptotic charge density of faraway particles produced in association with the particles treated quantum mechanically, whose influence is negligible otherwise. Alternately, recalling that $\rho(u) = \rho_+(u) - \rho_-(u)$, Eq. (1.7), we may interpret $\rho_{cl}(u)$ as the asymptotic charge density of particles in some initial state, from which the state of interest was produced by scattering.

A familiar example of the latter occurs in the standard treatment of radiative corrections to scattering and decay processes, which we call the retarded representation. Here well-defined particle momenta p_i and no infrared radiation are attributed to the initial state Φ,

$$\rho_{qu}^{in}(u)\Phi = \sum_i e_i \delta^3(\vec{u} - \vec{p}_i/m_i)u^0\Phi\,, \tag{7.19a}$$

$$a_R^{\mu\,in}(\hat{k})\Phi = a_z^{in}(u)\Phi = 0\,, \tag{7.19b}$$

so

$$[a_z^{in}(u) + i\rho_{qu}^{in}(u)]\Phi = i\sum_i e_i\delta^3(\vec{u} - \vec{p}_i/m)\Phi \tag{7.20}$$

and by Eq. (7.15)

$$[a_z^{out}(u) + i\rho_{qu}^{out}(u)]\Phi = i\sum_i e_i\delta^3(\vec{u} - \vec{p}_i/m)\Phi\,. \tag{7.21}$$

Thus, on states with well-defined outgoing particle momenta p_f, symbolically $\Phi = \int dp_f \Phi_{p_f}^{out}$, we have

$$a_z^{out}(u)\Phi_{p_f}^{out} = -i\left[\sum_f e_f\delta^3(\vec{u} - \vec{p}_f/m_f) - \sum_i e_i\delta^3(\vec{u} - \vec{p}_i/m_i)\right]u^0\Phi_{p_f}\,. \tag{7.22}$$

From this, we recover the familiar low-frequency bremsstrahlung formula, using Eq. (7.8),

$$a_R^{\mu\,out}(\hat{k})\Phi_{p_f}^{out} = \frac{1}{(2\pi)^{3/2}}\left(\sum_f \frac{e_f p_f^\mu}{E_f - \vec{p}_f\cdot\hat{k}} - \sum_i \frac{e_i p_i^\mu}{E_f - \vec{p}_i\cdot\hat{k}}\right)\Phi_{p_f}^{out}\,. \tag{7.23}$$

This standard approach does not allow superpositions of states with different incoming charged particle momenta, or, in other words, no wave packets of incoming charged particles.[33] Thus no localization of incoming charged particles is possible in the retarded representation. Although the retarded representation is most convenient for cross sections, the more general representation (7.18) must be used if localization is important.

There remains to show that Eq. (7.18) defines a subspace $\mathcal{S}[\rho_{cl}]$ of non-negative norm. Let $\mathcal{S}[\rho_{cl}]$ be expressed as a direct integral over subspaces where the (incoming or outgoing) charged particles have definite momenta $p = \{p_n\}$, symbolically,

$$\mathcal{S}[\rho_{cl}] = \int \mathcal{S}[\rho_{cl}, p]dp\,. \tag{7.24}$$

On each $\mathcal{S}[\rho_{cl}, p]$ we have

$$\rho_{qu}(u)\mathcal{S}[\rho_{cl}, p] = \sum_n e_n\delta^3(\vec{u} - \vec{p}_n/m_n)u^0\mathcal{S}[\rho_{cl}, p]\,, \tag{7.25}$$

where $\int \rho_{cl}(u)(u^0)^{-1}d^3u = \sum_n e_n$, so

$$a_z(u)\mathcal{S}[\rho_{cl}, p] = -i\left(\sum_n e_n\delta^3(\vec{u} - \vec{p}_n/m_n)u^0 - \rho_{cl}(u)\right)\mathcal{S}[\rho_{cl}, p]\,. \tag{7.26}$$

If $j^\mu(x)$ is a classical current with charge-scattering function

$$\rho(u) = \sum_n e_n\delta^3(\vec{u} - \vec{p}_n/m_n)u^0 - \rho_{cl}(u)\,,$$

and

$$U(j) = T\exp\left(-i\int j\cdot A d^4x\right),$$

then $U(j)^{-1}$ is a pseudounitary operator which maps $\mathcal{S}[\rho_{cl}, p]$ isometrically onto $\mathcal{S}_0$. Thus $\mathcal{S}[\rho_{cl}]$ is a direct sum of positive metric spaces. Hence $\mathcal{S}[\rho_{cl}]$ is a space of positive norm which may be completed in norm to a physical Hilbert space $\mathcal{H}[\rho_{cl}]$.

Note that a superposition of states with different eigenvalues $\rho_{cl}(u) = \rho_{qu}(u) - ia_z(u)$, is not, in general, a state of positive norm, and different functions $\rho_{cl}(u)$ label different superselection sectors.[33] These different superselection sectors are presumably physically equivalent since they differ only at arbitrarily low frequency or large distance.

VIII. CONCLUSION

We have presented a method of handling infrared divergences in quantum electrodynamics which is very close to the photon-mass method. In fact, a term by term comparison of the two methods is possible, although, in our case, the photon has a strictly zero mass. In practical calculations our method may have some advantage. In particular, the evaluation of the zero-frequency-photon kernel in the Appendix means that part of the sum over final-state photons is effected once and for all. The virtue of this approach as a practical tool will be exhibited in an accompanying article where the spectral composition of coherent bremsstrahlung radiation is calculated nonperturbatively. For virtual photons, Eq. (4.14) provides a convenient representation of the photon propagator which resembles analytic regularization.

The reader will have observed (Sec. VI and the end of Sec. IV) that in calculating S-matrix elements, the charged particles are first put on the mass shell trivially (by dropping external legs in Feynman diagrams, as in a theory with massive photons) before the integrations over virtual photons are effected. These limits are not interchangeable, for the on-shell limit of the off-shell Green's functions is anything but trivial. The present approach bypasses the intricacies of the other order of limits.

Finally, the author cannot resist speculating about whether the approach described here to deal with infrared divergences may also be helpful for ultraviolet divergences. The basic philosophy is to suitably extend the free-field theory, so divergences are not subsequently encountered, instead of applying a subtraction procedure to divergent graphs which represent the interaction. The philosophy is implemented by extension of the free-particle propagator. It is intriguing in this regard that the photon propagator (1.1b) renders the electron self-mass ultraviolet finite in lowest order, for the reason given in the Introduction, with the result,

$$\delta m = -(4\pi)^{-2} 3e^2[\ln(m_0^2/\lambda^2) - \tfrac{1}{2}]m_0 . \qquad (8.1)$$

The finite parameter λ cancels out of cross sections everywhere except in their dependence on the electron self-mass. In lowest order $Z_1 = Z_2$ is also finite.[22] Now the photon propagator (1.1b) does not, by itself, eliminate all higher-order ultraviolet divergences. At the very least the electron propagator must also be extended. However, the possibility suggests itself that, by a further development of the present method, the scale-breaking mechanism of Sec. III may allow a natural introduction of mass into a theory with a scale-invariant Lagrangian. Suppose the physical mass $m = m(m_0, \lambda)$ were an attractive fixed point m^* of the mass renormalization series, independent of m_0. In this case, if m_0 is chosen equal to m^*, then $\delta m = 0$, and Eq. (8.1) gives

$$m^* = \lambda \exp(\tfrac{1}{4}) . \qquad (8.2)$$

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APPENDIX: EVALUATION OF THE ZERO-FREQUENCY-PHOTON KERNEL

We will separate the kernel $K(u_1, u_2)$ of Eq. (5.33),

$$K(u_1, u_2) = \frac{1}{4}\frac{1}{(2\pi)^3}\int d\hat{k}\int_0^1 dv\, \ln[\tfrac{1}{4}(1-v^2)]\frac{\partial}{\partial v}\left[\frac{v(-u_1\cdot u_2)}{(u_1^0 - \vec{u}_1\cdot v\hat{k})(u_2^0 - \vec{u}_2\cdot v\hat{k})}\right], \qquad (A1)$$

into a Lorentz-invariant part $K_i(\psi)$, where $u_1 \cdot u_2 = \cosh\psi$, $\psi \geq 0$ and a frame-dependent part $K_f(u_1, u_2)$ obtained from $v = 1$. As will be seen, the frame-dependent part is just such that $\langle \rho, \rho\rangle + \langle \phi, \phi\rangle$ is Lorentz invariant although the separate terms are not. For this purpose it is convenient to introduce as a factor in the integrand $1 = \lim_{L\to\infty}\theta(L - u_1\cdot k)$ where $k^\mu = (1-v^2)^{-1/2}(1, v\hat{k})$, and integrate by parts on v. This gives

$$K(u_1, u_2) = K_i(\psi) + K_f(u_1, u_2), \qquad (A2)$$

$$K_f(u_1, u_2) = \frac{-1}{(2\pi)^3}\frac{1}{2}\int d\hat{k}\, \ln(u_1^0 - \vec{u}_1\cdot\hat{k})\frac{u_1\cdot u_2}{(u_1^0 - \vec{u}_1\cdot\hat{k})(u_2^0 - \vec{u}_2\cdot\hat{k})}, \qquad (A3)$$

$$K_i(\psi) = \frac{1}{(2\pi)^3}\frac{1}{2}\lim_{L\to\infty}\left(\int d\hat{k}\, \ln 2L\frac{u_1\cdot u_2}{(u_1^0 - \vec{u}_1\cdot\hat{k})(u_2^0 - \vec{u}_2\cdot\hat{k})} - \int d\hat{k}\int_0^1 dv\frac{v^2}{(1-v^2)^2}\theta(L - u_1\cdot k)\frac{u_1\cdot u_2}{u_1\cdot k\, u_2\cdot k}\right). \qquad (A4)$$

Each of the terms in the last expression is an invariant function[34,35] of u_1 and u_2.

We first evaluate the invariant kernel $K_i(\psi)$. This is effected most conveniently in the frame where

$$u_1 = (1, 0, 0, 0), \quad u_2 = (\cosh\psi, 0, 0, \sinh\psi),$$

$$K_i(\psi) = \frac{1}{(2\pi)^2}\frac{1}{2}\lim_{L\to\infty}\int_{-1}^{1} dx\left(\ln 2L\frac{1}{1-\beta x} - \int_0^V dv\frac{v^2}{1-v^2}\frac{1}{1-\beta v x}\right), \tag{A5}$$

where $x = \cos\theta$, $\beta = \tanh\psi$, and $V = 1 - \frac{1}{2}L^{-2}$. An integration by parts on v^2 yields

$$K_i(\psi) = \frac{-1}{(2\pi)^2}\frac{1}{4}\int_{-1}^{1} dx\int_0^1 dv\,\ln[\tfrac{1}{4}(1-v^2)]\frac{\partial}{\partial v}\left(\frac{v}{1-\beta v x}\right), \tag{A6}$$

$$K_i(\psi) = \frac{-1}{(2\pi)^2}\frac{1}{2}\int_0^1 dv\,\ln[\tfrac{1}{4}(1-v^2)]\frac{1}{1-\beta^2 v^2}. \tag{A7}$$

Observe that K_i satisfies

$$\frac{\partial K_i}{\partial\beta^2} + \frac{1}{2\beta^2}K_i = X, \tag{A8}$$

where

$$X = \frac{-1}{(2\pi)^2}\frac{1}{8\beta^2}\int_0^1 dv\,\ln\tfrac{1}{4}(1-v^2)\left[\frac{1}{(1-\beta v)^2} + \frac{1}{(1+\beta v)^2}\right],$$

$$X = \frac{1}{(2\pi)^2}\frac{1}{4\beta^3}\frac{1}{1-\beta^2}\ln\frac{1+\beta}{1-\beta}, \tag{A9}$$

which has the solution

$$K_i = \frac{1}{(2\pi)^2}\frac{1}{2\beta}\int_0^\beta dx\frac{1}{1-x^2}\frac{1}{x}\ln\frac{1+x}{1-x}. \tag{A10}$$

The lower limit of integration is fixed at zero, since K_i is regular at $\beta = 0$. With $\beta = \tanh\psi$, we have

$$K_i(\psi) = \frac{1}{(2\pi)^2}\frac{1}{\tanh\psi}R(\psi), \tag{A11}$$

$$R(\psi) \equiv \int_0^\psi d\psi'\frac{\psi'}{\tanh\psi'}. \tag{A12a}$$

The latter function is ubiquitous in radiative correction calculations. It may be expressed in terms of Spence functions, but $R(\psi)$ is convenient for our purposes. It has the asymptotic limit

$$\lim_{\psi\to 0}[R(\psi) - \tfrac{1}{2}\psi^2] = \pi^2/12. \tag{A12b}$$

Consider next the frame-dependent part $K_f(u_1, u_2)$, Eq. (A3). By rotational invariance it is a function of the three positive hyperbolic angles ψ_1, ψ_2, and ψ, Eq. (5.34b). Let τ be the unit vector $\tau = (1, 0, 0, 0,)$, so the three unit four-vectors u_1, u_2, and τ define the vertices of a hyperbolic triangle. Its sides are the three hyperbolic angles ψ_1, ψ_2, and $\psi \geq 0$,

$$u_1\cdot\tau = \cosh\psi_1, \quad u_2\cdot\tau = \cosh\psi_2, \quad u_1\cdot u_2 = \cosh\psi \tag{A13a}$$

and they satisfy the triangular inequalities

$$\psi_1 + \psi_2 \geq \psi, \quad \psi + \psi_1 \geq \psi_2, \quad \psi + \psi_2 \geq \psi_1. \tag{A13b}$$

Equation (A3) may be rewritten as

$$K_f(\psi, \psi_1, \psi_2) = \frac{-1}{(2\pi)^3}\frac{1}{2}\int d\hat{k}\,\ln\left(\frac{u_1^0 - \vec{u}_1\cdot\hat{k}}{\tau^0 - \vec{\tau}\cdot\hat{k}}\right)\frac{u_1\cdot u_2}{(u_1^0 - \vec{u}_1\cdot\hat{k})(u_2^0 - \vec{u}_2\cdot\hat{k})}, \tag{A14}$$

and we observe that K_f is an invariant function[35] of the three four-vectors u_1, u_2, and τ. We evaluate it in the frame where $u_1 = (1, 0, 0, 0)$, $\tau = (\cosh\psi_1, 0, 0, \sinh\psi_1)$, and $u_2 = (\cosh\psi, \sinh\psi\sin\alpha, 0, \sinh\psi\cos\alpha)$, and $\cos\alpha$ may be expressed in terms of the hyperbolic angles by $\tau\cdot u_2 = \cosh\psi_2 = \cosh\psi\cosh\psi_1 - \sinh\psi\sinh\psi_1\cos\alpha$, which is the law of cosines for hyperbolic triangles. This gives

$$K_f = \frac{1}{(2\pi)^3}\frac{1}{2}\int_{-1}^{1} d\cos\theta\int_0^{2\pi} d\phi\,\frac{\ln[\cosh\psi_1(1-\beta_1\cos\theta)]}{1-\beta(\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\phi)}, \tag{A15}$$

where $\beta = \tanh\psi$ and $\beta_1 = \tanh\psi_1$. After integration on ϕ we find

$$K_f = \frac{1}{(2\pi)^2}\frac{1}{2}\int_{-1}^{1} dx\,\frac{\ln[\cosh\psi_1(1-\beta_1 x)]}{[(\beta x - \cos\alpha)^2 - (1-\beta^2)(1-\cos^2\alpha)]^{1/2}}. \tag{A16}$$

On eliminating $\cos\alpha$ from the law of cosines, and introducing as variable of integration $y = \sinh^2\psi\sinh\psi_1 x - \cosh\psi(\cosh\psi\cosh\psi_1 - \cosh\psi_2)$, we find

$$K_f = \frac{1}{(2\pi)^2}\frac{1}{2\beta}\int_{y_-}^{y_+} dy\,\frac{\ln[(-y - \cosh\psi_1 + \cosh\psi\cosh\psi_2)/\sinh^2\psi]}{(y^2+\lambda^2)^{1/2}}, \tag{A17}$$

where

$$y_\pm = \pm \sinh^2\psi \sinh\psi_1 - \cosh\psi(\cosh\psi \cosh\psi_1 - \cosh\psi_2),$$

$$\lambda^2 = 1 - \cosh^2\psi - \cosh^2\psi_1 - \cosh^2\psi_2 + 2\cosh\psi\cosh\psi_1\cosh\psi_2. \tag{A18}$$

The square root is eliminated by introducing as new variable of integration u, from $y = \frac{1}{2}(u - \lambda^2/u)$, with the result

$$K_f = \frac{1}{(2\pi)^2}\frac{1}{2\beta}\int_{u_-}^{u_+}\frac{du}{u}\ln\{[-\tfrac{1}{2}(u - \lambda^2/u) - \cosh\psi_1 + \cosh\psi\cosh\psi_2]/\sinh^2\psi\},$$

where $u_\pm = e^{\pm\psi}[\cosh\psi_2 - \cosh(\psi - \psi_1)]$. Finally, with $u = [\cosh\psi_2 - \cosh(\psi - \psi_1)]v$, we find

$$K_f = \frac{1}{(2\pi)^2}\frac{1}{2\beta}J, \tag{A19}$$

where

$$J = \int_{v_-}^{v_+}\frac{dv}{v}\ln\left(\frac{[b - av][av + c]}{2av\sinh^2\psi}\right), \tag{A20}$$

$$v_\pm = e^{\pm\psi},$$

$$a = 2\sinh\tfrac{1}{2}(\psi_1 + \psi_2 - \psi)\sinh\tfrac{1}{2}(\psi + \psi_2 - \psi_1), \tag{A21a}$$

$$c = 2\sinh\tfrac{1}{2}(\psi_1 + \psi_2 - \psi)\sinh\tfrac{1}{2}(\psi + \psi_1 - \psi_2), \tag{A21b}$$

$$b = 2\sinh\tfrac{1}{2}(\psi + \psi_1 + \psi_2)\sinh\tfrac{1}{2}(\psi + \psi_2 - \psi_1). \tag{A21c}$$

To proceed further note that $\partial J(\psi, \psi_1, \psi_2)/\partial\psi$ is an elementary integral, with the value

$$\frac{\partial J}{\partial\psi}(\psi, \psi_1, \psi_2) = S[\tfrac{1}{2}(\psi + \psi_1 + \psi_2)] - S[\tfrac{1}{2}(\psi_1 + \psi_2 - \psi)] + S[\tfrac{1}{2}(\psi + \psi_1 - \psi_2)] + S[\tfrac{1}{2}(\psi + \psi_2 - \psi_1)] - 4S(\psi), \tag{A22}$$

where $S(\psi) = \psi/\tanh\psi$. This determines $J(\psi, \psi_1, \psi_2)$ to within an additive function of ψ_1 and ψ_2. The latter may be found by evaluating $K_f(\psi_1 + \psi_2, \psi_1, \psi_2)$, with the result

$$K_f(\psi, \psi_1, \psi_2) = \frac{1}{(2\pi)^2}\frac{1}{\tanh\psi}\times\{R[\tfrac{1}{2}(\psi_1 + \psi_2 + \psi)] - R[\tfrac{1}{2}(\psi_1 + \psi_2 - \psi)] + R[\tfrac{1}{2}(\psi + \psi_2 - \psi_1)] + R[\tfrac{1}{2}(\psi + \psi_1 - \psi_2)] - 2R(\psi)\}, \tag{A23}$$

and R is defined by Eq. (A12).

We thus find for the kernel, $K = K_i + K_f$ by Eq. (A11),

$$K(\psi, \psi_1, \psi_2) = (2\pi)^{-2}(\tanh\psi)^{-1}\times\{R[\tfrac{1}{2}(\psi + \psi_1 + \psi_2)] - R[\tfrac{1}{2}(\psi_1 + \psi_2 - \psi)] + R[\tfrac{1}{2}(\psi + \psi_1 - \psi_2)] + R[\tfrac{1}{2}(\psi + \psi_2 - \psi_1)] - R(\psi)\}. \tag{A24a}$$

As a special case we have

$$K(0, \psi_1, \psi_1) = (2\pi)^{-2}(\tanh\psi_1)^{-1}\psi_1. \tag{A24b}$$

Equation (A24) gives the desired form for the kernel $K(u_1, u_2)$ of the inner product $\langle\rho_1, \rho_2\rangle$. Let us verify that the inner product $\langle j_1, j_2\rangle = \langle\rho_1, \rho_2\rangle + \langle\phi_1, \phi_2\rangle$ is Lorentz invariant although the separate terms are not.[36] Note first that with $\phi^{\Lambda\mu}(k) \equiv \Lambda^\mu_\nu\phi^\nu(\Lambda^{-1}k)$,

$$\langle\phi_1^\Lambda, \phi_2^\Lambda\rangle = \langle\phi_1, \phi_2\rangle + \tfrac{1}{2}\int d\hat{k}\ln(\Lambda_0^0 + \Lambda_i^0\hat{k}^i)\times\phi_{1R}^{\mu*}(\hat{k})(-g_{\mu\nu})\phi_{2R}^\nu(\hat{k}). \tag{A25}$$

Here we have introduced the residue of the radiation-photon wave function defined by

$$\phi_R^\mu(\hat{k}) = \lim_{\omega\to 0}\omega\phi_R^\mu(\vec{k}). \tag{A26}$$

By Eq. (5.30), the radiation-photon residue may be expressed in terms of the zero-frequency-photon wave function

$$\phi_R^\mu(\hat{k}) = \frac{i}{(2\pi)^{3/2}}\int\frac{u^\mu}{u^0 - \vec{u}\cdot\hat{k}}\rho(u)\frac{d^3u}{u^0}. \tag{A27}$$

This allows us to write

$$\langle\phi_1^\Lambda, \phi_2^\Lambda\rangle = \langle\phi_1, \phi_2\rangle + \langle\rho_1, \rho_2\rangle_\Lambda, \tag{A28}$$

where

$$\langle\rho_1, \rho_2\rangle_\Lambda \equiv \int\frac{d^3u_1}{u_1^0}\frac{d^3u_2}{u_2^0}\rho_1(u_1)K_\Lambda(u_1, u_2)\rho_2(u_2), \tag{A29a}$$

$$K_\Lambda(u_1, u_2) \equiv \frac{1}{(2\pi)^3}\frac{1}{2}\int d\hat{k}\frac{u_1^\mu}{u_1^0 - \vec{u}_1\cdot\hat{k}}(-g_{\mu\nu})\frac{u_2^\nu}{u_2^0 - \vec{u}_2\cdot\hat{k}}\times\ln(\Lambda_0^0 + \Lambda_i^0\hat{k}^i). \tag{A29b}$$

On the other hand, with $\rho^\Lambda(u) \equiv \rho(\Lambda^{-1}u)$, we have

$$\langle\rho_1^\Lambda, \rho_2^\Lambda\rangle = \langle\rho_1, \rho_2\rangle - \langle\rho_1, \rho_2\rangle_\Lambda, \tag{A30}$$

because $K_i(\Lambda u_1, \Lambda u_2) = K_i(u_1, u_2)$ and

$$K_f(\Lambda u_1, \Lambda u_2) = K_f(u_1, u_2) - K_\Lambda(u_1, u_2), \tag{A31}$$

by Eq. (A3). Thus we have

$$\langle\rho_1^\Lambda, \rho_2^\Lambda\rangle + \langle\phi_1^\Lambda, \phi_2^\Lambda\rangle = \langle\rho_1, \rho_2\rangle + \langle\phi_1, \phi_2\rangle \tag{A32}$$

and Lorentz invariance is verified.

It is also possible to make a Lorentz-invariant decomposition into radiation-photon and zero-frequency-photon parts. Let the decomposition $K = K_i + K_f$ define a similar decomposition of $\langle\rho_1, \rho_2\rangle$ into a Lorentz-invariant and a frame-dependent part, in virtue of Eq. (5.32),

$$\langle \rho_1, \rho_2 \rangle = \langle \rho_1, \rho_2 \rangle_i + \langle \rho_1, \rho_2 \rangle_f . \tag{A33}$$

Insertion of the identity

$$\frac{\ln(u^0 - \vec{u}\cdot\hat{k})}{u^0 - \vec{u}_1\cdot\hat{k}} = \frac{1}{4\pi}\int d\hat{k}'\frac{1}{1-\hat{k}\cdot\hat{k}'}\left(\frac{1}{u^0-\vec{u}\cdot\hat{k}'} - \frac{1}{u^0 - u\cdot\hat{k}}\right) \tag{A34}$$

into Eq. (A3), gives

$$\langle \rho_1, \rho_2 \rangle_f = \langle \phi_1, \phi_2 \rangle_f , \tag{A35}$$

where

$$\langle \phi_1, \phi_2 \rangle_f \equiv \frac{1}{8\pi}\int d\hat{k}\, d\hat{k}'\, \phi_{1R}^{\mu *}(\hat{k})\frac{(-g_{\mu\nu})}{1-\hat{k}\cdot\hat{k}'} , \times [\phi_{2R}^{\nu}(\hat{k}') - \phi_{2R}^{\nu}(\hat{k})] , \tag{A36}$$

and we have used Eq. (A27). Then, with[37]

$$\langle \phi_1, \phi_2 \rangle_i \equiv \langle \phi_1, \phi_2 \rangle_f + \langle \phi_1, \phi_2 \rangle , \tag{A37}$$

the desired Lorentz-invariant decomposition into zero-frequency-photon and radiation-photon parts is provided by

$$\langle j_1, j_2 \rangle = \langle \rho_1, \rho_2 \rangle_i + \langle \phi_1, \phi_2 \rangle_i . \tag{A38}$$

The noncovariant decomposition given in the text, Eq. (5.37) is more convenient for applications because of the explicit form of the zero-frequency photon kernel, Eq. (5.35).

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[16]The Lagrangian which makes $\Pi^{\lambda\mu}$ a local field was presented previously by D. Zwanziger, Phys. Rev. D 17, 457 (1978), Sec. VII, in a study of a soluble model. The method developed in this reference to master the infrared divergences of the model, is applied, in the present work, to quantum electrodynamics.

[17]Our conventions are $\lambda, \mu, \cdots = 0,1,2,3$; $i, j, \cdots = 1,2,3$; $g^{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$; $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, $\bar\psi = \psi^\dagger\gamma^0$, $\hbar = c = 1$, $\not a = \gamma^\mu a_\mu$.

[18]An alternative is

$$\mathcal{L}_2' = K_\nu(\partial_\mu \Pi^{d\mu\nu} - \partial^\nu L),$$

where K_ν and L are new fields that are varied independently and $\Pi^{d\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu}_{\kappa\lambda}\Pi^{\kappa\lambda}$. It is less convenient because it does not cause $\Pi^{\mu\nu}$ to be derived from a vector field U^ν.

[19]To obtain a gauge which is not purely transverse, one may add

$$\mathcal{L}_3 = -B\partial_\mu A^\mu + \tfrac{1}{2}\lambda B^2 + I_\nu \partial^\nu S,$$

where B and S are new fields that are varied independently.

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[27]This is possible without loss of phase information because $\langle \rho_1, \rho_2\rangle = \langle \rho_2, \rho_1 \rangle$ is real so all the $U_z(\rho)$ commute, $U_z(\rho_1)U_z(\rho_2) = U_z(\rho_2)U_z(\rho_1)$.

[28]An estimate of the remainder in Eq. (5.9) is required to establish this.

[29]This appears to contradict the commutation relation $[a_\gamma(\phi), a_z^\dagger(\rho)] = 0$ for all ρ and ϕ. However, this commutation relation holds on the product space of radiation photons and zero-frequency photons, $\mathcal{I}_\gamma \times \mathcal{I}_z$. On this space, Eq. (7.8) is a subsidiary condition which characterizes the representation space of the free electromagnetic field as a particular subspace of $\mathcal{I}_\gamma \times \mathcal{I}_z$.

[30]This suggestion was made to me several years ago by Professor Francis Low.

[31]Elsewhere, D. Zwanziger, Phys. Rev. D 18, 3051 (1978), the contradiction between the Gupta-Bleuler and the natural infrared coherence conditions has been examined in the simple soluble model of coupling to a conserved classical current.

[32]An alternate possibility, Ref. 10, is to maintain the

natural infrared-coherence condition, Eq. (7.14) and modify the Gupta-Bleuler condition.

[33]The basic reason is that $s(u) \equiv a_z^{\text{in}}(u) + i\rho_{qu}^{\text{in}}(u)$ has its support at spatial infinity, and so commutes with all local observables, Ref. 10, p. 2572, and Ref. 8, Sec. III A. The selection rule for the retarded representation may also be understood in the photon-mass formalism. Assume the initial state is described by a wave function $\phi(p_i)$ and no photons, so the final state (with suppression of photon variables) is $\int S(p_f, p_i, \lambda)\phi(p_i)dp_i$, where λ is the photon mass, and the probability distribution for observing a final charged particle with momentum p_f is $P(p_f) = \int S(p_f, p_i, \lambda)S^*(p_f, p_i', \lambda) \times \phi(p_i)\phi^*(p_i')dp_i dp_i'$, summed over final-state photons. Due to virtual infrared divergences the first factor depends on the photon mass through $\exp[(\alpha/\pi) \times (1 - \psi_{fi}\coth\psi_{fi})\ln\lambda]$ and the second factor through $\exp[(\alpha/\pi)(1 - \psi'_{fi}\coth\psi'_{fi})\ln\lambda]$, whereas the sum over final photon states produces the factor

$$\exp[-(\alpha/\pi)(1 - \psi_{fi}\coth\psi_{fi} - \psi'_{fi}\coth\psi'_{fi} + \psi_{ii}\coth\psi_{ii})\ln\lambda],$$

where ψ_{fi}, ψ'_{fi}, and ψ_{ii} are, respectively, the hyperbolic angles formed by p_f and p_i, by p_f and p_i', and by p_i and p_i'. These do not cancel unless $\psi_{ii} = 0$, or $p_i = p_i'$. This divergence will be avoided if the initial wave-packet state has an appropriate photon train.

[34]$\int d\hat{k} \int dv\, v^2(1 - v^2)^{-2} = \int d^3k\,(k^0)^{-1}$ is the invariant volume element on the hyperboloid.

[35]The integral $I(p_1, \cdots, p_n) = \int d\hat{k}\, F(p_1^0 - \vec{p}_1\cdot\hat{k}, \cdots, p_n^0 - \vec{p}_n\cdot\hat{k})$ is an invariant function of $p_1, \cdots, p_n$ provided $F(x_1, \cdots, x_n)$ is a homogeneous function of $x_1, \cdots, x_n$ of degree -2. For the first term in Eq. (A4) $F(x_1, x_2) = (x_1x_2)^{-1}$. See Ref. 8, Eq. (3.10).

[36]See Ref. 10, Eqs. (A6)–(A19) for a more detailed discussion.

[37]In fact, $\langle\phi_1, \phi_2\rangle_i$ is the Lorentz-invariant inner product that was previously constructed by hand, precisely because it had this property, without the benefit of being derived from a local Wightman function, Ref. 10, Eq. (A19).